2 Focusing on Multiplication and Division

In grade 3, students are introduced to the formal operations of multiplication and division. The focus is on understanding the operations and on finding patterns and strategies related to certain factors, with the eventual goal of gaining fluency in using all basic multiplication and division facts. Students gain this understanding and fluency by being able to represent and interpret multiplication and division in a variety of ways, including equal groups, arrays, and area models, and by developing an understanding of the mathematical relationship between multiplication and division.

Instructional Progression for Multiplication and Division

The focus on multiplication and division in grade 3 is supported by a progression of related mathematical ideas before and after grade 3, as shown in table 2.1. To give perspective to the grade 3 work, we first discuss some of the important ideas that students focused on in grade 2 that prepare them for multiplication and division in grade 3. At the end of the detailed discussion of this grade 3 Focal Point, we present examples of how students will use the multiplication and division understandings and skills in later grades. For more detailed discussions of the "before" and "after" parts of the instructional progression, see the appropriate grade-level books, *Focus in Grade 2* and *Focus in Grade 4*.

Table 2.1 represents an instructional progression for the understanding of multiplication and division in grades 2 through 4.

Early Foundations in Multiplication

In grade 2, students have experienced skip counting, the beginning of the foundation of understanding multiplication as the joining of equal groups. For example, when students in grade 2 skip count groups of two things by 2s, they say 2, 4, 6, 8, 10. At this level, we emphasize that each number in this sequence is 2 more than the number before it. In grade 3, we want students to learn that as they count 2, 4, 6, 8, 10, the numbers they are saying are the totals for different numbers of groups of 2, or different multiples of 2. They are naming the total for one group of 2, two groups of 2, three groups of 2, four groups of 2, five groups of 2, and so on. This idea of "joining equal groups" is the basic meaning for whole-number multiplication.

Students in grade 2 also prepare for multiplication by observing patterns on the hundreds grid. For example, students can highlight the results of joining groups of 10, or skip counting by 10s, as in figure 2.1, to see the pattern of 0s in the ones place.

Table 2.1

Grade 3: Focusing on Multiplication and Division—Instructional Progression for Developing Understanding in Grades 2–4

Grade 2	Grade 3	Grade 4
Students use the base-ten numeration system and place-value concepts to represent numbers up to 1,000. Students use place value and properties of addition and subtraction to compose and decompose multidigit numbers.* Students use addition and subtraction to answer questions about joining and separating situations involving equal groups.* Students skip count by various numbers to build foundations for understanding multiples and factors.*	Students use equal-group, array, and area situations and models to represent and interpret multiplication and division and their relationship. Students use multiplication and division to solve problems. Students connect skip counting with the multiples of a number. Students use patterns in lists of multiples (e.g., in a multiplication table) to learn basic facts. Students use properties of addition and multiplication (e.g., commutative, associative, and distributive properties) and known facts to find related unknown facts. Students use the mathematical relationship between multiplication and division to view "finding a quotient" as "finding an unknown factor."	Students work toward quick recall of basic multiplication facts and related division facts. Students expand their application of multiplication and division to solve problems, including those involving scalar comparison and combination situations. Students use their knowledge of multiples to extend whole-number division to include division with remainders.** Students use place-value patterns to find multiples of tens, hundreds, and thousands. Students apply their understanding of representations for multiplication (i.e., equal groups, arrays, area models, and scalar comparisons), knowledge of basic facts, place value, and the distributive property to multiply multidigit numbers. Students use properties of multiplication and patterns in place value to estimate and calculate products mentally. Students use properties to apply the standard algorithmic approach to multiplication with multidigit numbers and use this approach to solve problems.

* Appears in the Grade 2 Connections to the Focal Points

** Appears in the Grade 4 Connections to the Focal Points

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Fig. 2.1. Pattern for tens on a hundreds grid

Work in grade 2 related to addition "across a decade" is also helpful for developing speed and accuracy in determining an unknown fact from a known fact. For example, in counting by 4s, students can go from 28 to 32 by thinking 28 + 4, 28 + 2 + 2, 30 + 2, 32. Similarly, when counting by 6s and going from 48 to 54, students can think 48 + 6, 48 + 2 + 4, 50 + 4, 54. Ease in making these transitions from twenties to thirties, to forties, to fifties, and so forth, will help students develop fluency in using basic multiplication and division facts.

Building Depth of Understanding in Multiplication and Division

Meaning of multiplication

Multiplication is first introduced as the joining of equal groups. Students learn that the total in 3 groups of 4, or 4 taken 3 times, is written as $3 \times 4 = 12$. The numbers being multiplied are the factors; the answer is the product. Figure 2.2 shows the equal groups model for multiplication. To model $3 \times 4 = 12$, students can draw 3 groups with 4 squares in each group. The product is the total number of objects.

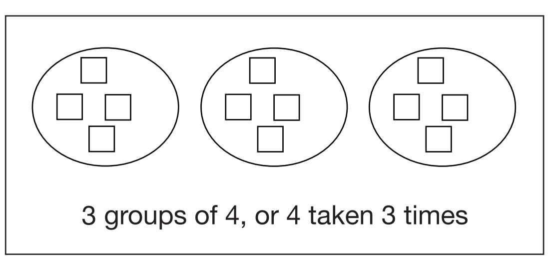

Fig. 2.2. Model of multiplication as joining equal groups

As students gain a deeper understanding of the equal-groups model, they can begin to move away from using objects to show equal groups and counting the objects to the more sophisticated model of writing the number 4 to represent each group and then using skip counting by 4 to determine the product. This move to a more abstract model of equal groups is an important one in students' conceptual development of the meaning of multiplication.

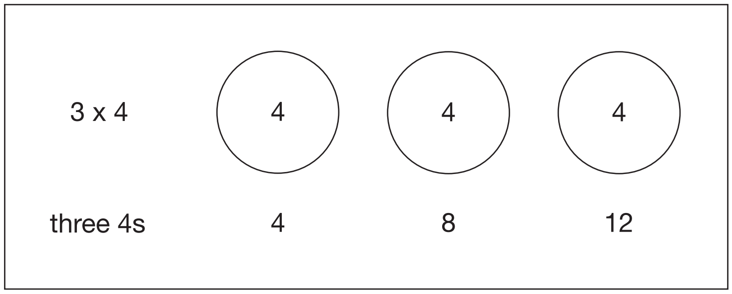

Fig. 2.3. Relating skip counting to a model of equal groups

Note that people in different parts of the world have different conventions about how to read 3 × 5. People in the United States have a nonmandatory interpretation of 3 × 5 as 3 groups of 5, but it could just as easily be read as "3 taken 5 times." Children coming from different places may use opposite conventions, so these different interpretations need to be recognized and discussed early on. To facilitate conversation about a particular equation, a helpful tactic might be to draw a circle around the factor that is being used to describe the size of the group. However, in the long run students benefit from being flexible in their interpretations of the two meanings of the factors.

Another way to model multiplication as the joining of equal groups is with an array. An array is a group of objects arranged in rows of equal length and columns of equal height. These rows and columns show the number of equal groups and the number in each group (the factors). The product is the total number of objects in the array. Students can interpret the array in figure 2.4 by looking at the rows to see 3 rows of 4 (3 groups of 4). By looking at the columns of the same array, students can also interpret the array as 4 columns of 3 (or 4 groups of 3). Students may skip count to find the total number of objects in the array. For example, in figure 2.4, they may skip count by the number in each row, 4, saying 4, 8, 12, and then know that 3 rows of 4 is 12, or 3 × 4 = 12.

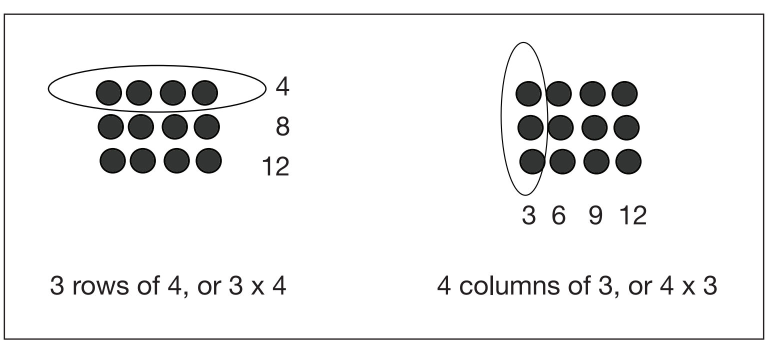

Fig. 2.4. Different interpretations of an array

Even though 3 groups of 4 are physically different from 4 groups of 3, the array model shows that mathematically, in multiplication, it does not matter which factor represents the groups and which factor represents the quantity in each group. This property of multiplication is called the *commutative property;* multiplying factors in any order results in the same product.

In the area model, the squares in an array are pushed together into a rectangle, as in figure 2.5. Students can also relate the area model to the equal-groups model by circling groups of rows or groups of columns.

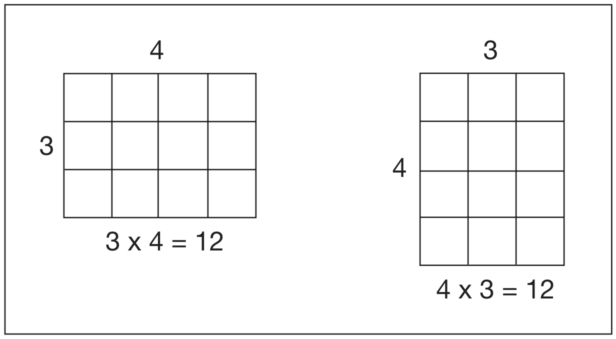

Fig. 2.5. Model of multiplication as area of a rectangle

Using area models is an effective way to demonstrate in general the commutative property of multiplication because the area model connects with both arrays and equal groups. By rotating the 3 × 4 rectangle having 3 rows of 4 squares, students can see that it is the same as a 4 × 3 rectangle having 4 rows of 3 squares. The rectangle itself did not change, so the total area is still 12 unit squares, which can be represented by 3 × 4 or 4 × 3.

Using a variety of models to represent multiplication and exploring connections among the models will enhance students' understanding of the meaning of multiplication. This understanding will facilitate their learning of basic multiplication facts.

Strategies for learning multiplication and division facts

A solid understanding of the meaning of multiplication will give students the foundation they need to learn, retain, and apply the basic multiplication facts. Many strategies involving the meaningful application of the properties of multiplication can be used to help students find products. The area models from figure 2.5 provide a pictorial example of the commutative property of multiplication. They show that 3 × 4 = 4 × 3 = 12. The commutative property of multiplication states that two factors can be multiplied in any order to get the same product. Applying this understanding of the commutative property to all numbers will help students learn their multiplication facts at a much faster pace because the number of facts that students need to learn is decreased almost by half (see table 2.2).

The associative property of multiplication also can be used by students to learn new facts. The associative property of multiplication, when combined with the commutative property, allows more than two factors to be multiplied in any combination of two factors in any order. In particular, the associative property can be used to apply the idea of doubling to learn new facts. For example, if students know that 2 × 7 = 14, they can use doubling to find 4 × 7. Students who know that 4 is double 2 can think of the product 4 × 7 as (2 × 2) × 7, or 2 × (2 × 7). Since 2 × 7 = 14, then 4 × 7 is two of that product, and 14 + 14 = 28. Figure 2.6 gives an example of doubling.

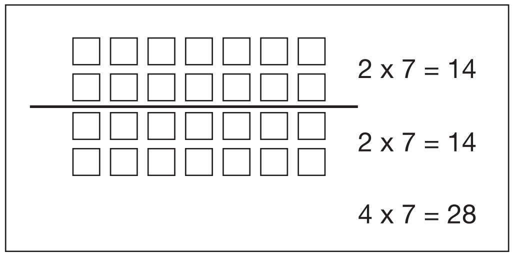

Fig. 2.6. Using the associative property of multiplication to double a known fact

In grade 3, students are introduced to the distributive property of multiplication over addition as a strategy for using products they know to find products they do not know. The distributive property of multiplication over addition is expressed mathematically as $a \times (b + c) = (a \times b) + (a \times c)$. In other words, a factor can be "taken apart" into addends and the addends can each be multiplied by the other factor to find partial products, and then those partial products can be added to find the total product. For example, in figure 2.7, to find 8 × 6, students who know the multiplication facts for 5 can think of 6 as 5 + 1. Then they can multiply 8 × (5 + 1). Through the distributive property, they can "distribute" the 8 over the 5 + 1 and rewrite the expression 8 × 6 as (8 × 5) + (8 × 1). Then they can use the facts they know, that 8 × 5 = 40 and 8 × 1 = 8, and can then add 40 + 8 to get 48. Thus, they have used the distributive property of multiplication over addition to determine that 8 × 6 = 48.

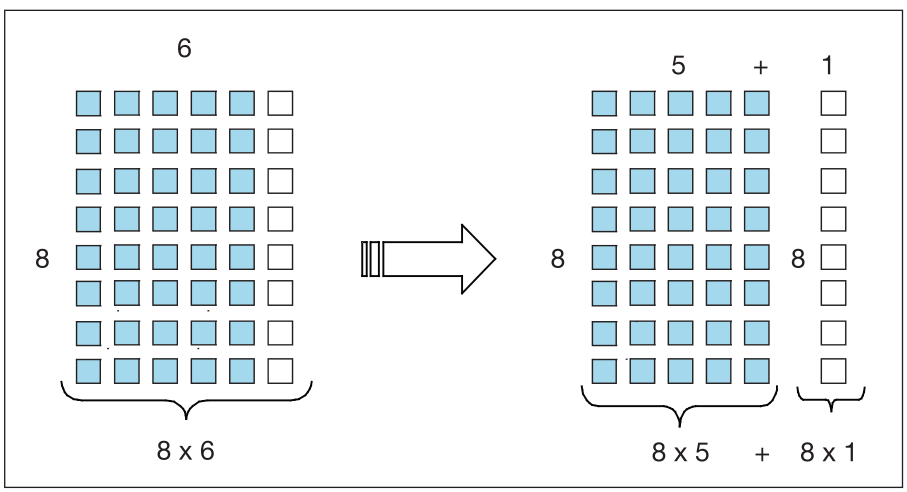

Fig. 2.7. Picturing the distributive property of multiplication over addition:
$8 \times 6 = 8 \times (5 + 1) = (8 \times 5) + (8 \times 1)$

Area models can also be used to illustrate the use of the distributive property of multiplication over addition to build unknown products from known products. For example, figure 2.8 shows a model of 7 × 8. The area model is divided into two sections: one section that represents 5 × 8 and another section that represents 2 × 8, because 7 can be thought of as the sum 5 + 2. Students can join the areas from both sections and add the areas to get a total product of 56. Thus, they can see from the model that 7 × 8 = (5 × 8) + (2 × 8) = 40 + 16 = 56. This strategy is more efficient than counting the squares in a 7 × 8 model to find the product of 56.

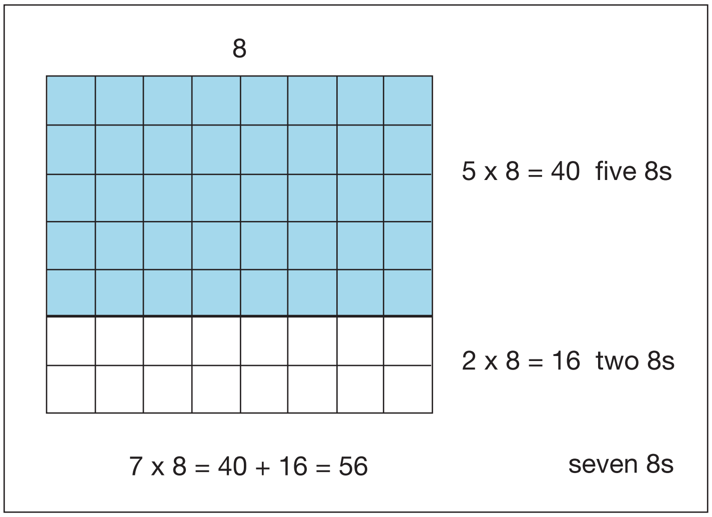

Fig. 2.8. An area model showing the distributive property of multiplication over addition

Patterns in a multiplication table can help students develop the conceptual groundwork needed for learning basic multiplication facts. For example, in figure 2.9, the multiples of 9 are highlighted. Students may notice how the multiples of 9 can be found across a row and down a column on a multiplication table. At first, students may identify patterns in the row and column—for example, the tens digit increases by one when reading across or down, and the ones digit decreases by one when reading across or down. Students may notice that the difference between adjacent values in the 10 column and the 9 column is the number that is multiplied by 10 to produce the value in the 10 column. For example, 70 = (7 × 10);
70 – 7 = 63 = 7 × 9.

x	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

Fig. 2.9. Multiplication table

Table 2.2 summarizes some of the strategies that can be used when multiplying with certain factors.

Table 2.2
*Some Strategies for Multiplying**

Multiplying by 0	The product is always 0. For example, 0 × 2 = 0.
Multiplying by 1	The product is always the other factor. For example, 1 × 5 = 5.
Multiplying by 2	You can skip count by 2s.
Multiplying by 3	You can multiply by 2, creating two groups, and add one more group. For example 3 × 7 is 2 × 7 plus one more 7.
Multiplying by 4	You can multiply by 2 and then double the product. For example, 4 × 7 = 2 × 2 × 7 = 2 × 14. Two 14s are 28.
Multiplying by 5	You can skip count by 5s. Or you can multiply by 10 and take half of the product. For example 8 × 5 is half of 8 × 10, or half of 80, which is 40.