

Planning for Instruction: Increasing Multilingual Learners' Access to Algebraic Word Problems and Visual Graphics

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This chapter connects to:

- Mathematical Practice #1 (MP #1): Make sense of problems and persevere in solving them.
- Mathematical Practice #3 (MP #3): Construct viable arguments and critique the reasoning of others.

The Common Core State Standards and Specific Demands for ELLs

The Common Core State Standards for Mathematics (CCSSM, National Governors Association Center for Best Practices [NGA] & Council of Chief State School Officers [CCSSO], 2010) emphasize problem solving, reasoning, and communication; however, little guidance is available to teachers regarding how to integrate these mathematical practices in general, and to support ELLs in particular. While ELLs are categorized in various ways and English proficiency is defined differently across states (Cook, Boals, & Lundberg, 2011), previous research shows ELLs typically need 4 to 7 years to become proficient in English for academic purposes (Hakuta, Butler, & Witt, 2000). Is it possible for ELLs who make up a growing 4.4 million students (9%) in U.S. schools (U.S. Department of Education, 2014), to meaningfully participate in a community of practice for developing “mathematical power” when they are acquiring English? How can teachers shift instructional practices so that ELLs are able to meet the CCSSM mathematical practices (MPs)? This chapter provides suggestions on how teachers can shift instructional practices so that ELLs have access and are better equipped to meet the CCSSM MPs, specifically MP #1 (Make sense of problems and persevere in solving them) and MP #3 (Construct viable arguments and critique the reasoning of

others). The CCSSM require that students comprehend, use, create, and respond to language in specific ways. CCSSM MP #1 expects that ELLs will:

. . . (explain) to themselves the meaning of a problem . . . make conjectures about the form and meaning of the solution . . . consider analogous problems . . . monitor and evaluate their progress . . . explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends . . . and they continually ask themselves, “Does this make sense?” (NGA & CCSSO, 2010, p. 6)

Similarly, CCSSM MP #3 calls upon ELLs to:

. . . understand and use stated assumptions, definitions, and previously established results in constructing arguments . . . make conjectures and build a logical progression of statements to explore the truth of their conjectures . . . analyze situations by breaking them into cases . . . recognize and use counterexamples . . . justify their conclusions, communicate them to others, and respond to the arguments of others . . . (make) plausible arguments that take into account the context . . . compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is . . . construct arguments using concrete referents such as objects, drawings, diagrams, and actions . . . listen (to) or read the arguments of others . . . and ask useful questions to clarify or improve the arguments. (NGA & CCSSO, 2010, pp. 6–7)

These language-dependent skills should be developed by teachers explicitly through cultural and semiotic mediation (Mariotti, 2009) that should be consistent, deliberate, authentic, and task-based facilitations to help ELLs comprehend tasks and persevere through mathematical objectives selected or designed to be meaningful to them. Moreover, MP #3 calls for teachers to build upon the effective foundation established as a result of MP #1, as ELLs create, recall, engage, apply, and respond to language and representations that will serve as the basis for both their own contribution toward a solution, as well as the platform from which they will evaluate, challenge, and seek clarity on their peers’ contributions.

Teachers of ELLs apprentice their students into the language of mathematics via their role as model mathematicians, practicing technical and everyday language and representations. This role should model mathematical and textual agreement as well as dissent with interactions involving student-to-student, teacher-to-student, and even teacher think-alouds, in order to create a learning environment welcoming of constructive critique as ELLs progress toward a shared understanding of the solution(s).

The discourse-oriented “community of practice” (Lave & Wenger, 1991) approach to teaching mathematics expected by the CCSSM MPs #1 and #3 draws heavily upon students’ abilities to comprehend language effectively and communicate precisely (Pimm, 2010). According to Schleppegrell (2007, 2010), among others, this not only involves receptive and expressive knowledge of specialized math vocabulary words or terms (e.g., *slope*, *linear*, *variable*) but also clauses and phrases at the sentence level to successfully comprehend word problems and communicate questions, clarifications, or justifications. Additionally, visual representations (e.g., arithmetic and geometric

symbols, coordinate and trigonometric graphing, data tables) have been a cornerstone of mathematics for centuries (Cajori, 1928). Drawing is an important constructive resource central to learning, because drawing enhances engagement, helps students learn to represent mathematics, helps students to reason mathematically, may be implemented as a learning strategy, and helps students to communicate mathematical ideas (Ainsworth, Prain, & Tytler, 2011). While students typically are asked to produce or to acknowledge a graphical depiction, they are “rarely asked to explain their representation, evaluate it, compare it to others, or produce different representations” (diSessa, 2004, p. 302). Ultimately, greater instructional focus on representations would enable students to: a) invent or design new representations, (b) critique and compare the adequacy of representations, (c) understand the purposes of representations generally and in particular contexts, (d) articulately explain representations, and (e) learn new representations quickly and with minimal instruction (diSessa, 2004).

Mathematical discourse (or literacy) requires the “interwoven grammars of language, mathematical symbolism, and visual images” (O’Halloran, 2005, p. 94) so that the student explaining or justifying his or her solution must make seamless shifts within and across all three semiotics, or meaning systems. To facilitate math discourse, language is generally used to introduce or describe the problem and its context. Students then need to visualize the problem in graphic or diagram form. Finally, math symbolism using different approaches (or, in other words,

. . . the recognition of patterns, the use of analogy, an examination of different cases, working backwards from a solution to arrive at original data, establishing sub-goals for complex problems, indirect reasoning in the form of proof by contradiction, mathematical induction . . . and mathematical deduction using previously established results [Stewart, 1999, pp. 59–60])

is used to solve the problem. The use of language to participate in math discourse has different purposes and hence requires a different register than the everyday communication styles associated with interpersonal discourse (Moschkovich, 2010). Additionally, teachers of ELLs should recognize language indicating uncertainty during discussions such as modal verbs (e.g., *could*, *should*, *might*) and hedging (e.g., “ummm . . .” or “like . . .”) as fillers that may indicate what Barnes (1976) called exploratory talk as opposed to final draft talk. Exploratory talk is used when students are thinking through what they know to come to new understandings and thus the need for modals to soften tone (i.e., less authoritative) and hedges to use as fillers while processing new insights. On the other hand, final draft talk indicates certainty with authoritative language used with confidence. There is a place for uncertainty within mathematical discussions (Meaney, 2006), specifically for ELLs as they may not only be processing novel mathematical insights but also how to express them in a second language.

We assert that the learning environment created by the teacher’s strategic task set-up, culturally-responsive instruction, and proactive dissection of the language and representations comprising mathematics problems help develop ELLs’ skills and confidence to persevere toward a solution while developing sound mathematical arguments that may require clarification, justification, or modification by and for their peers. Put another way, teachers are the catalysts to apprentice ELLs to make sense of and persevere through mathematics problems (MP #1) and create the text and visual graphic representations that will undergird viable solution pathways and provide the forum to dialogue with peers in pursuit of consensus and comprehension (MP #3). MP #1 requires

teachers to develop a shared context, comprehension, and conceptual understanding of the problem posed. Often, ELLs' experiences are not congruent with those of mainstream populations, thus the context surrounding tasks that include submarine sandwiches or a fundraising school bake sale, for example, must either be determined as familiar to all ELLs, or mediated to come to a shared understanding of them via discussion and drawing out of what the students know about the problems' context. This will maximize ELLs' awareness of the problem's contextual facts and nuances, and ultimately provide access to the task (Jackson, Garrison, Wilson, Gibbons, & Shahan, 2013).

ELLs' comprehension of word problems is accomplished by deliberate mediation of the language and representational features of math problems (Aguirre & Bunch, 2013; de Oliveira, 2013; Schleppegrell, 2007) once shared meaning for an authentic task's context is established. Next, conceptual understandings are furthered or reinforced using representational tools that are discussed with ELLs to further scaffold access to problems (i.e., making sense of them—MP #1). Finally, ELLs describing and explaining their representations in mathematical discussions to construct arguments and critique others' reasoning (MP #3) influences learning opportunities and creates space for new mathematical understandings (Wood, Williams, & McNeal, 2006). Collectively taking all of the factors related to MP #1 and MP #3 described above, ELLs can become well-versed in a set of access strategies to authentic, meaningful tasks (i.e., relevant context, contributions emulating teacher-modeled language and teacher-mediated representation features) that lead to greater understanding of mathematics and language, thus richer and more meaningful discussion, critique, and agreement of shared meaning and solutions to mathematics problems.

This chapter provides one way for teachers to proactively plan and be mindful about how they can facilitate ELLs' access to language and representations, and thus math content, using a community of practice approach. What is described is the result of a classroom-based intervention project that sought to increase teachers' and ELLs' knowledge of the math register in order to increase mathematical understanding and achievement (Secada & Avalos, 2010–2013). The intervention is premised on a community of practice approach and modifies a planning scheme by Smith and Stein (2011) to focus on language and communication for mathematics discourse. Additionally, the planning tool includes diSessa's (2004) framework to determine an instructional purpose for visual representations and also builds on the work of Jackson et al. (2013) to provide access to the context of math problems for diverse populations. Furthermore, math word problems are analyzed using a Functional Grammar framework (Halliday, 1978; O'Halloran, 2005; Schleppegrell, 2007) to identify possible obstacles and misconceptions for ELLs based on word, sentence, and text level confounds (de Oliveira, 2013).

In the next section, we provide a rationale for the need to be mindful of language used in and for math instruction. We explain a process and tool for planning discourse-based instruction, providing a couple of sample lessons. Finally, we provide some practical tips as to how a discourse approach could be implemented in a math classroom.

Rationale

ELLs come to school with diverse background experiences, languages, and ways of using language that may exclude them from becoming “insiders” in a math discourse community if teachers are not mindful of how to facilitate access to language. Purposeful lesson planning for mathematics with a

focus on language, visual representations, and problem contexts is advocated to level the playing field and minimize obstacles that ELLs may face when learning math. Frontloading information about how language is used and creating shared understandings of semiotic systems and problem contexts can widen and deepen opportunities for students to learn math content.

Algebra is known as the gatekeeper to advanced mathematics courses, but teachers can also be gatekeepers to discourse-, representation-, and context-rich content, or to learning environments less reflective of these qualities. The mathematics textbook serves as a proxy for the intended curriculum. When planning a lesson on functions, a teacher might rely on the textbook's index in order to increase ELLs' access to the content. The index can be used to map mathematical terms, concepts, procedures, representations, and interconnected concepts. A typical textbook's index will list functions, along with a series of related topics including: comparison of; definition of; domain of; evaluating; families of; graphing of; as main idea; linear; nonlinear; notation of; quadratic; range of; representations of; vertical line test; and writing. Mathematics teachers could use this textual map to holistically plan their ELLs' introduction to, application of, development of, and representation of mathematical terms, concepts, and topics, thereby acknowledging the need to continuously mediate evolving definitions, criteria, uses, and extensions.

Functions are often defined as a relationship between input variables and their unique output variables. Typical definitions of functions state that they are generally represented by graphs, tables, equations, or words, with each representation preferred in particular situations. ELLs will commit such definitions to memory, as they do for so many other subjects in school, only to face the unmediated evolution and redefinition of mathematics terms like "ordered pairs" as "input, output," then as "independent, dependent," then as " (x, y) ," and then as "(domain, range)." Teachers also must help students understand nested and embedded mathematical definitions such as *linear function* which also is a *linear equation*, which is a *first-degree equation*, which is an *equation whose variables are raised to the first power (or an exponent of 1) and no higher*, which happens to be an *equation in slope-intercept form (or $y = mx + b$)*, which is similar to the function notation $f(x)$ of slope-intercept form (or $f(x) = mx + b$).

Finally, with regard to mathematical definitions found in mathematics textbooks, teachers need to help students reconcile definitions that may appear to be conflicting or otherwise ambiguous. For example, linear functions may be defined by linear equations; but students might incorrectly identify an equation in the form of $x = k$ as being a linear function because it looks like it fits the numerous definitions that have been provided. Someone needs to explain to the student that the above linear equation is not a linear function because it does not pass at least one of the tests for linear functions: the "vertical line test," whereby no vertical line anywhere along a graph can include two or more points of the graph. In fact, textbooks provide several other ways to determine if a linear function exists including checking to see if every x -value has *only* one corresponding y -value and/or using tables to determine if differences between sequential numbers are constant throughout (resulting in a pattern).

Many teacher editions for mathematics textbooks will provide "big [or main] idea" or "math background" sections that attempt to document overarching principles that students are expected to understand by the time the unit is completed. Teachers of ELLs should foreground, highlight, and

reiterate these big, background ideas as explicit statements for all their students, but particularly for those students who might otherwise miss these nuances. In the case of functions, it is important that teachers repeatedly note that the many representational forms of functions (as verbal descriptions, tables, equations, and graphs) are essentially the same thing: a rule or direction that relates or that maps an input (starting) variable to a unique output (end) variable. Another important idea is that the representation of input and output variables as x and y , respectively, is only one of many ways to represent these variables. For students who rely on and replicate textual samples before they fully understand them, this note is an important one that helps students carry out the CCSSM MPs that are central to this chapter.

In addition to the planning tool, we describe in our next section how high school mathematics teachers can purposefully plan for ELLs' access to relevant mathematics language, representations, and concepts using their math textbooks in a number of other ways:

Representations Mediated by Language

1. Recognize and develop the diverse representation aspects embedded within concept and vocabulary lists (e.g., dependent and independent variables; ordered pairs; function; input; output; geometric relationship; linear function; perimeter) to build upon students' prior funds of knowledge (Moll, Amanti, Neff, & Gonzalez, 1992) and math content knowledge. Figure 1 asks teachers and students to focus on language that describes, clarifies, creates, compares, and questions visual representations. Conversely, focus on clear, original, comparable visual representations that support this kind of language development.

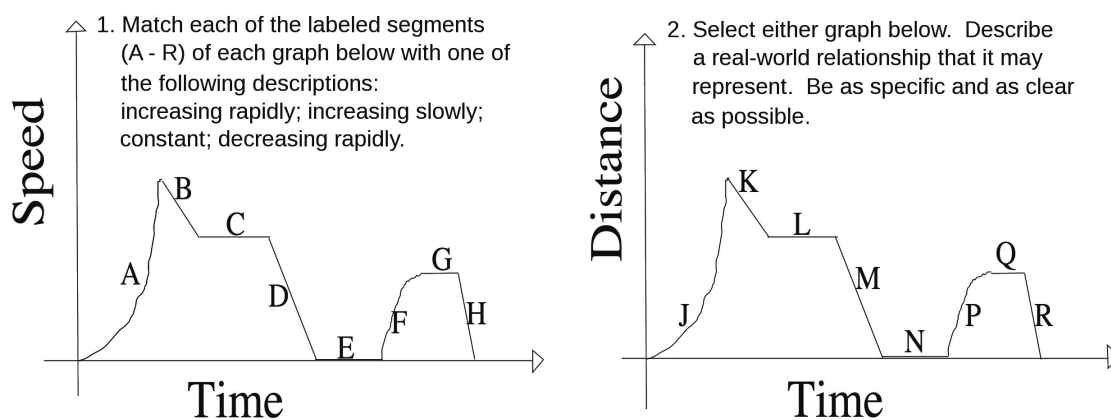


Figure 1. Example #1 of a Dynamic Interplay Between a Visual Graphic Representation and Related Mathematics Language

The first question asks students to match language to a pair of representations, while the second question asks students to generate language that aligns with the mathematics represented in the graph. Teachers can help ELLs access the mathematical rationale for phrases such as “increasingly slowly” and “decreasing rapidly” through an explicit, proactive discussion of this graph, while discussing the nuances of matching four verbal descriptions to eight different sloped segments on the graph.

2. Given that a number of the world's languages, including Chinese, Arabic, Urdu, Hebrew, Japanese, and Korean, are written either right-to-left and/or vertically top-to-bottom, teachers should acknowledge the differences in directionality of reading and writing that many ELLs may bring to the mathematics classroom, which may differ from Western assumptions regarding left-to-right number lines, graphs, inequalities, and ordinality.
3. Use language to describe the mathematizing of word problems and/or to contextualize symbolic or algorithmic mathematics problems.

Representations Mediated by Mathematics

4. Plan on eventually making explicit the mathematics that is implicit in representations, including the mathematics that is not necessary to solve a problem. In the case of functions, and as shown in Figure 2, this might include noting that the “common difference” between terms of a function is both the subtraction of an earlier number in the series “from” the subsequent number in the series, as well as the subtraction of the latter number “minus” the prior number, within cells along the same variable of a table (row or column) instead of between these rows or columns. Furthermore, this “common difference” may mislead students into thinking that it is always a subtraction and exclude the possibility that there is a common addend. This common addend or subtrahend is, of course, only true of arithmetic sequences (which are linear functions),

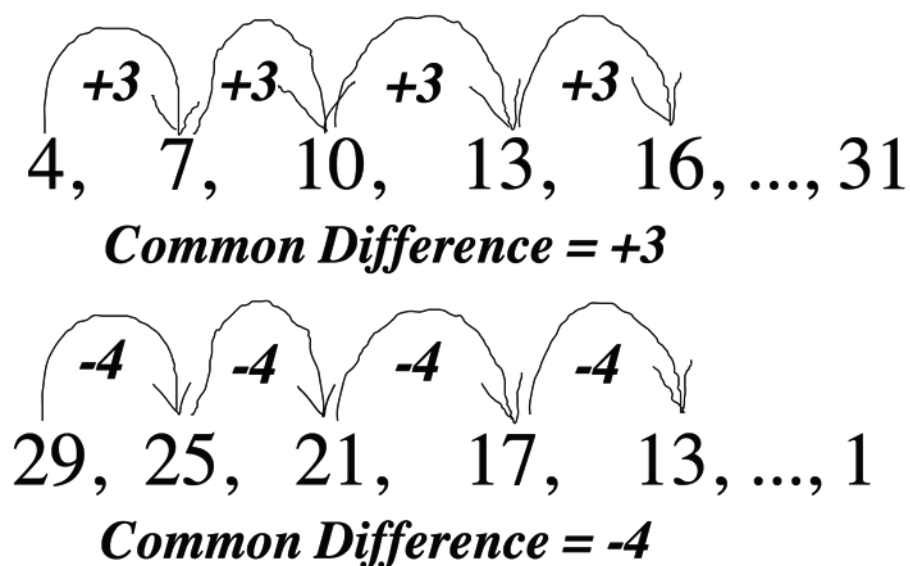


Figure 2. Example #2 of a Dynamic Interplay Between a Visual Graphic Representation and Related Mathematics Language

Representations ask students to determine if a common difference exists between consecutive elements in a sequence using right-pointing curved arrows with both positive and negative differences. Teachers can help ELLs access this concept by reconciling that “difference” here goes beyond a mathematical subtraction of the first term from the second term (as indicated by the arrows’ direction), to ask whether a common addend exists between each consecutive term.

and not of geometric sequences (which rely on a common factor or divisor) that may still be (nonlinear) functions. Discuss mathematical representations that may be purposefully deceptive or inadvertently incomplete or replete (with extraneous information). Typical function problems will ask students to sketch a graph of a variable (e.g., height) of an item (e.g., an elevator) over time while providing (irrelevant, but alluring) information (e.g., about who got on and off the elevator at various floors). When plotting the variables on a graph, it is common to see students graph the number of elevator passengers against the height of the elevator, completely ignoring the relevant factor of time and despite its specific mention in the problem's directions. Similarly, tables, graphs, and keys may be labeled such that important or relevant details are not as evident. Foreground synonymous and analogous mathematics concepts traditionally relegated to the background or summary components of mathematics units. Be explicit about commonalities shared by different topics.

5. Allow for alternate representations that might expose misconceptions or evidence comprehension. Figure 3 shows two identical-looking graphs differentiated by their y-axes. In support of MP #1, teachers of ELLs can ask students to contextualize these graphs (potentially resulting in situations involving hills including bike riders, runners, and rollercoasters, which all resemble the hill shapes of the graphs). Students may then match and/or critique real-life situations related to these graphs, whereby developing important MP #3 skills—an activity that may prove as informative for teachers as it is for their students.
6. Similar to the cloze activities common in reading and writing activities, mathematics teachers could purposefully omit aspects of a representation that are critical to understanding and solving the problem. ELLs developing MP #1 skills who are encouraged to complete or correct partial representations should find this exercise instructional as well as engaging. Extend the use of graphic organizers to include representations both in problems and solutions. For example, add a Draw/Represent component to Read-Write-Think cards (readwritethink.org). Also, extend Venn diagrams beyond characteristics and terms to include representations (such as linear and nonlinear function graphs, discrete and continuous graphs, or arithmetic and geometric sequences).
7. If using a calculator (e.g., to solve and graph functions), MP #1 goals would be supported by the discussion of the rationale for: the keystrokes and sequence entered into the calculator; the proposed and final values of X_{\min} , X_{\max} , Y_{\min} , and Y_{\max} in the graphing window; the use of the “CALC” feature and buttons like “INTERSECT” when solving function problems; and how to check the calculator's solution using the calculator as well as noncalculator methods.

The participatory, problem-solving nature of mathematics suggests that math could be an ideal platform upon which to develop ELLs' representational competence and semiotic practice: “Math is not a spectator sport. In order to think mathematically, students need to do mathematics, actively and vocally. This necessitates solving problems, real problems. . . . A real problem is a question to which the answer is not immediately apparent” (Allen, 2011, p. 5). Increasingly, and commonly

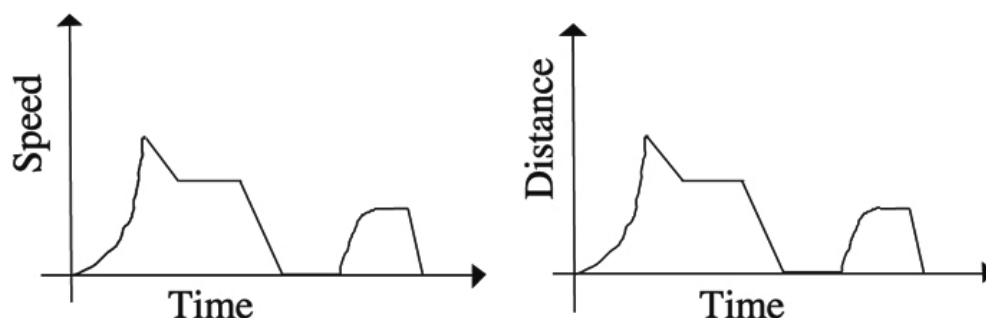


Figure 3. Alternate Representations

Visual graphics are used to challenge students to evaluate, critique, and create mathematical representations that may be inconsistent with the mathematics concepts they purport to convey. Teachers of ELs may use these to encourage their students to not always believe everything they see in a reference source, while encouraging them to be authors and artists (as well as readers and thinkers) of mathematics.

found in CCSSM expectations, math problems are asking students to generate equations or exercises that represent a written description of a context. The struggles that many students experience when working with and through these word problems are because students see them, the problems, as “neither interesting nor relevant to their life experience” (Allen, 2011, p. 5). Allen (2011) echoes Pólya’s (1945) sentiment that “students need to experience authentic problem solving” (p. 5). Moreover, when teachers incorporate mathematical modeling and/or graphic representations within instruction, ELLs’ knowledge of language beyond the sentence level is needed because multiple, complex concepts can be encapsulated within these representations (O’Halloran, 2005), especially at the secondary level.

Pedagogical Practice: Language in Math Planning Tool

We now describe a planning tool and process designed to provide ELLs with access to math problem solving, as well as to increase their repertoire for the math discourse and discussion expected by MPs #1 and #3. It should be noted, however, that implementing a community of practice in your classroom is another topic for which space restrictions do not allow us to properly address in this chapter. Briefly, a community of practice requires that the mathematics teacher establish appropriate norms for participation in classroom discourse. These norms should be shared and applied consistently so that ELLs have plenty of opportunities to participate; in other words, a community of practice does not develop by taking one day a week for this approach, or trying a lesson every so often. ELLs’ background experiences and the strengths they bring with them to school need to be valued and should be drawn upon as much as possible in order to successfully engage them in mathematics discourse (Hansen-Thomas, 2009).

In developing this planning tool, we built on the work of Smith and Stein (2011), who provide in-depth information on how to select word problems that will provoke “mathematical power” for students and guide teachers to proactively plan for math discussions by identifying common misconceptions and predicting possible solution paths. Our work modified the planning tool to focus

on the context of the math problem (Jackson et al., 2013) and the language used in the problem (Cocking & Mestre, 1988; Schleppegrell, 2007). Unfamiliar contexts and overly complex language may create obstacles and thus inhibit opportunities for ELLs to gain access to the content and to participate in in-class discussions.

Language features that differ in math from those found in oral, interpersonal communication patterns should be targeted for instruction (Table 1). Features of mathematical language different than interpersonal communication registers include the use of passive voice, complex noun phrases and strings of words, and symbols or mathematical notation (Schleppegrell, 2007). At times, problems also have vague directions, making it difficult for the reader to determine the problem to be solved (Spanos, Rhodes, Dale, & Crandall, 1988), providing even more reason for students to focus on any decontextualized numbers that may be found in the rest of the problem. Moreover, symbols are used to represent words (e.g., $>$ = greater than), which creates even more compact texts and requires the reader to remember the omitted words via representations or symbols within the specific context of the math problem; for example, an equal sign can contain identity as in $(x + 1)^2 = x^2 + 2x + 1$ or an equality as in $f(2) = 5$ (where f is a given function). In addition, math has specific vocabulary with three different categories of words: 1) words that have the same meaning in everyday language (words that are used to contextualize mathematics), 2) words that have a meaning specific to mathematical language (*coefficient*, *hypotenuse*, *rhombus*), and 3) words that have different meanings in math than in everyday language (*difference*, *mean*, *value*, *odd*, *even*; Lee, 2006, p. 15).

The Language in Math Planning Tool is designed to assist with planning for access to the math content of word problems by focusing on the language and context of the problem, as well as instructional purpose for visual representations, math discussions, and possible misconceptions or solution pathways students may use to solve the problem. A planning tool with an algebra problem has been completed to demonstrate how this may work (Appendix A), and a blank planning tool for photocopying and personal use is provided in Appendix B.

The problem should be written in the first row along with the content objectives. To provide ELLs with access to the problem, and thus to equip them to meet MPs #1 and #3, the context of the problem (usually found in the first sentence or two of the problem) should be analyzed to determine if students may have experiences or understanding of the context. The problem in Appendix A may be even more confusing because the context is embedded with the information needed to solve the problem. The mathematical relationships or foundational understandings needed to solve this problem are noted here to keep in mind what students should already know in order to successfully solve the problem.

Language features emphasized for the lesson are from analyses of the problem at the text level (overall problem organization), sentence level, and word level. At the text level, is the request or problem to be solved clear? Common word and sentence level language features that are problematic for ELLs are provided in Table 1; analyze problems and make note of these to discuss and clarify with students how language is used.

The Accountable Talk Moves (Michaels & O'Connor, Williams Hall, & Resnick, 2010) section is where open-ended question stems or discourse prompts are identified for targeted use by

Table 1. Examples of Math Language Features That May Require Instructional Support for ELLs

Language Feature	Example
Passive voice Making the object of an action into the subject of a sentence; that is, whomever or whatever is performing the action is not the grammatical subject of the sentence.	<ul style="list-style-type: none"> • <i>x is defined to be greater than or equal to zero;</i> • <i>When 15 is added to a number the result is 21. What is the number?</i>
Complex noun groups Used to make language precise, complex noun groups often contain both pre- and postmodifiers and can include additional embedded clauses that often further define or identify other bits of relevant information.	<ul style="list-style-type: none"> • <i>Find the pair of bars in which the bar for the pizza looks three times as tall as the bar for hot dogs.</i> • <i>Cell phone Company A charges a base rate of \$3.00 per month plus 5 cents a minute that you're on the phone.</i>
Complex strings of words or phrases (specific to math) Two or more words that together create specific math concepts and in other contexts are not generally linked or used together in this way.	<ul style="list-style-type: none"> • <i>Least common denominator</i> • <i>negative exponent</i> • <i>place value</i> • <i>ascending order</i> • <i>average number</i> • <i>stem-and-leaf plot</i>
Symbols and mathematical notation The symbols used to create meaning in math.	<ul style="list-style-type: none"> • <i>= , ≠ , ~ , ≤ , < , ≥ , > , Σ , \$, : , % , - (negative sign), . (decimal point)</i> • <i>Abbreviations for measurements (e.g., cm, lbs)</i> • <i>Variables representing a known or unknown number (e.g., x, y, mx + a = b)</i>
Vague referring words Referents allow the writer to establish cohesive links to prior information (and sometimes upcoming information). Referents also contribute to the organization, and are typically realized in pronouns (she, they) and demonstratives (this, that)	<ul style="list-style-type: none"> • <i>Ms. Smith teaches a karate class every Monday at 4:00 p.m. Initially, 26 students registered for her class . . .</i> • <i>The moon is about 3.84×10^5 kilometers from Earth. Which of the following represents this number in standard notation?</i> • <i>. . . Compared to the other values, 200 is extremely high. So it is an outlier.</i>
Vagueness within the problem or directions Usually found at the sentence level. This results in confusion about what information is needed to solve the problem or what answer or what type of answer is required to solve the problem.	<p><i>"Parking takes up 8% of the average person's commute time. An average person takes up to 9 hours a month looking for parking. How much is spent on parking?"</i></p> <ul style="list-style-type: none"> • <i>It is unclear if the problem should be solved for each month, year, or how much time needs to be factored into the equation.</i> <p><i>"Write the answer which best represents the equation."</i></p> <ul style="list-style-type: none"> • <i>It is unclear in what form the answer is to be provided.</i>

continued on page 16

Table 1. (Continued)

Vocabulary	
Technical vocabulary Vocabulary words or terms needed to express concepts efficiently in math (specific to math).	<ul style="list-style-type: none"> • <i>Additive inverse, binomial, coefficient, monomial, graph, data, plot, value, average, maximum, minimum, grid, length, ordered pair, order, pictograph, sum, dataset, place value, ascending, vertical, horizontal, round, mean, digit, discrete, origin.</i> • Any <i>unit of measurement</i> (e.g., foot/feet, inch, yard, meter, etc.)
Natural language confounds Polysemous words or phrases that could be confused with “everyday” language terms that have other meanings (not specific to math). Often these are homonyms or homographs.	<ul style="list-style-type: none"> • <i>Sum</i> (some), <i>table</i> (as in dining room vs. means to organize data), and <i>mean</i> (as in cruel vs. average), or <i>square, power, equality, and rational.</i> • <i>Bar, key, scale, points, value, age, plot, digit, set, order, place, feet, yard, figure, model, left.</i>
Synonymous words and phrases Words/phrases used interchangeably that could be confusing due to the many labels attached to one concept.	<ul style="list-style-type: none"> • “The mean of heights was 24 inches. This <i>average</i> was higher than expected” • For addition, <i>add, plus, combine, sum, more than, and increase by</i> are all synonymous. • For subtraction, <i>subtract, minus, differ(ence), less than, and decreased by</i> are all synonymous. • Others: <i>less vs. less than; the square vs. the square root; divided by vs. divided into.</i>

the teacher or students during the lesson. We have found that talk moves, such as, “I disagree because . . .” or “So are you saying . . .” provide stable routines to scaffold ELLs into appropriate norms for disagreeing, clarifying, or challenging others’ solutions and/or representations, developing MP #3 practices. They also assist teachers with facilitating discussions by extending opportunities to contribute (e.g., “Who can add on to that?”), pressing for deeper reasoning (e.g., “Can you say more about that?”), or making connections to previous lessons and experiences (e.g., “How does this connect to. . . ?”). We gradually introduced these moves to students, identifying two or three every couple of weeks to post and model, reminding students to use them when opportunities came up during discussions. In about one month’s time, the students had internalized a repertoire of Accountable Talk Moves. While incorporating them at first was a bit awkward and stilted during our discussions, eventually we didn’t need to remind students to use them as they came naturally; this may not always need planning once students internalize the moves and are using them fluently.

The section for Academic Conversation Focus is to identify what aspect(s) of students’ problem-solving presentations may need more time to develop when introducing students to a discourse-based approach so that they receive the necessary MP #1- and MP #3-related scaffolds that lead them to independence in solving problems and making their presentations. For example, when students are learning how to present their solutions, more time may need to be spent during the discussions on making sure there is a model representing the problem within the solution, or preparing students to explain their models to the class. In other words, the process for presenting

problem solutions is given to students holistically using multiple problems every few days, but the Academic Conversation Focus unpacks how this process is carried out over several weeks with a different instructional focus to allow for scaffolding and release to independence, according to how the students take up the process and the teacher's expectations for each stage. This process is described in more detail under the Practical Tips section at the end of the chapter.

Finally, the Visual Representation section complements the discussion by considering the affordances, challenges, and pedagogic possibilities that drawings, equations, tables, graphs, symbols, diagrams, photos, and other representations may provide for ELLs and their teachers. Building on the work of diSessa (2004) and others in Metarepresentational Competence (MRC), this section prompts teachers to anticipate and plan how visual graphic representations help ELLs access the mathematics concepts, terms, and procedures from a variety of perspectives. Teachers are asked to plan for instruction in support of MP #3 practices that includes representations (either textbook or teacher provided, or student generated) that give students the opportunity to invent, design, critique, compare, explain, learn, and judge the suitability of multiple, related, complementary representations. It is worth noting that sequencing anticipated representations in the order in which they will most typically be created or discussed helps foreground representations as part of a successful solution pathway.

Also, teachers are encouraged to consider using technological representations (graphing calculators, interactive whiteboards, virtual manipulatives, etc.) when they supplement and verify student work completed without the assistance of technology, in addition to using technology when the mathematics involved would simply be better taught and learned with technology. Furthermore, teachers are asked to think about how their instruction might elicit the purpose of these representations and how their form and function might increase access to mathematics. The column for Possible Solution Paths should include ways ELLs may solve the problem, including correct and erroneous solutions. According to Smith and Stein (2011), this assists teachers in planning for the unknown during discussions by anticipating both correct and incorrect solutions. This may take some time at first and it may not be possible to anticipate all solutions, but the process does prepare teachers to facilitate discussions by predicting ways ELLs could approach the problem-solving process and, over time and with experience, teachers become more accurate and more fluent in making these predictions.

The Actual Solution Paths column is for you to make notes after task setup while observing what students are actually doing to solve the problem. This also helps plan for the discussion by noting the different solution pathways students are taking to sequence these during their presentations, which is what should be written in the final column. It is important to orchestrate presentations in different ways—sometimes having students with errors or misconceptions leading to incorrect solutions present first, sometimes in the middle, and sometimes last so that the first presenters don't always have the “wrong” answer—to create interesting and engaging discussions. Also, this creates a record to better understand ELLs' learning needs and perhaps highlight basic procedures or concepts that may need to be reviewed or taught in different ways.

In sum, ELLs need specific supports to transition to a discourse based approach for mathematics teaching and learning. Purposeful planning and analysis of the context and language used for the problems is needed to provide access to content and language learning expected by the CCSSM.

Authentic problems that are relevant to real-life applications are important to engage students in the problem-solving process (Smith & Stein, 2011). Deep teaching at the conceptual level is needed to give students something other than procedures to talk about during math discussions; yet procedural fluency as a primary goal of teacher-facilitated math discussions is equally important. Anticipating how students may understand/misunderstand problem solution processes will assist in understanding possible directions discussions may take. This also helps you become aware of possible misconceptions to purposefully order student presentations.

The problem-solving and discussion processes are facilitated by Accountable Talk Moves® and the Academic Conversation focus to lead students to independent participation and support them in developing reasoning and thinking skills as valued by the discipline of mathematics. Finally, visual representations could be clarified when an instructional purpose for them has been identified and discussed. When mathematics teachers increase the amount of discourse in their classrooms, ELLs become engaged in negotiating meaning, discussing ideas and strategies, and appropriating mathematical language as their own. However, teachers will need to initiate and share in the discourse, and to manage the process so that students become more and more proficient in continuing the discourse so that, ultimately, they (the students) become independent problem-solvers and mathematics authors and designers.

Reflection Questions and Action Plans

We include the following questions to help teachers reflect upon the ELLs served by their schools in preparing to use the planning tool.

1. Do you have many emergent speakers (i.e., recent arrivals) in your class? If so, they may not be comfortable speaking English in front of the class, and other accommodations should be made for them to present solutions (e.g., a buddy could translate if they do not speak the language). In our experience, if emergent speakers with strong basic math skills are in math discourse classrooms, they are soon participating *if* the classroom culture established by the teacher is encouraging and inviting, and norms for participation do not focus on right vs. wrong answers, but rather what can be learned from errors or making mistakes.
2. Did your ELLs experience more traditional education contexts in which the teacher is the classroom authority? If so, might they misunderstand or misjudge the reasons behind a math discourse-based approach? If this is the case, you should be explicit with your goals when communicating with these students daily, as well as with their parents.
3. Are your ELLs more accustomed to the metric system or different algorithms to solve problems? If so, are you aware of this and what the algorithms are so you will not jump to conclusions when observing ELLs solve problems?
4. Are your ELLs struggling to describe visual graphic representations? If so, is this a result of a language proficiency or mathematics content challenge, or a combination of both?
5. Similarly, are your ELLs struggling with mathematical persistence? If so, is this a result of a language proficiency or mathematics content challenge, or both?

To apprentice ELLs successfully to a discourse approach, a process or protocol for solving and presenting solutions should be introduced with a specific instructional focus on different stages over time; we include a protocol we found helpful when working with ELLs in Table 2.

The Academic Conversation Focus section of the planner will provide a record of where students are in gaining independence; it may take focusing on each stage for two or three problem-solving sessions (or more) for them to really understand the purpose of and expectations for each stage. The idea is to present and model the protocol from Table 2 holistically and as teachers would expect students to complete it, but scaffold details and specific expectations over time (Table 3). This is so that ELLs fully understand and can meet teacher expectations while being introduced to a discourse approach without teachers spending a great deal of time on these details, which detracts from math problem solving and stifles productive discussions. Moreover, if implementing the problem-solving process with small groups working together, a focus on the different stages can ensure that each group member is accountable for the group's work and can eventually carry out problem solving independently. While these stages should be explained and presented initially to all students, a deeper instructional focus on each stage will ensure ELLs understand how this actually looks and feels as they are apprenticed into thinking, speaking, and representing in math classrooms.

It is important to stress that each stage should involve ELLs speaking and interacting prior to writing or recording on paper; this allows for students to think through the stages and work together in supporting each other when solving problems, as well as providing an initial focus on problem solving over written communication. The stages to be emphasized begin with the final stage of the presentation process (Justify and defend solution, an MP #3 expectation) and work backward to the beginning stage of the protocol (Identify the problem to be solved, an MP #1 expectation) to first focus on establishing a community of respect among students, which is foundational for productive discussions.

While the sequential stages of focus emphasized by the teacher in this section should take the bulk of time when implementing and apprenticing ELLs into a discourse approach, it is not suggested that teachers ignore other stages or areas if immediate feedback is needed, or if students are

Table 2. Possible Protocol to Apprentice Students Into Problem-Solving and Math Discourse Approach

In small groups, and eventually independently, all students should be able to:

1. Rewrite the problem to identify and explain the request or problem to be solved.
 2. Represent the problem visually.
 3. Make a plan to solve the problem (may use words or mathematical notation).
 4. Recount and accurately explain solution pathway to a partner (how problem was solved).
 5. Justify solution and defend reasoning (why problem was solved in such a way).
-

Table 3. Academic Conversation Foci to Apprentice Problem-Solving Independence and Discussion Participation

Teachers instructionally emphasize the stages in this order during implementation:

1. Justify solution and defend reasoning (why problem was solved in such a way).
 2. Recount and accurately explain solution pathway to a partner (how problem was solved).
 3. Make a plan to solve the problem (may use words or mathematical notation).
 4. Represent the problem visually.
 5. Rewrite the problem to identify and explain the request or problem to be solved.
-

struggling to understand what is expected in different stages; rather, emphasizing stages over time allows for teachers to focus on specific stages in order to explain and lay out detailed expectations for students. Teachers should explain the process to students in the order found in Table 2 with rewriting the problem to identify what is to be solved as the first step, representing the problem visually as the second, making a plan to solve the problem as the third, and explaining and justifying the solution as the fourth and fifth steps. The following suggestions begin to unpack the discussion process for teachers to build upon and modify what works best for them and their students when they (the teachers) emphasize specific stages during instruction.

Justify and Defend Solution (MP #3)

This first stage should be completed orally prior to asking students to write their justifications. As stated previously, deep conceptual understandings will assist students with justifying and defending their solutions. Emphasizing respect among and for all students should be the main thrust at this stage, with zero tolerance of snickering, laughing, smirking, or any remark that may inhibit participation in math discussions. The Talk Moves® will be important in apprenticing ELLs to respectfully challenge and disagree with others' solutions and reasoning. Modeling and posting the Talk Moves® to refer to during discussions can be helpful to remind students to use them, and, eventually, students will acquire and use them without your having to post or prompt the moves.

Explain Solution to a Partner (MP #3)

Before writing an explanation of the group's solution process, students should orally explain how the problem was solved so that they are able to clarify the process followed, as needed, asking questions of the partner prior to writing or explaining in front of the class. Teachers should position this stage as providing a way for students to practice their presentations and underscore the possibility that anyone could be called upon to present their group's solutions to the class (though the teacher will have an order in mind while observing the students during problem solving). This is also a difficult stage for students to initially grasp because it is common for details and steps to be missing in the explanations. Again, Talk Moves will be important here as teachers and students prompt with questions and encourage students to think more carefully about how solutions came about to fill in the missing pieces of the process. Once all students have explained the solution

process to a peer, a written explanation should be completed prior to the student presentations; the written explanations will help guide student presentations to the class.

Making a plan (MP #1)

Creating a plan is important to show how ELLs are making sense of the problem and/or visual representation. This stage could be the most difficult for students to agree upon if working in groups as they come to consensus and decide on a way to solve the problem; however, they should be able to go back and modify their plan if, once solved, all agree the solution doesn't make sense. It may be that teachers would like students to include words and mathematical notation in the plan, so emphasizing what is expected and highlighting student plans that meet these expectations are important when presenting solutions. This enables the teacher to point out what is considered exemplary as far as following the appropriate format for making a plan.

Visually Represent the Problem (MP #1 & MP #3)

The purpose of visual representation allows students to gain or communicate another perspective of the problem. Furthermore, the teacher can formatively assess students' understandings by looking at how the problem is represented visually as many misconceptions are evident at this stage. A focus on precise labeling and evaluation of diagrams, tables, lines, and so on is important for teachers to emphasize at this stage. Teachers should facilitate the interplay between text and other visual representations so that multiple representations of a mathematics problem (including equations, graphs, tables, and drawings) are as consistent with each other as they are accurate.

Identify What Is to Be Solved (MP #1)

During stage one, rewriting the problem to identify what is to be solved helps students focus on what is requested of them. At this stage, teachers should help students identify where the requests of problems are generally found (within last sentences), and model the format ELLs should use in rewriting the request. For example, teachers may expect students to include any technical vocabulary, units of measurement, or multiple requests (if more than a one-step problem) when rewriting the problem's request.

In closing, language and visual representations used for mathematics and expected for MPs #1 and #3 can confuse ELLs and inhibit their opportunities to learn and successfully participate in math discussions. The textbook index serves as a starting map for how teachers can sequence and mediate concepts, terminology, and procedures that evolve throughout the textbook and semester. Visual graphic representations provided in textbooks and by teachers of ELLs are positioned to enrich class discourse that describes, evaluates, changes, or creates these and other representations, while also challenging students to identify, convey, apply, complete, and solve the mathematics embedded within these visuals.

Language used in mathematics problems may result in visual graphic representations that help teachers gauge where ELLs are on the comprehension spectrum, either reinforcing and supplementing the mathematics prompted by the word problems, or serving as evidence that ELLs need modified and additional instruction in order to correctly align what they're being asked with how they are representing their solution pathways. Techniques typically foregrounding text and discourse,

such as graphic organizers like Venn diagrams and concept maps, are repurposed to include representations as elements of these graphic organizers that may further assist ELLs in learning secondary mathematics content and concepts. With purposeful planning, teachers can scaffold the processes necessary for ELLs to meet CCSSM MPs #1 and #3 and thus become proficient in mathematical communication, reasoning, and problem solving.

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