CHAPTER

3

Making sense of addition and subtraction algorithms

Case 12
The pink way
Lynn
Grade 2, May

Case 13
Subtraction and invented algorithms
Lynn
Grade 2, April (one year later)

Case 14
Partitioning subtraction problems
Nadine
Grade 4, February

Over the last several years, Lynn, a second-grade teacher, has been thinking hard about the issues explored in chapters 1 and 2. As she tries to sort out what her second graders need to learn and as she reviews what she has done in different years, she poses big curricular questions for herself. What do second graders need to learn about adding, subtracting, and place value? What kinds of mathematical tasks will help them learn these things? And what is the role of the standard algorithms conventionally taught in the United States?

In the first of her two cases, Lynn reviews what she has done in her first two years of teaching second grade and shares her current thoughts. In the second case, written one year later, she has different ideas.

In case 14, Nadine relates one lesson with her fourth graders in which she, too, is addressing issues of subtraction, place value, and the U.S. standard algorithm.

Before taking on Lynn's curricular questions for yourself, first examine the thinking of the students in the cases. As you read, take notes on these questions:

- How do the two groups of students in case 12 experiment with different addition algorithms?
- In case 13, consider second-grader Fiona's work: Where does she get stuck and how does she sort herself out?
- In case 14, in what ways are Hallie's and Janet's subtraction procedures the same and how are they different?

After reading the chapter, reread this introduction.

case 12

The pink way

Lynn
GRADE 2, MAY

As the end of my second year of teaching second grade approaches, I find myself consumed again with questions about what the children are thinking when they add and subtract with regrouping. These questions seem broad and deep, and range from the level of the individual child—"What is he thinking?"—to the level of math education in the United States—"What is the role of traditional algorithms when the focus of teaching is on student understanding?"

Last year I used trading games (chip trading and rods trading) to get at the issues I thought were involved in regrouping. I taught the children how to play the games (roll the dice, take tens rods and ones rods to correspond to your roll, and represent your accumulating total with rods), making sure they knew to trade in 10 ones for a ten rod and 10 tens for a hundred rod. Later I gave them word problems that involved regrouping, thinking they would apply their new knowledge to solve them. I was dismayed to discover that very few, if any, children made the connection between trading ones rods for tens rods and adding two-digit numbers when the ones column totaled more than ten. Some of them did learn the algorithm—I think because they were able to remember the steps and wanted to do as they were told.

I am not convinced that any of them really understood how quantities combine. Some of the children did not learn the procedure, despite an obvious desire to do well and please their teacher. Some children, when faced with a problem that involved regrouping, just stopped. Some added on, counting by ones. Some ignored the dilemma of having, for example, 15 in the ones column, and came up with solutions that did not make sense.

This November I taught the rods trading game again. I did not dare *not* to. I was determined to find an effective way to help the children *connect* the game to the addition it is meant to represent.

It did not work. Every way I thought of to make the connection clear was so confusing to the children that they could not even understand how to "do" the paper, much less say, "Oh! This adding is just like trading rods!" or "Oh! This rods trading is the same thing as adding!" It was frustrating for me and bewildering for the children.

A significant difference this year was that I included lots of word problems along with rods trading throughout the whole year. I did not try to get the children to use the algorithm to solve the problems and I did not try to push the connection to trading rods. Instead I asked children how they solved the word problems—because they all did—and recorded their procedures on audiotape and on posters.

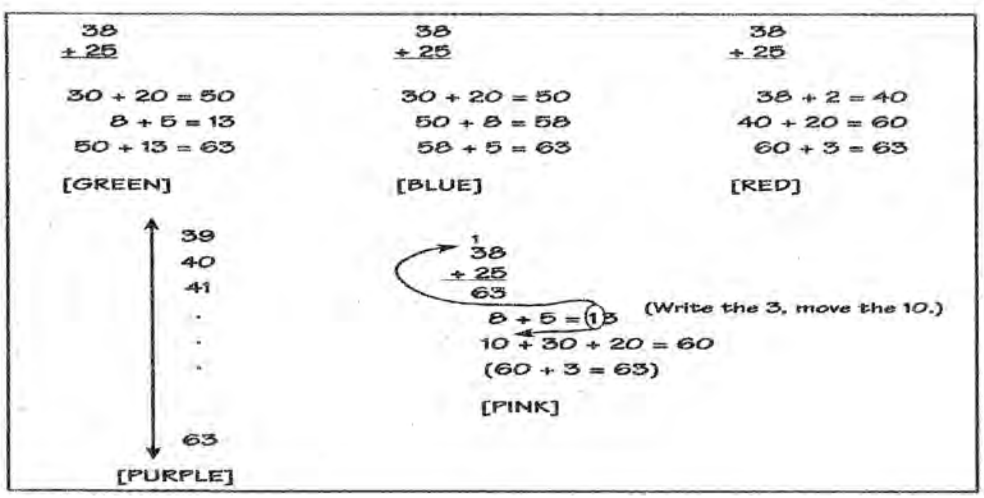
Trading rods had no discernible impact on how the children thought about addition and subtraction problems. Some children, who could answer questions about how many tens, how many ones, location of the tens and ones, and so on, would still count by ones when solving two-digit addition problems. Those children who apparently had a deeper understanding of place value still added the way they always had—tens first and then the ones. They would occasionally use the words *trade in* to describe how they dealt with having more than 10 ones. Some children could not answer questions about where the tens and ones were or did not understand what counting by tens really meant.

For a while we left trading games and regrouping as topics and worked on other things for the next several months, including geometry and many, many word problems.

Then it was May and I felt the need to somehow revisit the algorithm. My thinking went smething like this: Third grade approacheth [sic]. It is not up to me to say that we will all abandon the conventional algorithm. The children will be expected to know it. If I don't teach it to them, someone else will, perhaps wondering why I didn't do my job, which means now they have to, and now the kids are way behind and they'll test poorly the next year in fourth grade.

At some point I would like to have a discussion with my colleagues about the algorithm and the ways kids solve problems. I fantasize that we will all agree that children should be encouraged to think flexibly and solve problems in ways that make sense to them and in ways they can explain without saying, "and then you do this because my teacher said to."

Meanwhile, I am still facing the end-of-year dilemma over the algorithm, so last week I prepared to approach addition with regrouping again. I made a poster showing five different methods of adding 38 + 25, each method written in a different color. The first four were methods that second graders had articulated when explaining their thinking in November; three of these (the green, blue, and red ways) involved adding the tens first. The purple way showed counting on from 38 by ones. And, last, the poster showed the traditional algorithm in pink.



I met with a group of ten children who could solve regrouping problems pretty easily. I gave them the following word problem, asking them to pay attention to how they solved it and to please write something that would explain their thinking:

> Ms. Kosaka and Ms. Rivest were watching kids playing outside. They counted 38 children in and around the climbing structure. They saw 25 kids playing freeze tag on the field. How many children did they see?

The children got right to work and solved the problem quickly. They were eager to explain their thinking, anticipating what I usually ask them to do. They seemed slightly miffed that, no, I did not want to hear their ideas right off; I wanted to show them something first. However, when I told them that I had listened to a tape of second graders explaining their thinking earlier in the year in order to make a poster of different ways kids solve problems, they felt sufficiently represented to listen. When I showed them the poster, they excitedly said, "I did it that way!" or "I solved it the green way!" This was actually a change from earlier in the year when each child seemed invested in having his or her way be a little bit different from everyone else's.

We looked at the methods shown on the poster, trying to understand what was happening in each. Out of the ten children, eight solved the problem by adding the tens first, so we started with those three methods. Then we turned to the pink way, the traditional algorithm. One child, Wayne, had solved the problem using this algorithm. He did not know that algorithm in November. When parents help children with homework, they often teach them the algorithm; so I assume that's how Wayne had learned it. Zack, who usually used that algorithm, did not this time. Another child, Eric, said that his grandmother had tried to show him the pink way. So we looked at the pink way and tried to make sense of it.

Wayne described how he used it. He said he started with the ones, adding 8 and 5 to get 13. Since you can't write 13 in the ones, you write the 3 there, and put the l over the tens. Then you add the tens, which is 60, so the total is 63. When I asked him why you couldn't write 13 in the ones, he said because "fifty-thirteen" doesn't make sense. Jamal added that 513 looks like five hundred thirteen. I asked, "What is this 1?" indicating the "carried" digit. Jamal said it was the ten from 13.

The other children in the group seemed to be following the discussion and were making comments. There was consensus in the group that this method was a weird way. Adam said, "That's way harder. Why would anyone do it that way?" The group agreed, and there were murmurs of "I'm just not going to learn it" and "Me either."

This was a painful decision point for me. I had gone into the lesson with fairly clear goals. I wanted the children in this group to think flexibly and solve these problems several different ways, one of which should be the conventional algorithm. I wanted them to know the algorithm for third grade and beyond, although I did not want to place more value on it than on the other methods. But the very fact that I was thinking they should solve problems lots of different ways, one of which had to be "the pink way," automatically placed much more value on this new, weird method, one that did not come from the students. I now realized I had conflicting goals between validating all the different methods of adding and wanting to be sure that everyone learned the algorithm.

I ended up saying something like it was wonderful how they had ways of thinking that made sense to them and that they were able to stretch their brains to understand how someone else was thinking. I added that I did want them to stretch their brains further and try to figure out this new method, partly because some adults and teachers would expect them to know it. Their response was interesting. They wondered why some teachers thought this was the “best” way, a translation they made, of course, despite my delicate phrasing. They decided that it was because this way is harder and, therefore, “more math-y.” Now that they were older and smarter, went their reasoning, they should do things a harder way. I often tell them that I give them hard work because it stretches them, and they are able to do it; when work is just hard enough, they do their best learning.

Anyway, next I gave the children another word problem and told them I wanted them to solve it using at least two different methods. They attacked it with gusto. Most of them tried the pink way as one method. Many of the children used all the methods shown on the poster. What amazed me was that they could all make some sense of the pink way.

I am not saying that the whole group now knows and understands that algorithm, but during this math period, they were all able to use it to add 49 + 39. They talked to each other and helped each other. Some children wrote the numbers side by side instead of in vertical columns and got confused. Others wrote the numbers vertically, but left out the line that separates the addends from the sum. Yet they all found a way into the procedure and how it works.

I remember despairing last year because the children had been pretty consistently perplexed by what I wanted of them during trading games and this adding process. They had been hard-working and conscientious children in general, but most of them did not understand how this algorithm relates to addition or place value or trading.

I think a very important difference this year was that by the time these ten children were exposed to the traditional algorithm, they had successfully constructed their own understanding of addition with regrouping. They were comfortable thinking of numbers in terms of tens and ones; this had meaning for them. Therefore, their task was different. Now I was asking them to reconcile a new method with what they already knew. Last year, on the other hand, I had been wanting the class to construct an understanding of tens and ones, how numbers are made up, and how numbers combine, all at once—using one particular method that made no sense to them.

And what about the other twelve children in my class this year? I have many children whose grasp of tens and ones and of place value is less developed than it is for the ten children I met with initially. Given the same word problem, several of the remaining kids started at 38 and counted on 25 by ones. Two made Unifix cube collections of 38 and 25 and counted the whole thing by ones. A couple of them successfully added the tens and added the ones but were frozen when faced with 13 ones. All these children can identify the tens place in a two-digit number. They can say how many tens and how many ones there are. They understand counting by tens at some level. But they do not use this knowledge when adding two-digit numbers.

Heather was laboriously counting on by ones to solve a word problem that involved 49 + 39. Someone near her suggested that she “use tens and ones.” Heather said, “Oh, yeah!” and added the tens. Then she added the ones. Then she combined them. She did not seem confident or comfortable, but she did it. I wonder, though, what she was thinking. Her initial impulse—to count on—made sense to her because that was what the problem suggested to her. She would probably have been sure that what she was doing represented the story in the problem and that her result was the right answer. It seemed as if her friend’s suggestion triggered her memory of a procedure, but I am not at all convinced that she was sure the “tens and ones” procedure matched counting on or the story problem in any way.

Many of my children are still working on constructing a system of tens and ones. Until they do, the traditional algorithm will not make sense to them.

Every time we work on it, I think my appreciation for the complexities of subtraction increases.

case 13

Subtraction and invented algorithms

Lynn

GRADE 2, APRIL (ONE YEAR LATER)

My students have been solving word problems involving two-digit addition and subtraction both with and without regrouping. They have been working on explaining and recording their processes for solving the problems. Our work with addition has been interesting and satisfying and, as I find every year, subtraction proves to be more problematic than addition for seven- and eight-year-olds.

One word problem involved pigeons in the park: First there were 39 of them, and then a dog came along and 17 flew away; how many pigeons remained? Children solved this problem in an even wider variety of ways than I anticipated. As they worked and described their thinking, and as they tried to understand one another's thinking, the issue of how to keep track of what was going on kept arising.

Many children counted back from 39 to solve the problem. When they did so, several of them had to pause along the way. Isabel counted on her fingers and at first she didn't know when to stop. She seemed to lose sight that the 17 she was counting back represented the departed birds, and therefore wasn't sure when she got to the remaining birds. She was, however, able to start over with a little more clarity and figure out how to use her strategy successfully.

Some children counted up from 17 to 39. At least one child, Sabrina, then pronounced the answer to be 39 rather than 22. At least part of her confusion could result from not being able to hold in mind all at once both the original problem and the meaning of her numbers and procedures. When children count up to combine two numbers, the last number they say is the answer; that's probably why Sabrina thought 39 was the answer.

Children who used more complex strategies also seemed to have trouble keeping in mind both the meaning of the numbers and the problem context. They also had trouble keeping track of numbers they had taken apart for their calculations. Fiona worked on a variation of the word problem that involved regrouping (of 37 pigeons, 19 flew away). She dropped the 7 from the 37 for the time being. She then subtracted 10 from 30. Then she subtracted 9 more. She puzzled for a while about what to do with the 7 now that she had to put it back somewhere. Should she subtract it or add it? I asked her one question, "Did those 7 pigeons leave or stay?" She said they stayed, and added the 7.

37 – 19
30 – 10 = 20
20 – 9 = 11
11 + 7 = 18

It was interesting to me that Fiona needed only that one question to clear up her confusion and I think, for the most part, she subtracts this way and keeps it straight. While Fiona goes through the steps in her algorithm, she is able to keep track of when to add and when to subtract. The 7 gets subtracted (from 37) and then added again (to 11 in the last step). Fiona has decomposed the 19 into 10 + 9, and now the 9 is subtracted because she needs to subtract all of the 19. The 7 is part of what is being subtracted from. The 9 is part of what is being subtracted. It is a complicated process, and it is amazing to me that a second grader can make sense of it for herself. Last year I had a student who struggled with this very issue for weeks and never figured it out.

Paul also took numbers apart to subtract. To solve 39 – 17, he took the 17 apart in three steps as follows:

39 – 10 = 29
29 – 4 = 25
25 – 3 = 22

Paul kept track of the 17 while breaking it into familiar chunks. Many children wondered where he got the 10, 4, and 3. How did he know what to subtract? How did he know when he was done?

Interestingly, Paul himself had questions for Nathan about how Nathan knew which numbers to put together for his answer. Here is Nathan's process for 39 – 17:

17 + 3 = 20
20 + 10 = 30
30 + 9 = 39
3 + 10 + 9 = 22

Nathan nearly always adds, even for a straightforward separating situation like birds flying away. After Nathan explained how he solved this problem, Paul said, "But how does he know what numbers to add up at the end?"

I thought a little bit about the conventional algorithm. A few children use it sometimes, ever since I gave word problems for homework. Clearly, a family member had shown it to them. If a child memorizes the procedure, there is no real "keeping track." They must learn the steps, but they do not need to keep track of what the 3 in 37 means or how much of the 19 they have subtracted so far. If they get confused or forget a step or go out of order, children using this procedure tend not to go back to make sense of the numbers or the problem or to try to keep track of what is going on. I am not sure if this is unique to this algorithm or if it happens because they have learned a procedure with no meaning in it.

Finally, an almost unrelated observation: This year for the first time I have never seen a single child “subtract up” in the ones column if the bottom number is greater than the top one. In other years, I have always had many children do this:

$$\begin{array}{r l} 37 & \\ \underline{-\ 19} & \\ 22 & \text{because } 3 - 1 = 2 \text{ and } 9 - 7 = 2 \end{array}$$

I am not sure what to make of this, but I hope it is because this year the children carry more of the meaning of the problem with them now that they are allowed to construct their own ways of solving it.

case 14

Partitioning subtraction problems

Nadine
GRADE 4, FEBRUARY

Today I worked with a group of eleven students who are still sorting out the complexities of subtraction. I posed the following problem:

$$\begin{array}{r} 387 \\ \underline{-\ 149} \end{array}$$

Teacher: There are different ways to solve this problem, but all of them start with taking the problem apart in some way. Why do we take the problem apart?

Nathan: Well, it would be really hard to just do 387 minus 149 in one step. I don't think anyone could do that. So you have to find smaller problems to make it easier.

Teacher: How would you take this problem apart to make it easier to solve?

Janet: I'd just work with each column separately.

Teacher: So what would your parts be?

Janet: 300, 80, 7 and 100, 40, 9.

Teacher: I know when Janet solves the problem, she doesn't write the parts out separately, but I'm going to do that to help us see what's happening with this strategy. [*I wrote the following on the board.*]

$$\begin{array}{rrr} 300 & 80 & 7 \\ \underline{-\ 100} & \underline{40} & \underline{9} \end{array}$$

Teacher: Who has a different way of pulling the problem apart?

Stephen: I'd keep the 387 together and only pull apart the 149. I'd do that just like Janet did—100, 40, 9.

Naomi: I'd do 380, 7 and 140, 9.

Juanita: I'd do 380, 7 and 100, 49.

I wonder how Naomi and Juanita will deal with their 7 and 9 or 7 and 49, but decide to let the students decide which way of partitioning they want to examine first. At this point, we have written down the following:

$$\begin{array}{r} 387 \\ \underline{-\ 149} \end{array}$$

Janet

$$\begin{array}{rrr} 300 & 80 & 7 \\ \underline{-\ 100} & \underline{40} & \underline{9} \end{array}$$

Stephen

$$\begin{array}{rrr} 387 & & \\ 100 & 40 & 9 \end{array}$$

Naomi

$$\begin{array}{rr} 380 & 7 \\ 140 & 9 \end{array}$$

Juanita

$$\begin{array}{rr} 380 & 7 \\ 100 & 49 \end{array}$$

Teacher: Which way of breaking the problem apart will help you solve it? Which way would make it easiest for you?

Albert: I'd use Stephen's way.

I recorded as Albert presented his steps:

387 – 100 = 287
287 – 40 = 247
247 – 9 = 238

At the last step Albert paused as he thought through the answer to 247 – 9.

Teacher: Can someone else talk us through Stephen's way? I noticed that Albert's last step was the hardest. Is there a way to make it easier?

As Mary explained her strategy, I took the opportunity to record it in a different format as follows:

$$\begin{array}{r} 387 \\ \underline{-100} \\ 287 \\ \underline{-\ \ 40} \\ 247 \\ \underline{-\ \ \ \ 7} \\ 240 \\ \underline{-\ \ \ \ 2} \\ 238 \end{array}$$

Nathan: Instead of subtracting 9, I'd subtract 10 and do 247 – 10 = 237. Then I'd have to add 1.

Teacher: I know you already know the right answer, so it's easy to see that you have to add 1, but I wonder if you can explain it another way. Why do you have to add 1 at the end?

Nathan: Well, you can't just put 1 on the 9 and leave it like that. You have to do something with it at the end. This is subtraction so you have to add it at the end. If this was addition, you'd subtract it at the end.

Teacher: Can anyone else explain why Nathan has to add 1 at the end?

Nathan: He took away too much, so he has to put 1 back.

Teacher: Does anyone want to explain how taking the problem apart in a different way will make the problem easier to solve?

Hallie: I'd do it Janet's way.

As Hallie explained her strategy, I recorded it as follows:

$$\begin{array}{rrrl} 300 & 80 & 7 & \\ \underline{-\ 100} & \underline{\ \ 40} & \underline{\ \ 9} & \\ 200 & +\ 40 & -\ 2 & = 240 - 2 = 238 \end{array}$$

We talked briefly about where her –2 came from.

Teacher: Hallie said she was going to do it Janet's way, but I see Janet shaking her head. Hallie's way certainly works, but it's not the way Janet had in mind.

Janet: I start with the ones. You can't take 9 from 7.

We paused to note that Hallie just did take 9 from 7, but Janet didn't want to get into negative numbers. She began her explanation again.

Janet: You can't take 9 from 7 without getting negatives, so I need to take 1 from the 8. No, I need to take 10 from the 80 and make it a 70. Then I give the 10 to the 7 and make it 17. Now I can subtract the 9.

$$\begin{array}{rrrl} & 300 & {}^{70}\cancel{80} & {}^{1}7 \\ - & 100 & 40 & 9 \\ \hline & 200 & +30 & +8 = 238 \end{array}$$

I noted to myself that seeing the numbers written out in expanded notation, Janet corrected herself as she began to say that she took 1 from the 8. She knows she actually took 10 from 80.

Other students nodded their heads as they watched what Janet did. I know most of these students don't use the U.S. standard algorithm successfully, but many of them try it. I saw this as an opportunity to build more foundation for understanding it and to raise questions about Janet's method. I gave a new problem:

$$\begin{array}{rr} & 503 \\ - & 247 \\ \hline \end{array}$$

Students described how to break it apart using Janet's way, written as follows:

$$\begin{array}{rrrr} & 500 & & 3 \\ - & 200 & 40 & 7 \\ \hline \end{array}$$

Hallie: You can't do 3 minus 7 if you're doing it Janet's way. So you need to get 10 more to put with the 3. There aren't any tens numbers. So you have to start by making the 500 a 400.

Teacher: What will you do with the 100 Hallie took out of the 500?

There were lots of confused looks. I let the students think and then asked again. Responses included those below:

You could make it 100 ones.

You could make it 50.

You could make it 60.

In response to the latter suggestions, I asked, “And just throw away the rest? Is that equal to Hallie’s 100?” Then I pushed students to tell me how to get what we need to subtract the 7.

Naomi: You could put 90 and then give the other 10 to the 3 and make it 13.

With Naomi’s suggestion, we finish the problem:

$$\begin{array}{rrr} \overset{400}{\cancel{500}} & \overset{90}{} & {}^{1}3 \\ -\,200 & 40 & 7 \\ \hline 200 & +\,50 & +\,6 = 256 \end{array}$$

Now the session was over. Stephen asked, “Why did we use only two of the ways on the chart instead of all four?”

I answered, “Because we’re out of time. Maybe we can return to these next time.”

I liked working with this group separately. I sense that with the slower pacing and focused work, many of them were figuring things out for themselves as we went along. In spite of my use of wait time in whole-class discussions, things can still move too quickly, and too many ideas get introduced for some of these students to keep up. They are also more willing to offer their own ideas in this group.

In this session we never did explore Naomi’s and Juanita’s ideas for how to break apart the problem. Maybe we will return to them as Stephen wants us to. Trying out ideas that don’t work easily or don’t work easily for a variety of problems is important in helping students evaluate strategies.