Chapter

1

Geometry:
The Big Ideas and Essential Understandings

Geometry, as an integral area of mathematics, often loses its way in the middle school grades. An increasing emphasis on early algebra, as well as on working with variables in deriving general formulas, can make what is essentially geometric about a situation fade into the background or even disappear. For this reason, we have opted to start this chapter with an assertion about the centrality of geometry, both as an area of study in its own right and as a means of providing insight and understanding for other areas of mathematics. We then turn to the importance of imagery, the nature of geometric figures, and the influence of tools and geometric activities.

Four big ideas and several smaller, more specific essential understandings provide the structure of this chapter. The big ideas and all the associated understandings are identified as a group below to give you a quick overview and for your convenience in referring back to them later. Read through them now, but do not think that you must absorb them fully at this point. The chapter will discuss each one in turn in detail.

Big Idea 1. Behind every measurement formula lies a geometric result.

> **Essential Understanding 1a.** Decomposing and rearranging provide a geometric way of both *seeing that* a measurement formula is the right one and *seeing why* it is the right one.

> **Essential Understanding 1*b*.** In addition to decomposing and rearranging, shearing provides another geometric way of both *seeing that* a measurement formula is the right one and *seeing why* it is the right one.

Big Idea 2. Geometric thinking involves developing, attending to, and learning how to work with imagery.

Essential Understanding 2*a*. Geometric images provide the content in relation to which properties can be noticed, definitions can be made, and invariances can be discerned.

Essential Understanding 2*b*. Symmetry provides a powerful way of working geometrically.

Essential Understanding 2*c*. Geometric awareness develops through practice in visualizing, diagramming, and constructing.

Big Idea 3. A geometric object is a mental object that, when constructed, carries with it traces of the tool or tools by which it was constructed.

Essential Understanding 3*a*. Tools provide new sources of imagery as well as specific ways of thinking about geometric objects and processes.

Essential Understanding 3*b*. Geometric thinking turns tools into objects, and in geometry the process of turning an action undertaken with a tool into an object happens over and over again.

Big Idea 4. Classifying, naming, defining, posing, conjecturing, and justifying are codependent activities in geometric investigation.

Essential Understanding 4*a*. Naming is not just about nomenclature: it draws attention to properties and objects of geometric interest.

Essential Understanding 4*b*. Definition can both generate and reflect structure: definitions are often dependent on a specific classification.

Essential Understanding 4*c*. Conjectures can emerge out of a problem-posing process that generates claims that need to be justified.

The Tangled Relationship between Measurement Formulas and Geometry: Big Idea 1

Big Idea 1. *Behind every measurement formula lies a geometric result.*

Measurement (and the use of formulas to produce numerical answers to measurement tasks) can seem to be the primary focus of geometry in the middle grades. For some people, however—including the authors on most days—most measurement lies outside geometry—despite the "-metry" in the word *geometry*!

The whole is the sum of its parts

Essential Understanding 1*a*. *Decomposing and rearranging provide a geometric way of both* seeing that *a measurement formula is the right one and* seeing why *it is the right one.*

When we are comparing the areas of different shapes, our first impulse is often to think numerically: do the measurements, in some square units of measure, yield the same number? Measurements may spring to mind quickly, because we have powerful formulas for determining them, such as the one for the area of a triangle.

In the ancient Greek approach to such a question, numbers would be completely absent. The area of a triangle, say, would be compared with the area of another geometric shape. But which shape? That would depend on the context. One very powerful means through which such a comparison might occur involves decomposition and rearrangement. Reflect 1.1 involves rearrangement with tangram pieces.

Reflect 1.1

Rearrange the seven pieces of a tangram into different designs. Start with a square. Then make other shapes. Convince yourself that your shapes all have the same area.

The shapes in figure 1.1 look quite different, and your first impression might be that some of them cannot possibly have the same area. But each one is composed of the exact same seven tangram pieces, and this means that they must be equal in area. A key fact about area is that it is *additive*: the area of the composition of two figures is the sum of their areas. This assures us that each one of the shapes in figure 1.1 has to have the same area as every other one. Thinking about area in this way focuses attention on comparing areas through decomposition and rearrangement, rather than by calculating numerical values. Reflect 1.2 invites you to compare areas of triangles and quadrilaterals.

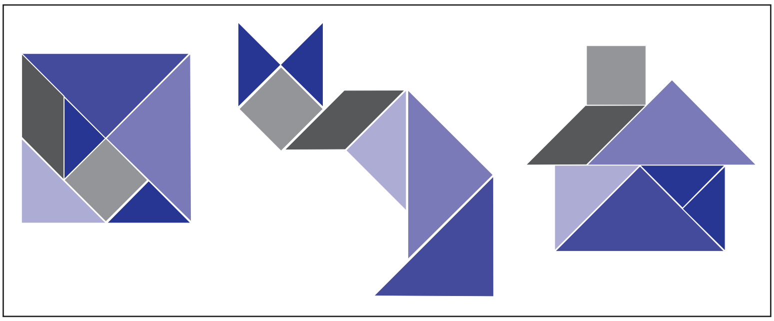

Fig 1.1. Shapes made with tangram pieces

Reflect 1.2

Draw a triangle. Now draw a rectangle that has the same area as your triangle. Can you draw a different quadrilateral (a non-rectangular one, perhaps) that also has the same area as the starting triangle? What about a polygon with more than four sides?

We could do the very same kind of work with triangles as with the tangram shapes, although slightly less playfully, to say something about their areas. In figure 1.2, an arbitrary triangle *T* is first decomposed into two parts *N* and *M* by the dotted line representing an altitude of the triangle. A second dotted line that is parallel to the base and passes through the midpoint of the altitude leads to a further decomposition of the original triangle into four parts, which we have labeled *A*, *B*, *C*, and *D*. By rearranging *C* and *D* (by rotating them 180 degrees about the points of intersection of the halfway line and the respective sides of the triangle), we transform triangle *T* into rectangle *R*. The triangle in this figure is acute. Any triangle, whether acute or obtuse, always has a longest side (though in the case of an isosceles or equilateral triangle this side is not unique). If we consider a rotation of an obtuse triangle—a change that does not alter area—we can think of that obtuse triangle as oriented as in figure 1.2.

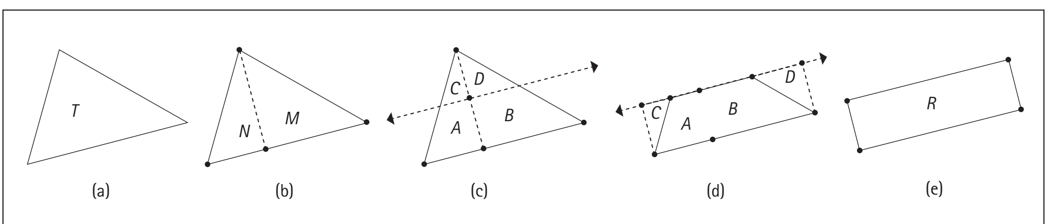

Fig. 1.2. Decomposing and rearranging a triangle into a rectangle

The diagrams in figure 1.2 tell a compelling story. But we can make the story even better by explaining why shape *R* is a rectangle. Figure 1.3a zooms in on just the left side of the configuration in figure 1.2c. Because the line passing through the midpoint of the altitude is parallel to the base, it cuts the altitude at right angles. The right angle marked in figure 1.3a is at one "corner" of *R*. How do we know that the other corner of *R* (the lower left vertex in figure 1.3b) is also a right angle? Because we are doing a half-turn, we know that the half-altitude of *C* (indicated by the double hash marks on one leg of triangle *C*), which was perpendicular to the base of *T* to start with, will still be perpendicular to the base of *T*, and parallel to the original altitude. A similar argument works for triangle *D* in relation to *B*.

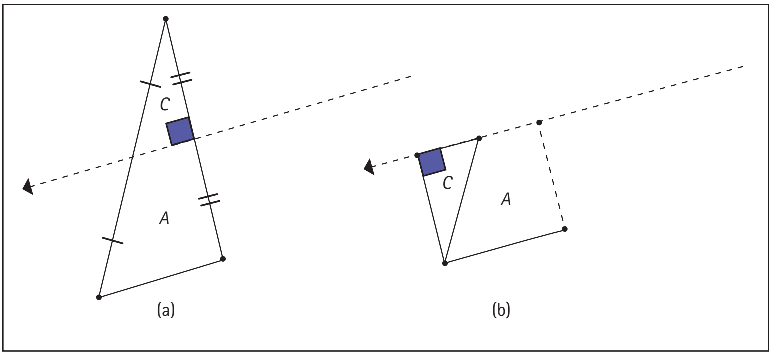

Fig. 1.3. Rotating *C* to create the left edge of the rectangle *R*

In this way, we can say that the area of *T* is the same as the area of *R*, so we have found the area of a triangle in terms of the area of an equivalent rectangle. We note that the length of the rectangle is the same as the base of the original triangle and that its width is exactly one-half of the height of the triangle. In this manner, we can compare the area of the triangle with the area of a rectangle, leading to the well-known formula for the area of a triangle: $A = \frac{1}{2}bh$.

If *T* is an obtuse triangle, we can still undertake an identical process, provided that we choose the longest side as the base of the altitude (see the alternative shown in fig. 1.4). However, we could also create a comparable dissection argument for this obtuse triangle by choosing a different altitude from the one shown in figure 1.4. We would still create the rectangle of base times half-height. The paired triangles in figure 1.5d and 1.5e are the same in that one is a rotation through one-half the angle of rotation of the other (this claim still needs a formal proof), but the sequence of decomposition and rearrangement in figure 1.5 explains *why* the same formula holds even for an obtuse angle triangle (though notice that the *base* and the *height* are different here—why does this not seem to matter in the formula?).

The word *height* sometimes means the same thing as *altitude* and sometimes is usefully distinguished, referring to the *measure* of the altitude. Furthermore, *height* is sometimes interpreted as vertical height, whereas an altitude can lie in any direction but is always perpendicular to a corresponding base.

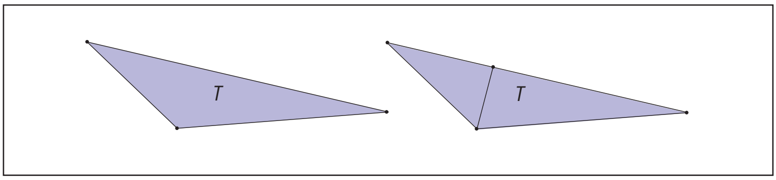

Fig. 1.4. Picking the height that corresponds to the longest side of the triangle

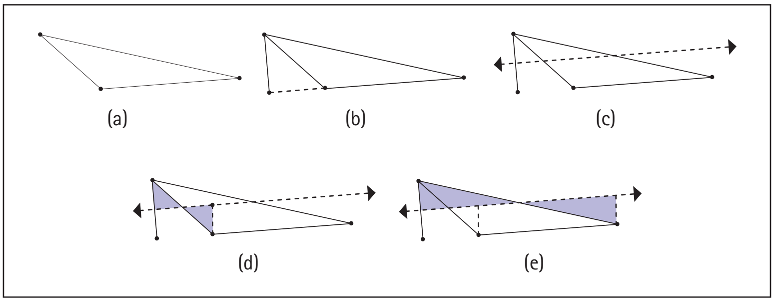

Fig. 1.5. The case of the decomposition of an obtuse-angled triangle, showing area is base times half-height

Through a different process of decomposition and rearrangement, we can also decompose the triangle *T* in figure 1.2 into two different triangles, *N* and *M* (see fig. 1.6). But this time, we draw the dotted line parallel to the base and passing through the third vertex, so that it intersects the two lines perpendicular to the base and passes through its endpoints to produce a rectangle containing copies of triangles *N* and *M*. Consequently, this rectangle will have twice the area of *T*, and this means that it can be split into two congruent rectangles, each having the same area as *T*. This immediately shows that the area of *T* is one-half of the rectangle's base times its height.

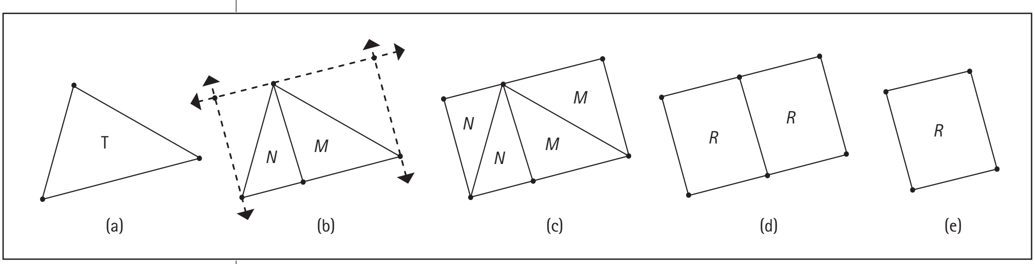

Fig. 1.6. Decomposing and rearranging a triangle into a different rectangle

Further, if the split occurs along the line perpendicular to the base and through the third vertex of *T*, then we get two copies of rectangle *R*, each of whose length is equal to half the base and each of whose width is equal to the height of the triangle, leading to the formula $A = (\frac{1}{2}b)h$. As was the case with the previous approach to decomposition, an obtuse triangle can be handled by taking the longest side to be the base. We offer you the challenge of creating a set of diagrams showing this decomposition approach for an obtuse triangle, in the manner of figure 1.6.

We chose to compare triangle T with rectangles in both cases, and for a good reason. Using our principle of decomposition and rearrangement, we might also have chosen to compare T with a variety of other shapes, as shown in figure 1.7. The first one (fig. 1.7a) shows a different way of rearranging C and D from the one shown in figure 1.2c, producing a heptagon instead of a rectangle. Figure 1.7b shows a division of the large rectangle (composed of two copies of R) into two right triangles—thus showing the area of T by comparing it with another triangle. Figures 1.7c and 1.7d show alternate ways of halving the rectangle—into trapezoids and hexagons, respectively. The fact that all these shaded shapes have the same area, which is the area of T—as well as the area of the rectangle in figure 1.2e *and* the rectangle in figure 1.6e—is a consequence of the decomposing and rearranging.

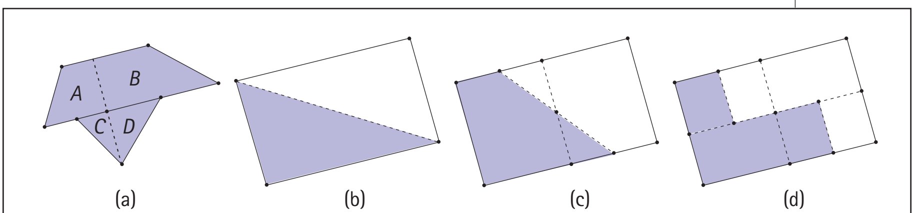

Fig. 1.7. Decomposing and rearranging a triangle into other shapes

Let's call the rectangle in figure 1.2e R_1 and the rectangle in figure 1.6e R_2. In comparing T with both R_1 and R_2 by using the triangle area formula, we notice the rather surprising fact that although the two rectangles R_1 and R_2 have distinct shapes, they have the same area (see fig. 1.8). This, of course, is something that the formula itself asserts, since algebraically, $(\frac{1}{2}b)h = b(\frac{1}{2}h)$, where b and h are now seen as numbers—length measures of sides. A third algebraically equivalent form is $\frac{1}{2}(bh)$. This formula has a geometric interpretation as one-half the area of a rectangle, as suggested by figures 1.7b, 1.7c, and 1.7d. However, bear in mind that the algebraic results arose from quite *different* ways of *seeing* the decomposition or rearrangement of a triangle geometrically. Algebra frequently masks distinctions that are clear and distinct in the geometry.

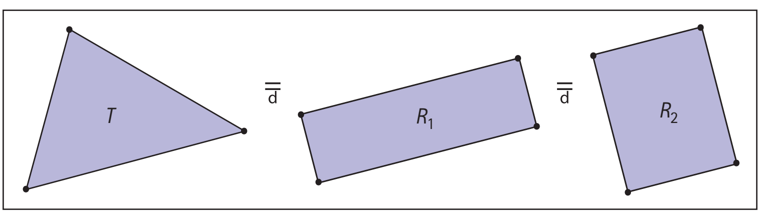

Fig. 1.8. Triangle T and rectangles R_1 and R_2 have the same area

The symbol $\underset{d}{=}$ is sometimes used to indicate that two geometric figures are "equivalent by dissection," which means, in particular, that they have the same area.

This is not the only sense in which the algebra asserts a sameness that the geometry does not so obviously show. In figure 1.2, one side of triangle T was chosen as the base, but any of the three sides *could* have played that role, with one of three different, corresponding segments then being called the *height*. Working with this first approach to decomposition and rearrangement, we obtain rectangles Q_1, R_1, and S_1, all different from one another but each having the same area as triangle T (see fig. 1.9a). Working with our second approach, we obtain the triangles R_2, Q_2, and S_2 (see fig. 1.9b). The triangle area formula is powerful because it works for any of the pairs of bases and corresponding heights that we choose and because it asserts that these six seemingly different rectangles are all equal in area to one another—and to T. This result is why we can get away with being casual about the form of the area formula for a triangle, asserting simply that the area is one-half base times height.

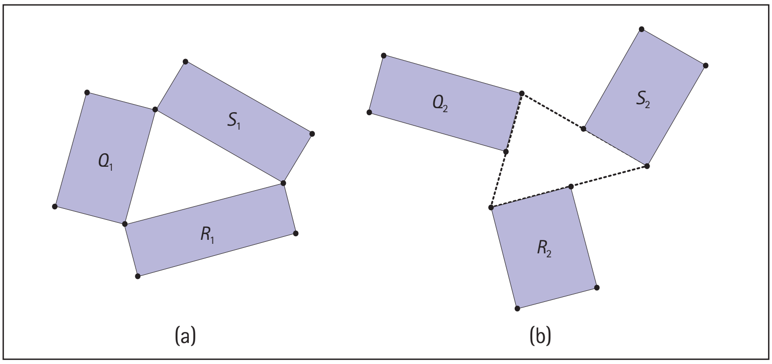

Fig. 1.9. Decomposing and rearranging a triangle into rectangles with different bases and corresponding heights

Figure 1.9 shows six rectangles with the same area as one another and as triangle T. Reflect 1.3 asks you to consider parallelograms that have exactly twice the area of a triangle.

Reflect 1.3

Given a triangle T, how many different parallelograms can you find that are exactly twice the area of T?

In textbooks, the area of a triangle is often compared with the area of a parallelogram, since any parallelogram can be divided into two congruent triangles just by drawing in one of its diagonals. Equivalently, any triangle can be rotated around the midpoint of one of its sides to produce another triangle that can be composed with the original triangle to form a parallelogram. So the area of any triangle can easily be compared with the area of a parallelogram. Dwelling on this idea by exploring the different ways in which one can produce a parallelogram out of a triangle helps to emphasize the idea of area as a *comparison between two shapes* rather than as a number. A premature shift to the algebraic formula can get in the way of developing the geometric insights that underlie any measurement formula.

In the Expectations for grades 6–8 of NCTM's Geometry Standard (NCTM 2000), we find, "All students should ... create and critique inductive and deductive arguments concerning geometric ideas and relationships such as congruence, similarity, and the Pythagorean relationship" (p. 232). As our final example of decomposing and rearranging in relation to Essential Understanding 1*a*, we look at the most well-known theorem of school mathematics—namely, the Pythagorean theorem. Here, and again at the end of the next section in connection with Essential Understanding 1*b*, we explore ways of seeing *why* this theorem is true. Our exploration begins with Reflect 1.4.

Essential Understanding 1*b*
In addition to decomposing and rearranging, shearing provides another geometric way of both seeing that *a measurement formula is the right one and* seeing why *it is the right one.*

Reflect 1.4

Below are two slightly different ways in which some people state the Pythagorean theorem:

1. For any right triangle in the plane, the square on the hypotenuse is always equal to the sum of the squares on the other two sides.
2. For any right triangle in the plane, the square of the hypotenuse is always equal to the sum of the squares of the other two sides.

What difference do you notice and what mathematical significance, if any, does this difference have?

As you might expect from our inclusion of the Pythagorean theorem in our discussion of Big Idea 1, we are interested in this theorem as a theorem of geometry, as opposed to its common algebraic interpretation. Further, here we are interested in attempting to see *why* the theorem is true, by using decomposition and rearrangement. The difference between the two statements in Reflect 1.4 is the choice of preposition (*on* versus *of*) with regard to the relationship between the square and the sides of the triangle. In talking about the square *on* a side of the triangle, we emphasize that a geometric object is placed in relation to another geometric object, as shown in figure 1.10.

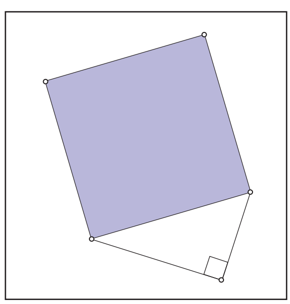

Fig. 1.10. The shaded square is "the square on the hypotenuse."

The square *of* the hypotenuse refers to the numerical operation of squaring a number (representing the *length*

Big Idea 1

Behind every measurement formula lies a geometric result.

of the hypotenuse). Choosing this preposition is enough by itself to put us into an arithmetic or algebraic frame of mind, in which the Pythagorean relationship is conceived as an equality of numbers rather than as a sameness of areas. If we are thinking about sameness of areas, decomposing and rearranging the square on the hypotenuse so that it fits exactly into the other two squares is a plausible avenue for an informal (leading to a formal) proof. The question then becomes, "How should the square on the hypotenuse be decomposed? If we were going to cut the square into pieces, how do we know where to cut?"

One broad heuristic is to overlap the relevant interior regions as much as possible and then concentrate on what is left. Figure 1.11 shows first one, then a second, reflection of squares across a side of the original right triangle (transforming *ACDE* to *ACD′E′* and, subsequently, *BCGF* to *BCG′F′*). The third construction involves dropping a perpendicular from *D′* onto *CH*.

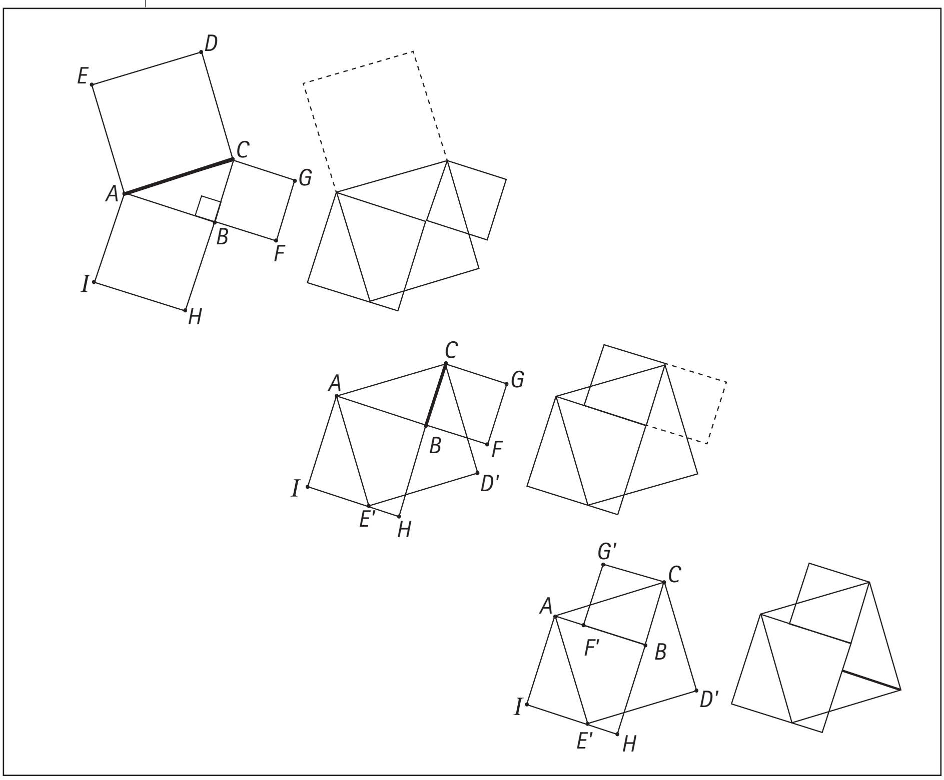

Fig. 1.11. A sequence of three constructions involved in a dissection proof of the Pythagorean theorem

Figure 1.12 shows how the five polygons into which the square on the hypotenuse has been cut in figure 1.11 can then be rearranged to fit exactly onto the two squares on the other sides of the right triangle. This proof is still informal, in the sense that we have not yet shown the exactness with which these pieces fit together, nor do we, for instance, yet know for certain that one vertex of the square on the hypotenuse (point E in fig. 1.11) will always fall precisely onto the side of one of the other squares (E'). However, other than this uncertainty, the precise points at which the cuts are to be made are determined by the constructions.

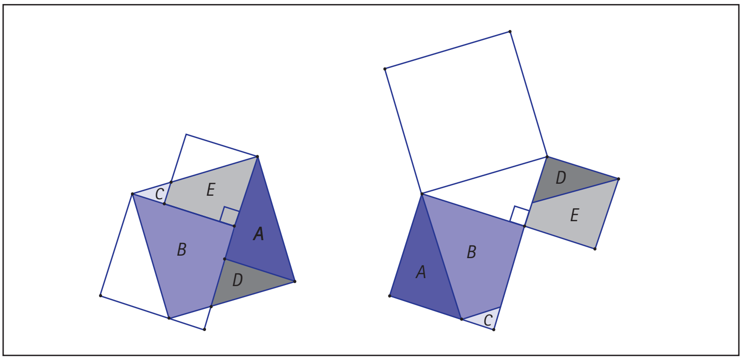

Fig. 1.12. Rearranging the pieces of the square on the hypotenuse to fit into the two squares on the sides of the right triangle

Further empirical as well as heuristic evidence can be found by constructing these figures in a dynamic geometry environment (DGE), such as The Geometer's Sketchpad or Cabri Geometry, and, for example, dragging the vertices of the original right triangle to see how the configuration and its decomposition hold together. In particular, looking at special cases, as when the right triangle is isosceles or when one leg is very small, can help provide a continuous sense of cases.

In our discussion of Essential Understanding 1*a*, we have focused on area formulas relating to triangles, but similar points underlying measurement formulas hold for all such results taught in school geometry, both in the plane and in three dimensions. For three-dimensional shapes, decomposing and rearranging solids can accomplish the work of making the formulas geometrically meaningful, though this can prove much more challenging.

To start to *see* the volume of a square-based pyramid, for instance, one can begin with a simple decomposition of a cube into three identical and specific square-based pyramids, as shown in figure 1.13. From this decomposition, one can see that the volume of each pyramid is exactly one-third of the volume of the original cube. For this decomposition, we have had to use a very particular type of square-based pyramid: it has the same height as the sides of the square base and it is right-angled at one corner. Both these attributes are required so that parts of these pyramids match the outside faces and corners of the starting cube. Consequently, we have shown this relationship only for this very specific type of pyramid. In the discussion that follows of Essential Understanding 1*b*, we will work toward showing that the same volume formula holds for a general pyramid.

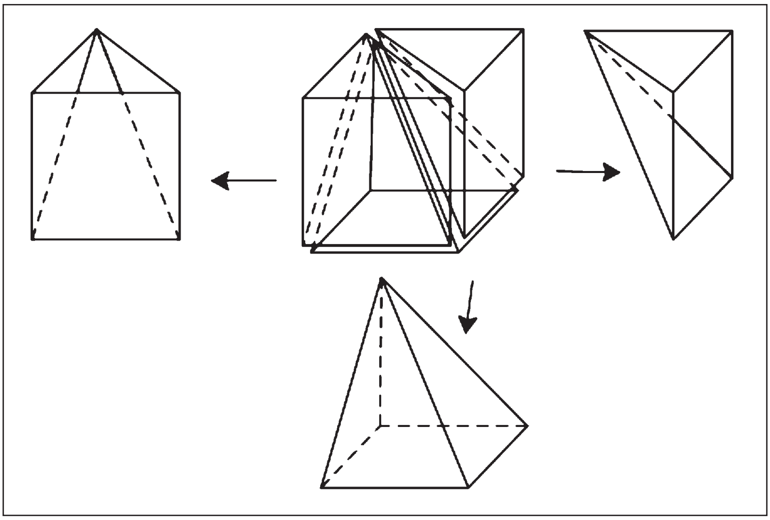

Fig. 1.13. Decomposition of a cube into three identical square-based pyramids

Continuous decomposition and infinite rearrangement

Essential Understanding 1*b*. *In addition to decomposing and re-arranging, shearing provides another geometric way of both* seeing that *a measurement formula is the right one and* seeing why *it is the right one.*

Shearing is a geometric transformation that makes a continuous and systematic change to a figure in such a way that it does not alter the figure's area. With Reflect 1.5, we begin our exploration of shearing by focusing on how we might know that two triangles have the same area.

Reflect 1.5

Draw five different triangles that have the same area. How do you know that their areas are the same?

Working from the formula, you might have generated pairs of numbers that have the same product and then used these numbers to build the sides of different triangles. Or perhaps you worked more geometrically by starting with rectangles that have the same area