Patterns and Functions

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THE PROBLEMS in this chapter aim explicitly at encouraging students to look for patterns in data, tables, and graphs and make generalizations, first in words and then in symbols. As students look for patterns among quantities in the problems and seek ways to represent those relationships, they encounter variables naturally as a way to make sense of problem situations. If we encourage students to generalize—first in words, tables, and graphs and then in symbols—the symbols become more than just symbols: they represent the reasoning and patterns that the students used in solving the problem. Writing algebraic expressions to represent relationships, judging the equivalence of algebraic expressions, and judging equivalence among tables, graphs, and algebraic expressions all provide a solid basis for studying algebra.

Problems 1–6 in this chapter begin with explorations of some important relationships found in beginning algebra courses—linear, quadratic, cubic, and exponential functions. Problems 7–9 explore important features of functions more generally.

You can use each problem with students with various levels of algebraic experience. For example, for beginning algebra students, an appropriate approach might be to generalize the relationships embedded in the problem in words, tables, and graphs, and then move to simple symbolic statements if students are developmentally ready. Problems that require more complicated symbolic statements are appropriate for students who have a bit more experience with symbolic manipulation. Students can solve many such problems using tables or graphs to represent and reason about the relationships. The Teacher's Notes that accompany a given problem discuss the specific mathematics embedded in that problem.

Problem 1: The Race—Linear Functions

(Adapted from *Patterns and Functions,* investigation 1, pp. 55–57 [Phillips et al. 1991])

Pat and his older sister, Terri, run a race. Pat runs at an average of 3 meters every second, and Terri runs at an average of 5 meters every second. In a 100-meter race, Pat gets a 40-meter head start because he runs at a slower pace. Who wins the race? Explain your reasoning.

Teacher's Notes

This problem can introduce linear functions and, in particular, the use of tables, graphs, and symbols to represent and communicate a linear relationship's salient features.

Solution to Problem 1

The race ends in a tie.
See the discussion in the Teacher's Notes for examples of students' strategies.
Some students may use a table as shown in figure 4.1.

	Distance (meters)	
Time (seconds)	**Terri**	**Pat**
Start	0	40
1	5	43
2	10	46
3	15	49
?		
10	50	70
?		
20	100	100
t	$5t$	$40 + 3t$

Fig. 4.1. Distances for Terri and Pat

Common Core Mathematical Practices
When students choose various ways to represent their reasoning about the problem numerically, with graphs, tables, or equations, they learn to ***use appropriate tools strategically.***

Let groups of students share their solutions. Some may just use an arithmetic argument to figure the time each person takes to reach 100 meters. For example, if Terri runs 5 meters per second, then it will take her 100 meters divided by 5 meters per second, or 20 seconds, to run the race. A similar argument for Pat shows that he also takes 20 seconds to run the race. In this example, each person takes 20 seconds to reach the finish line, so the race ends in a tie. Some students may look at the distance between the two racers. For example, at the start, they are 40 meters apart. After one second they are 38 meters apart; after two seconds 36 meters; and so on. At 20 seconds, they are 0 meters apart. A few students may use a table as shown in figure 4.1. Some students may use equations if they have had some previous experience with algebra.

Extensions

Ask more questions about the race, such as the following:

- Who wins the race if it is 150 meters? By how many meters?
- How long has each person been running when Pat reaches the 75-meter mark?

Use the students' table or create one (fig. 4.1) with the class to open up the discussion and highlight important mathematical aspects of linear functions. Start by asking the following questions:

- What patterns do you observe in the table?
- What are the variables?
- What is the relationship between the two variables?
- What do these relationships look like on a graph? In an equation?

Students may have several observations. As the time increases by 1 second, Terri's distance increases by 5 meters and Pat's distance by 3 meters. To determine Pat's distance, students can also successively add 3 meters to the previous entry for Pat's distance. The constant rate of change between the two variables identifies this relationship as a *linear function*. As the independent variable (x, or t in this example) increases by a constant amount, the dependent variable (y, or D in this example) also changes by a constant amount.

Students should know that distance equals rate times time and be able to write those patterns in symbols. If t represents time in seconds, D_T represents the distance Terri runs after t seconds, and D_P represents the distance that Pat runs after t seconds, then $D_T = 5t$ and $D_P = 40 + 3t$. The running rate for each person is the coefficient of t in the equation. The 40-meter head start is the constant term in the equation that represents Pat's relationship. It is called the y-intercept on that linear function's graph—the place where the graph of the equation $D_P = 40 + 3t$ intersects the y-axis. The y-intercept for the equation $D_T = 5t$ is 0. This equation's graph intersects both axes at 0 (fig. 4.2). In general, one can represent a linear function by the equation $y = b + mx$, where b is the function's y-intercept and m is its slope.

Answering the question about how a graph represents the linear relationship, students might respond that the graph would be a straight line. This question presents a good opportunity to connect the related patterns of change that the students have expressed in words or observed in a table or symbolic statement. For example, pick two successive entries in the table (e.g., $t = 1$ and $t = 2$) and use them to find the corresponding two points on the graph. As time goes from 1 to 2, distance goes from 5 to 10 for the data representing Terri's relationship. These two entries in the table correspond to the points (1, 5) and (2, 10) on the graph for Terri's relationship. On the graph, the horizontal distance between these two points is 1 and the vertical distance is 5. The ratio, $^5/_1$, of these two distances is called the *slope*. The ratio is the same for any two points on the line or corresponding entries in the table. The slope is also the coefficient of the x in the general equation for a linear function, $y = mx + b$. The graph in figure 4.3 illustrates

Common Core Mathematical Practices

When students observe particular patterns of change between two variables embedded in the problem, they learn to ***reason abstractly and quantitatively,*** and ***look for and make sense of structure.***

this relationship by the two points (0, 0) and (10, 50) and the two points (20, 100) and (35, 175).

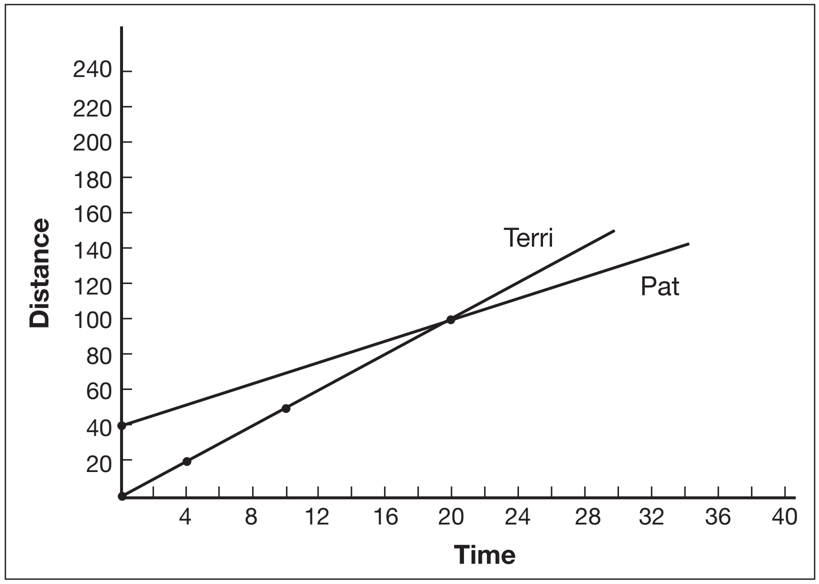

Fig. 4.2. Results of the race between Terri and Pat

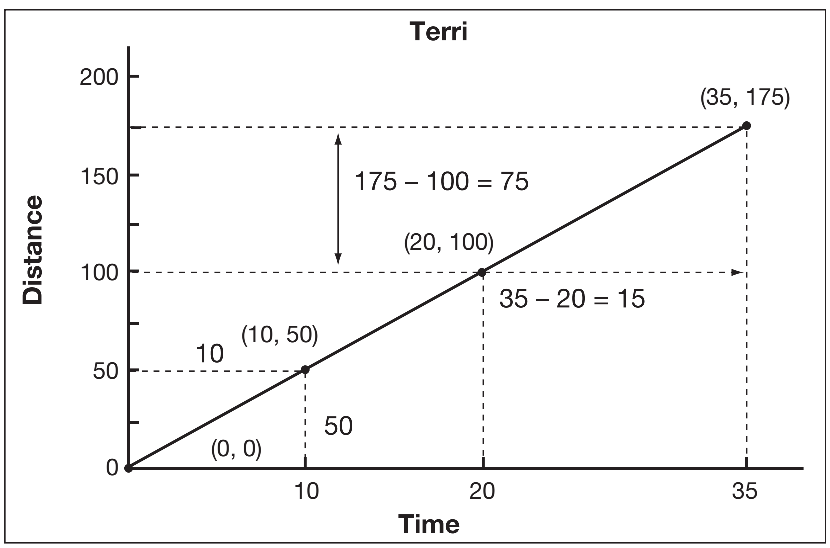

Fig. 4.3. The slope of the line that represents the relationship for Terri's race

Notice that for the points (0, 0) and (10, 50), the slope is the ratio

$$\frac{\text{vertical distance}}{\text{horizontal distance}} = \frac{50}{10} = \frac{5}{1}.$$

For the points (20, 100) and (35, 175), the slope is the ratio

$$\frac{\text{vertical distance}}{\text{horizontal distance}} = \frac{75}{15} = \frac{5}{1}.$$

Several of this chapter's following problems will revisit linear functions.

Problem 2: Rumors—Exponential Functions

(Adapted from *Patterns and Functions,* investigation 1, pp. 6–8 [Phillips et al. 1991])

In Bugsville, USA, Tara, a student at Swat Middle School, decides to start a rumor that the town of Bugsville will declare September 14 as National Bug Day and will close all the schools for the day. She tells two students the rumor with the instructions that each of those students is to repeat the rumor to two more students the next day and that each of those students will tell two more students on the third day, and so on.

- How many students will know the rumor on day 10?
- If Tara starts her rumor on September 1 and 8000 students attend school in the district, will all the students hear the rumor before September 14? Explain your reasoning.

Teacher's Notes

You can use this problem to introduce students to exponential relationships and how to represent the growth pattern associated with those relationships in tables, graphs, and symbols.

Before students explore the problem, be sure that they understand how the rumors will spread. For example, at the start of the first day (or day 0), 1 student, Tara, knows the rumor. At the end of the first day, 2 more students know the rumor for a total of 3. At the end of the second day, 4 more students know the rumor for a total of 7. At the end of the third day, 8 more students know the rumor for a total of 15, and so on. At this point, ask the class if they think that all the students will hear the rumor before September 14. Many students will conjecture that not all the students in the district will hear the rumor before September 14.

Students may propose a variety of ways to find the answer. As you discuss the answers, look for one that used a table, like in figure 4.4, or generate that table with the class. As you generate the table, ask students to predict the next entry. Stop the pattern on day 14. Once you have generated the table, you can promote algebraic understanding by asking many interesting questions about the patterns in the table:

- On what day will approximately half the student population hear the rumor?

Common Core Mathematical Practices

When students explain their reasoning about whether all the students will hear the rumor before September 14, they learn to ***make sense of problems and persevere in solving them*** and to ***construct viable arguments and critique the reasoning of others.***

Solution to Problem 2

See the Teacher's Notes for a discussion of student-devised strategies. Some students may use a table to guide their reasoning (see fig. 4.4).

Day	The Number of New People Who Hear the Rumor on a Given Day	Total Number of People Who Have Heard the Rumor on a Given Day, Including Tara
Start of 1st day or Day 0	1	1
1	2	3
2	4	7
3	8	15
4	16	31
5	32	63
6	64	127
7	128	255
?	?	?
10	1024	2047
11	2048	4095
12	4096	8191
13	8192	16383
N	2^n	$2^{n+1} - 1$

Fig. 4.4. An analysis of the rumor problem

- If the rumors continued, on what day would 65,536 students hear the rumor?
- On what day would a total of 524,287 students hear the rumor?
- What two quantities (variables) are changing? Describe the pattern of change between the two quantities.

As the number of the day increases by 1, the number of new people who hear the rumor doubles or increases by a factor of 2. This relationship is called an *exponential relationship*. In the second relationship, the total number of rumors on any given day, as the day's number increases by 1, the total number of rumors more than doubles. It does not increase by a constant factor. The second relation-