Chapter

2

Connections: Looking Back and Ahead in Learning

Geometry and measurement help us understand the space in which we live. Because they also connect closely with other areas of mathematics, teachers of children of *all* ages, regardless of the mathematical content that they are actually teaching, must understand some key ideas in geometry and measurement. This comes as no surprise.

Seeing Connections with Geometry at Other Age Levels

Each of the big ideas and essential understandings elaborated in chapter 1 helps teachers understand how children's geometric thinking develops across the grade bands. It is useful to begin by looking back at the earliest years.

Locating and visualizing: The earliest geometry

From birth, children possess remarkable competencies in observing and moving within their spatial world. Infants can focus their eyes on objects, and soon they begin to follow moving objects with their eyes. Toddlers use geometric information about the overall shape of their environment to solve location tasks; this is the intuitive basis of Big Idea 2. For example, figure 2.1 shows a baby and a mother, as seen from above, with the baby in a highchair and × marking a familiar toy in the baby's field of vision in the space between them.

Big Idea 2

Geometry allows us to structure spaces and specify locations within them.

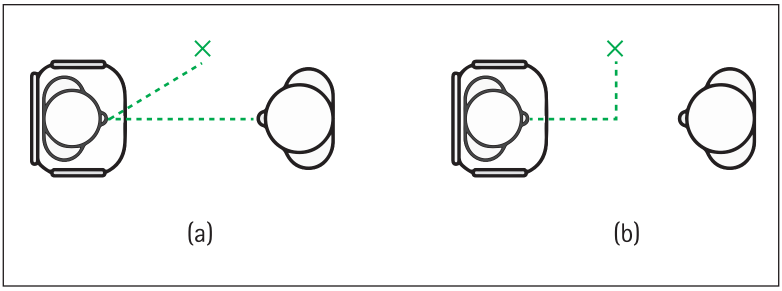

Fig. 2.1. As seen from above, a baby in a highchair opposite a mother, with × marking the location of a toy between them

The dotted lines in figure 2.1a indicate the baby's perception of the toy as about 20 degrees left of the mother and slightly closer than she is—without, of course, knowing anything about degrees or measuring distances. Yet, the baby does have a sense of orientation in space and of comparative distance.

The baby certainly does not locate the toy by thinking in x- and y-coordinates: "so much distance ahead and then so much distance directly to the left," as suggested by the dotted lines in figure 2.1b. A child thinks much more naturally in terms of direction and distance to a goal (polar coordinates) than of distance left-right and forward-backward (Cartesian coordinates).

The "geometry in action" that the baby does in locating the toy lays an intuitive foundation for a variety of mathematical topics, such as work with paths and polar coordinates that he or she will encounter much later. Very young children start to develop the foundation for these topics as they remember a location or route to a location through a pattern of movements associated with a goal.

Later, they learn entire paths. They remember locations as distance and direction of their own movements and landmarks found along that path. Crawling from the kitchen to the playroom, the child learns the location of the door, the turn, the mirror that marks the playroom's door. They build these competencies from an internal reference system—the reference is the self, moving through space. They lay the developmental foundation for understanding the concepts of geometric paths, straight paths, paths with bends (angles), distance and length, and, eventually, differential geometry (the study of the geometry of curves and surfaces).

Toddlers are also building experience with externally based reference systems. These systems and the experiences gained by using them in these early years support understanding of coordinate systems. The beginnings are spatial relationships within and between environmental structures and landmarks. Such landmarks are initially objects that are familiar and important. For example, the child might remember last seeing a toy under a couch against a certain wall and, moreover, might recall that the toy was closer to the end of the couch that is by the door. Such competencies develop into "mental maps" because they build knowledge of locations from distances and spatial relationships among environmental landmarks, structuring space as captured in Big Idea 2.

Children later may recall that the sand shovel was buried about halfway out from the wall of the garage and "about this far" from the edge on the street side of the sandbox, a remembered location that shows the development of precision central to Essential Understanding 2*b*. This is an early use of intuitive geometry that will later be articulated as Cartesian coordinate systems, as described in the discussion of that essential understanding.

At first, children naturally see most objects in their environment from many points of view, and they code all those distinct images as the same object without attending very much to how the different viewpoints affect the image. Their frequent early confusion of **p**, **b**, **d**, **q** (reflections and rotations of one another), as discussed in chapter 1, illustrates this point, which applies equally well to objects and pictures. Early geometric learning involves beginning to *notice* that the position of an object can change the way it looks. Children know that it is the same object, but they now also notice how the appearance can change. For example, the rim of a paper cup can seem either circular or elliptical or even appear to be a straight line, depending on how the cup is tilted. It looks circular when they stare down on it, straight into the cup, and looks like a straight line when they see the cup from the side, with their eyes level with the top.

The opposite occurs, too. Because we have special neurons devoted to recognizing *vertical* lines (presumably to help us remain upright ourselves), we privilege features that are "on top and bottom" or that suggest vertical lines. As a result, young children *see* the two shapes in figure 2.2 quite differently. They are most attuned to the vertical and horizontal lines of the first, and most attuned to the vertically and horizontally aligned *corners* of the second. A young child will actually *draw* the second figure not as four lines, but as four corners, not necessarily even making straight-line connections between those corners.

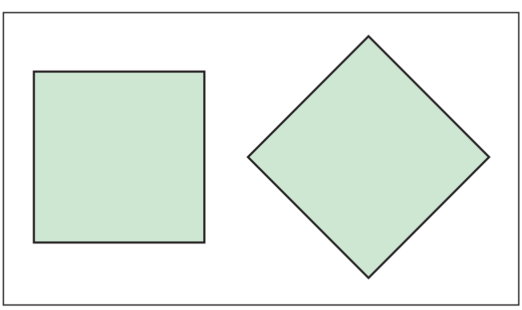

Fig. 2.2. A square in different orientations

Big Idea 2

Geometry allows us to structure spaces and specify locations within them.

Essential Understanding 2*b*

Geometry and measurement can precisely specify directions, routes, and locations in the world—for example, navigation paths and spatial relations—with precision. Given a reference point and an orientation, we can label position with numbers.

It is no surprise that they, and we, refer to the second shape as a "diamond" or at least as a "turned square" (or "rotated square"). Even after we know that the two are "the same shape," they don't look the same. These two squares are, by the way, the same size, but often people see the "diamond" as larger. So, sometimes we need to notice that things really are different, although we "see" them naturally as the same; and sometimes we need to notice that things really are the same, although we see them as different. Both require learning.

Further, children learn spatial and geometric vocabulary—terms that signify position, such as *in, on, under, up, down, beside, between, in front of, behind,* and later, *right* and *left*. This vocabulary is the linguistic basis for connecting children's early, intuitive ideas with the refinements and extensions that we know as mathematics. Intuitions become more precise models of everyday situations when we use mathematical ideas of number and shape, mathematical actions such as measuring or transforming shapes, and structural relationships among these ideas and actions. Mathematics involves both systematizing (refining, extending, and relating) these ideas and actions, and using the resulting models to solve problems. Learning mathematical language and using it are essential to this process of *mathematizing*.

Essential Understanding 2*b*

Geometry and measurement can precisely specify directions, routes, and locations in the world—for example, navigation paths and spatial relations—with precision. Given a reference point and an orientation, we can label position with numbers.

Bringing together ideas related to location and transforming shapes, children can mathematize their experiences with navigation and spatial relationships as they use and create simple models and maps. Block building (see fig. 2.3), including making models and maps of the classroom or playground, capitalizes on many of these experiences and has meaning when viewed by teachers who have grasped Essential Understanding 2*b*. Building with blocks also relates to students' later experiences with coordinate systems and the spatial structuring that underlie the measurement of area and other topics both in and out of geometry. For example, making a floor for a building with square blocks can be the beginning of spatial structuring. The ability to organize objects into rows or columns, or into distances from axes, can begin with such activity and is meaningful to teachers with a firm grasp of Essential Understanding 2*a*.

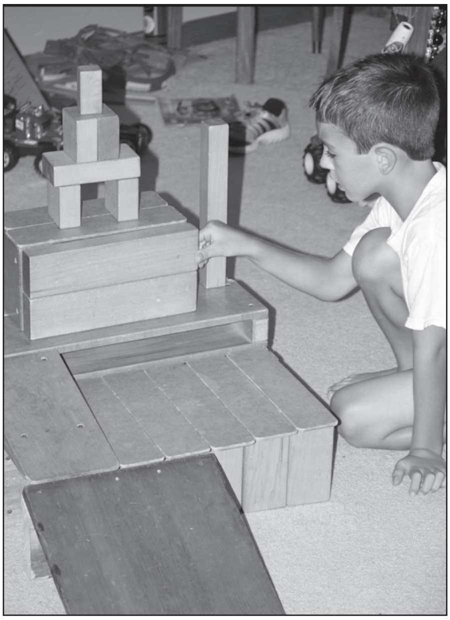

Fig. 2.3. A child building with blocks

Ideas about geometric shapes have their beginnings in early geometry. Shape is central to young children's understanding of the world. Children's attention to shape goes far beyond their learning names of familiar shapes such as circles, although that is important. When children learn new words for objects (such as "salt shaker"), the shape of that object is the main feature that they use, rather than the color, size, sound, or other attributes (Smith et al. 2001).

The shape of almost every object is a combination of "basic" shapes. In fact, that is what we mean when we *call* a shape "basic": it is the *basis* of many familiar but more complicated shapes. Think of the common simple representations of people with circular heads and rectangles for many other body parts, or the house represented as a triangle on top of a square. Such idealizations of shape are not restricted to children's immature ways of drawing. Artists are often taught to "see" complex forms geometrically and to lay out the forms with purely geometric guidelines, as in figure 2.4, before filling in the details that make the artwork look more "real."

Essential Understanding 2*a*

To describe a location, we must provide a reference point (an origin) and independent pieces of information (often called coordinates) indicating distance and direction from that point.

Fig. 2.4. An example of an artist's use of simple geometry figures

"Basic" shapes, just like complex forms, can also be decomposed, and children gain a head start in understanding some of these decompositions when they combine, for example, two blocks that are triangular prisms to make a prism that is square or rectangular, or a larger triangular prism. As when children lay down an array of blocks to make a floor, such shape compositions build the conceptual foundation for understanding geometric compositions, length, area, volume, and coordinate systems.

Building to geometry in grades 3–5

Big Idea 1

A classification scheme specifies for a space or the objects within it the properties that are relevant to particular goals and intentions.

The geometry learned during the early childhood years forms a critical foundation for geometry and measurement in grades 3–5. Early learning of shape becomes more systematic, with students *classifying* shapes explicitly on the basis of properties, and *drawing inferences* about the shapes on that basis (indicative of early work related to Big Idea 1). For example, students can now use a rectangle's four right angles to make inferences about the relationships among its sides, pairs of which must be parallel and the same length. Hierarchical inclusion—which involves recognizing that some classifications are subsets of other classifications—also begins more explicitly. For instance, students become aware that squares are special cases of both rectangles and rhombi. They can compare different ways to define a shape category (for example, defining a rectangle as a quadrilateral with all right angles or as a parallelogram with a right angle).

Big Idea 2

Geometry allows us to structure spaces and specify locations within them..

Children in grades 3–5 build on their early work with length measure to address precision and subdivisions of units. Their knowledge of length measurements, origins, units, and distances becomes the foundation for their learning of a Cartesian coordinate system, which is one way to structure space and specify locations (Big Idea 2).

Early work with sliding, flipping, and turning objects provides students with experiences that they later consider systematically as the geometric motions of translations, reflections, and rotations, respectively. Each of these transformations has particular characteristics and specific geometric attributes that it does or does not preserve. To describe the characteristics or attributes of a rotation, for example, one must specify a point around which the rotation occurs—the *center of rotation*—and an amount of rotation (an angle, taken to be counterclockwise by convention but typically specified in elementary school with *both* an angle *and* a direction clockwise or counterclockwise, for clarity). In the later grades, students study such characteristics explicitly. Further, the implications of these transformations—what does or does not change under a particular transformation—are also made explicit and studied. The mathematics of these transformations is discussed in chapter 1 (Big Idea 3), but most are developed in a school curriculum for students in grades 3 or above.

Big Idea 3

We gain insight and understanding of spaces and the objects within them by noting what does and does not change as we transform these spaces and objects in various ways.

Students in the intermediate grades extend earlier shape composition to reflect on spatial structuring, area, and volume. The next section discusses these measurement topics.

Making Connections with Other Topics

The concepts that are specific to geometry and measurement are complemented by essential mathematical habits of mind, many of which are shared by other mathematical domains, illustrating both the coherence of mathematics and the centrality—and therefore the usefulness—of these habits of mind for educators and students. Connections between geometry and the rest of mathematics both illustrate the elegance of mathematics and help students see that one domain can imbue another with new meaning.

Using number lines

Spatial, geometric, and measurement competencies connect directly with one of the basic models used in number and arithmetic: the number line. For mathematicians, just as for school children, the number line is a way of visualizing numbers both as locations and as distances between those locations. Each point on the number line is uniquely identified with a number, just as a house is uniquely identified with its address along a street. Houses generally use only whole numbers as addresses; likewise, number lines for the youngest children typically show only whole numbers.

By contrast, a mathematician's concept of a number line includes *all real numbers*—positive ones to the right of zero, and negative ones to the left, and all fractions and all other numbers as well, filling in the spaces. The number line, as children first see it in school, is generally shown as a horizontal line, with a point designated as zero and equally spaced points labeled 1, 2, 3, 4, … representing the whole numbers. Pictures of the number line come, as all pictures do, in various sizes. But whether the space between consecutive numbers is an inch or a centimeter, as it might be on a ruler, or a much larger space, as wall models usually require, *that distance* between two consecutive numbers on *that line* is considered to be "the unit" for that number line. This allows us to say that the distance between 7 and 10 is 3. The line segment from 0 to 1 is the conventional, or standard, example of the unit segment, and the number 1 is also called the unit, with numbers serving both as locations and as distances. Once we have determined this, all the whole numbers are fixed on the line.

For more details about connections between the number line and measurement, see *Developing Essential Understanding of Number and Numeration for Teaching Mathematics in Prekindergarten–Grade 2* (Dougherty et al. 2010).

Rulers are portable, physical representations of finite parts of number lines, so they can be readily used to find distances. Each ruler has its own unit—inches, centimeters, or, for special purposes, other units (or no *named* unit at all, as on wall number lines, with only the regular spacing serving as the unit). Thus, when we report a length (distance) that we measure with a ruler, we must also specify the unit of that particular ruler: the distance between the 7 and the 10 on an *inch*-ruler is 3 *inches*.

The number $^1/_2$ is exactly halfway between 0 and 1. We call the distance from 0 to $^1/_2$ "one-half." We see that exactly three of those distances to the right of 0 is a point exactly halfway between 1 and 2, which can be called "three halves" ($^3/_2$) because it is three *halves* from 0. We can also call that point "one and one-half" ($1^1/_2$) because it is 1 *and* $^1/_2$ units from 0. (This is an example of number serving as location and distance.) Rulers marked in inches often subdivide each unit into halves, quarters, and even smaller subunits. We can subdivide units on the number line any way we need, to find thirds, or eighths, or tenths, or seventeenths.

The number line clearly connects directly with linear measurement. It represents numbers—whole or rational or irrational (numbers that can't be expressed as 10ths or 17ths, or any other *n*ths)—as the length or distance from 0 to that number. Thus, both measurement and associated number line models can serve as tools for mathematics: number (including fractions and decimals), number comparison (any number "to the right" of another number is larger than that number), arithmetic (subtraction uses the distance between numbers as a way to compare them), and estimation (finding numbers "in the neighborhood"—that is, not too distant).

Number lines are especially important for understanding rational numbers, including fractions and their decimal representations. They can be used to illustrate the relationship between fractions and whole numbers and to demonstrate that fractions *are* numbers, that fractions include numbers greater than 1, that fractions can be added on a number line model in a way that closely mirrors adding whole numbers on a number line, and so forth. In all these ways, measurement can support building "mental number lines," which research suggests play a critical role in understanding mathematics (e.g., Elia, Gagatsis, and Demetriou 2007; Geary et al. 2008; Ramani and Siegler 2008; Rodriguez, Parmar, and Singer 2001; Vanbinst, Ghesquiere, and Smedt 2012).

Composing, decomposing, and unitizing

The composition and decomposition of shapes are core processes in geometry. Such processes, and the conservation of area, are useful conceptual tools for solving problems involving tessellations, area, and so on. However, these tools are also useful for understanding number. For instance, although the number line serves best to show fractions *as numbers*, fractions also describe amounts *of* something, and the decomposition of a geometric whole into parts of equal area can serve as a useful model of that use and aspect of fractions. Consider an example. Suppose that we choose the area of the equilateral triangle in figure 2.5 as the unit with which to measure the area of the larger shape. The area measure is then 18 units. It takes 18 of the units to cover the region.

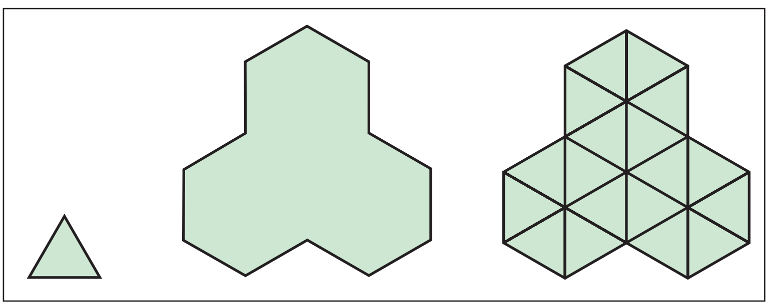

Fig. 2.5. An equilateral triangle used as the area unit to measure area of a figure

If we choose, instead, to measure the same region with a unit that is exactly 6 times the area of the original unit—that is, with the area of a regular hexagon in place of the equilateral triangle—we will now use $^1/_6$ as many units as before. We will now report the area as 3 units. Figure 2.6 provides examples for $^1/_3$ $^1/_9$, and $^1/_{18}$. This experience exactly parallels the experience with different size units for measuring length and helps establish a general truth about measurement, not just a specific kind of measurement.

In *Developing Essential Understanding of Rational Numbers for Teaching Mathematics in Grades 3–5*, Barnett-Clarke and colleagues (2010) use number line, unit, and iteration to interpret rational numbers as measures.

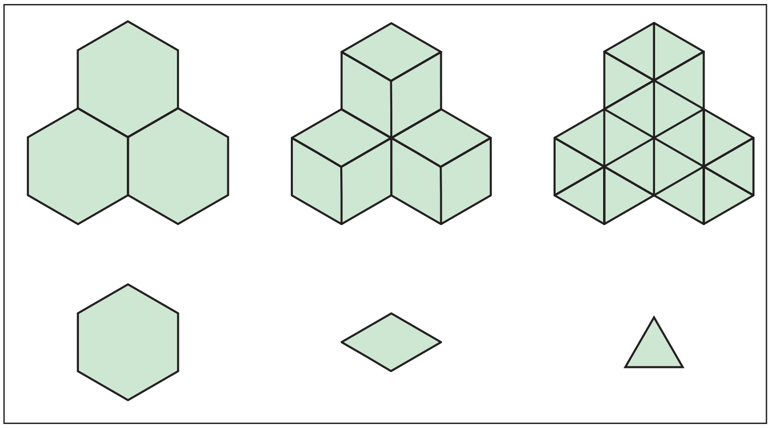

Fig. 2.6. Finding the area of same shape with different units

If the area unit is the area of the hexagon, the area is 3 units. If the unit is $1/3$ that size—the area of the rhombus—the area is 3 times as many units, or 9 units. If we change units to use units of one-third the size, we will need three times as many of them to measure the area: $1/3$ is the multiplicative inverse of 3. If we choose instead to measure the area with a unit that is $1/2$ the size of the rhombic area unit, this time using the area of the triangle, we will need two times as many of them to cover the region: $1/2$ is the multiplicative inverse of 2. Working with the area of the triangle as the unit, we will report the area as 18 units. The *number* of units that we report—the area measure—depends on the *size* of the units that we use, and the relationship is an inverse one under multiplication: the smaller the unit, the larger the measure.

The inverse relationship between the size of the unit and the number of units that compose a given shape or quantity is thus modeled in geometric shape composition and in area measurement. It also mirrors the inverse relationship between the size of the counting numbers (increasing as 1, 2, 3, 4, ...) and unit fractions (decreasing as $1/2$, $1/3$, $1/4$, ...).

Further, all numbers are structured by composition related to a specific unit. The decimal place-value system groups by tens. Base-ten blocks are designed to embody this structure. If we choose the length of one edge of a small cube to represent “one,” then the length of the rod formed by the lengths of the edges of ten cubes represents “ten,” as in figure 2.7. This is similar to the composition of shapes that are combined to form a larger shape that is conceptualized as a new shape (a rectangle, say) as well as a composite unit of smaller shapes (say, squares). This reflects the importance of viewing a number flexibly as “10 tens” and “1 hundred.”

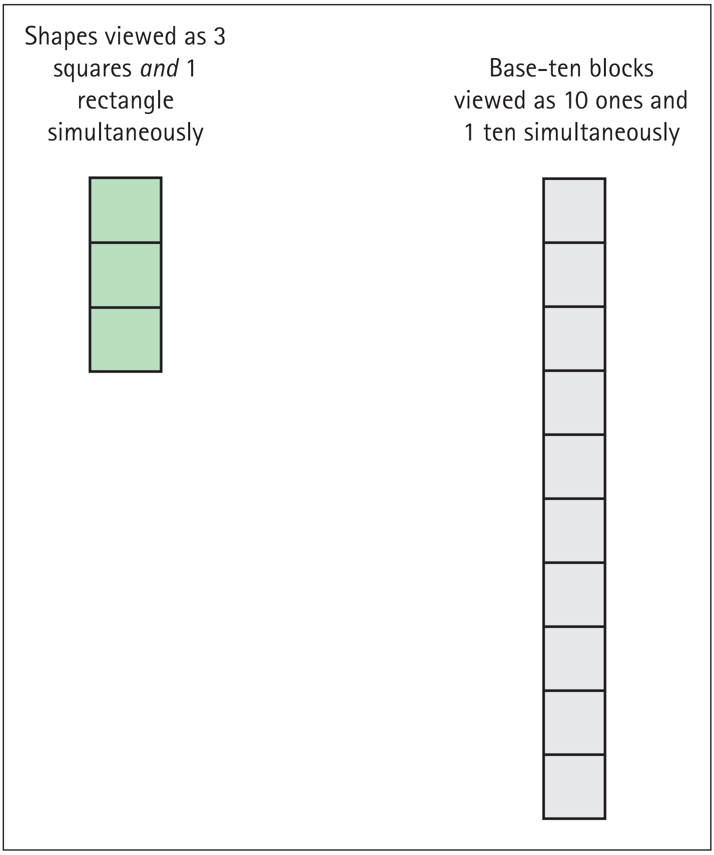

Fig. 2.7. Viewing shapes as a composition of units

Having a firm grasp of composition and decomposition in geometry and measurement broadens students' understanding of parts and wholes and provides concrete models for numerical operations that combine, separate, or compare numbers.

Working with spatial structures

A particularly important example of composition and unitizing is in spatial structuring. One way to view the area of a rectangle is as partitioned into identical square units organized in rows and columns, as illustrated in chapter 1 (see fig. 1.26). Such decompositions, organized in rows and columns, lend a mathematical interpretation to the formula *Area* = *length* × *width*. Each row contains the same number of square units, and the number of rows in the rectangle equals the number of squares in a column.

Another connection between geometry and arithmetic involves rectangles and area. The commutative property of multiplication (by which, for example, 5 × 3 = 3 × 5) connects with the notion that one

See *Developing Essential Understanding of Number and Numeration for Teaching Mathematics in Prekindergarten–Grade 2* (Dougherty et al. 2010) for an extended discussion of unit and place value.

can group the squares in the columns together into units, and then the number of squares in a row equals the number of those units. Alternatively, one could consider that a geometric motion, a 90-degree rotation of the rectangle, "switches" the rows and columns.

The rotation of a rectangle is also an application of conservation of area. As is true of any shape, the area of a rectangle does not change with a rotation. However, young children may think that if a rectangle is rotated so that it is "taller" than the original rectangle (see fig. 2.8), it will have more area.

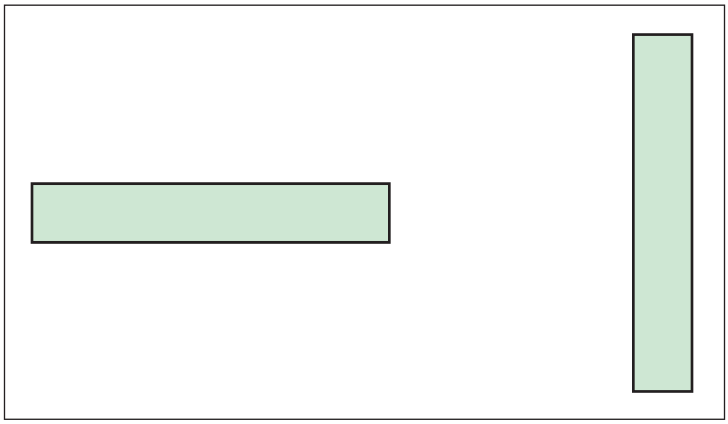

Fig. 2.8. A rectangle rotated to produce a "taller" rectangle

For a discussion of the meaning of multiplication and properties of multiplication as illustrated in array models and other models of multiplication, see *Developing Essential Understanding of Multiplication and Division for Teaching Mathematics in Grade 3–5* (Otto et al. 2011).

Area models for multiplication are important for another reason. Using arrays of discrete objects (for instance, 4 rows of pennies with 8 in each row) to model multiplication works well for whole numbers, but not in general for rational numbers. For example, 2.5 rows containing 5.1 pennies each makes no sense. Because area models based on measurement are continuous models and can be partitioned into units of any size, they lend themselves well to rational numbers and thus to the multiplication of fractions and decimals.

Just as the area of a rectangle tiled by unit squares can be measured by counting unit squares per row and multiplying by the number of rows—that is, multiplying rows by columns, height by width—the volume of a rectangular solid can be measured by counting the unit cubes in a layer (multiplying rows by columns in that layer) and multiplying by the number of layers. The formula *Volume = length × width × height* is explained by this spatial structure.

Thinking of a plane as an infinite array is a different spatial structuring. This structuring relates to a Cartesian coordinate system, in which an ordered pair of numbers identifies each location, or point, in the plane and indicates its distance from two coordinate axes.

Conclusion

The sophistication of geometric concepts required in later grades builds on the experiences that students have in the early grades. Understanding spatial relationships is one of the most important outcomes from children's experiences. Children learn to structure and understand space in new ways. It is through spatial relationships that children come to understand transformations of shapes in planes and space, discerning what is or is not changed by the transformations.

Measurement concepts are closely aligned with transformations as children consider iterations of units to form measurement tools. This leads to an understanding of how the size of the unit affects the measurement count.

Even though geometry is often not stressed in school mathematics in the same way that number and operations are, the topics centered on geometry and measurement clearly link to and support development across other mathematical areas. Teachers who understand how these topics support learning beyond geometry can use geometric concepts to support students in building stronger knowledge related to other topic areas.