into practice

Chapter 1
Attending to Invariance in Ratios and Proportions

Big Idea

When two quantities are related proportionally, the ratio of one quantity to the other is invariant as the numerical values of both quantities change by the same factor.

One big idea underlies ratios and proportions, as identified in *Developing Essential Understanding of Ratios, Proportions, and Proportional Reasoning for Teaching Mathematics in Grades 6–8* (Lobato and Ellis 2010). This is the idea of invariance among quantities in ratios and proportional relationships. A critical aspect of students' understanding of ratios and proportions is the notion that as one quantity in a proportional relationship changes, so does the other quantity—and by the same factor.

For at least the past two centuries, proportional relationships have been regarded as a critical topic for students to understand, and proportional reasoning as an essential skill for them to develop, in school mathematics. Cohen (2003) notes that books used by students in the nineteenth century often ended with the "rule of three." This was a procedural mechanism enabling students to attend to the invariance in a proportional relationship without understanding that invariance conceptually. Stated simply, the rule of three was an algorithm that "instructed the student to set down the three known numbers in a particular order, and then multiply two and divide the product by the third" (Cohen 2003, p. 51). The following is an example of a problem to which students might apply the rule of three (Carter et al. 2013, p. 41):

> Cans of corn are on sale at 10 for \$4. Find the cost of 15 cans.

Determining the ratio of 15 to an unknown dollar amount that is the same as the ratio of 10 to \$4 solves the problem. That is, by setting up an equation of fractions to indicate equivalent ratios,

$$\frac{10}{4} = \frac{15}{x},$$

and solving for x, students find that the cost of 15 cans is \$6.

This problem and others like it would fall under what Nicolas Pike (1809) characterized as the "rule of three direct": "If more require more, or less require less, the question belongs to the Rule of Three Direct. But if more require less, or less require more, it belongs to the Rule of Three Inverse" (p. 101).

In contrast to this rule-driven problem solving, the approach of Warren Colburn's early nineteenth-century textbooks more closely reflects that of recent NCTM publications. For example, one of Colburn's textbooks presented the following problem (1821, p. 44):

> A man had forty-two barrels of flour, and sold two sevenths of it for six dollars a barrel; how much did it come to?

Cohen (2003) notes, "Under the old arithmetic texts, this was a problem for the rule of three calling for careful written work, but Colburn expected students could reason it out without benefit of formula or paper" (p. 58). Cohen remarks that textbook authors such as Colburn (1826, p. 7) recognized the potential for the procedural use of the rule of three to interfere with the development of students' conceptual understanding of ratios, proportions, and proportional relationships:

> Those who understand the principles sufficiently to comprehend the nature of the rule of three, can do much better without it than with it, for when it is used, it obscures, rather than illustrates, the subject to which it is applied.

Thus, as early as 1826, U.S. textbook authors were drawing attention to proportional reasoning and the role of ratios and proportions in developing students' understanding of underlying concepts of invariance and scalability.

Along with building on the Big Idea and the related essential understandings outlined by Lobato and Ellis (2010), we hope to illustrate ways of working with invariance, scaling, and rates to develop students' understandings of ratios, proportions, and proportional relationships. This chapter focuses on invariance, the Big Idea of ratios and proportions, to establish a foundation for the chapters that follow.

Working toward the Big Idea of Ratios and Proportions

Providing your students with opportunities to engage in rich mathematical discussions engages you in two activities:

- Selecting, adapting, and designing tasks that will allow you to interpret your students' work

- Making thoughtful instructional decisions to build on your students' understanding

This work requires you, the teacher, to constantly challenge yourself to develop the specialized knowledge that will allow you to provide such opportunities for your students.

Developing this specialized knowledge involves examining the mathematics in the tasks that you select for your students and then using your mathematical and pedagogical content knowledge to predict the representational and computational strategies that they might use in responding to the tasks. Consider problem 1.1, for example:

Problem 1.1

Jamie solved the following problem for homework:

> Three-fifths of the sand went through a sand timer in 18 minutes. If the rest of the sand goes through at the same rate, how long does it take all the sand to go through the sand timer?

Jamie got 30 minutes for her answer. Show two different ways that Jamie could have solved the problem.

Reflect 1.1 asks you to examine problem 1.1 in the context of the Big Idea's focus on invariance. Later, Reflect 1.2 and 1.3 follow up on this work by asking you to analyze the work of three students with regard to the Big Idea.

Reflect 1.1

In what ways does invariance arise in problem 1.1?

How do you predict that students in grades 6–8 would attend to invariance in their solutions to the problem?

Problem 1.1 confronts students with the relationship between a quantity of sand (three-fifths of the sand) and the time for that quantity of sand to go through a given timer, assuming that the remainder of the sand will go through the timer at the same rate. That is, the ratio of "quantity of sand to time" for the remaining two-fifths of the sand to go through the timer is assumed to be equivalent to the ratio of "quantity of sand to time" for the initial three-fifths of the sand.

Underlying these two equivalent ratios is the tacit assumption that the ratio of *any* quantity of sand to time does not vary from the original ratio, *three-fifths to*

18 minutes. A further assumption is that for other timers to exhibit similar invariance in the ratios of quantity of sand in the timer to elapsed minutes for it to go through the timer, those other timers must be essentially identical to *our* timer.

If students approach the problem solely from the standpoint of the two equivalent ratios, they can reason that if $^3/_5$ of the sand takes 18 minutes to go through the timer, then $^2/_5$ of the sand must take 12 minutes to go through the same timer. They can determine that $^2/_5$ of the sand takes 12 minutes because they know that two-fifths is *two-thirds* of three-fifths, and *two-thirds* of 18 is 12. As figure 1.1 illustrates, by reasoning in this way, students assume only that the equivalence of the ratios is invariant: *three-fifths to 18* is equivalent to *two-fifths to the time for the remaining sand to go through the timer.*

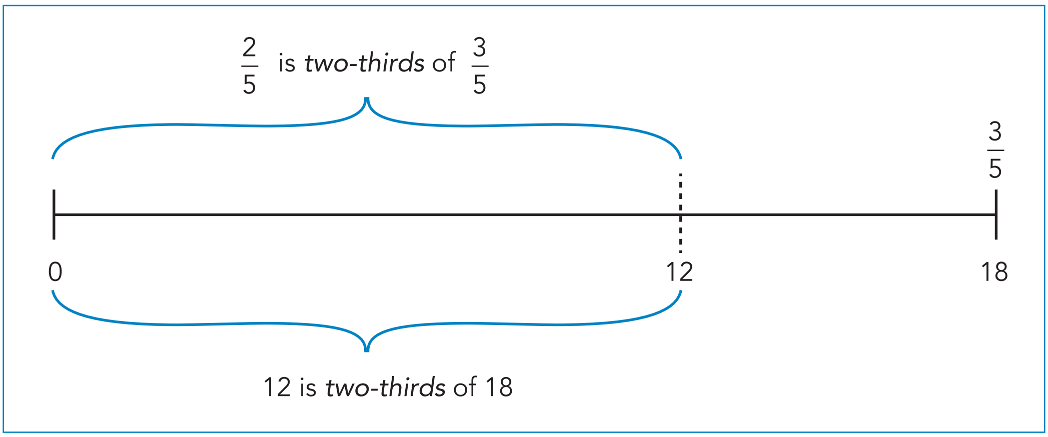

Fig. 1.1. Invariance in the relationship between the quantity of sand that has gone through a timer and the quantity of sand that remains to go through it, with respect to time required to go through the timer

The relationship between the original quantity of sand that went through the timer and the quantity of sand that remains to go through it should be the same as, or invariant from, the relationship between the original time taken and the remaining time needed. Namely, this invariant relationship is *two-thirds*. That is, the remaining quantity of sand, which is two-thirds of the quantity of sand that already went through the timer, should take two-thirds of the corresponding elapsed time. By reasoning about this relationship in this way, students can discern that the entire quantity of sand (five-fifths) should take the sum of the elapsed times for the respective amounts (three-fifths and two-fifths)—namely, 30 minutes—to go through the timer.

However, if students approach problem 1.1 by assuming that the first three-fifths of the sand went through the sand timer at a constant rate for any measurable amount of time, then the other invariant relationships will become evident. For example, given the ratio of $^3/_5$ of sand through the timer to 18 minutes, students may determine another ratio: $^1/_5$ of the sand through the timer to 6 minutes. Figure 1.2 illustrates one way in which students might determine the ratio $^1/_5$ to 6.

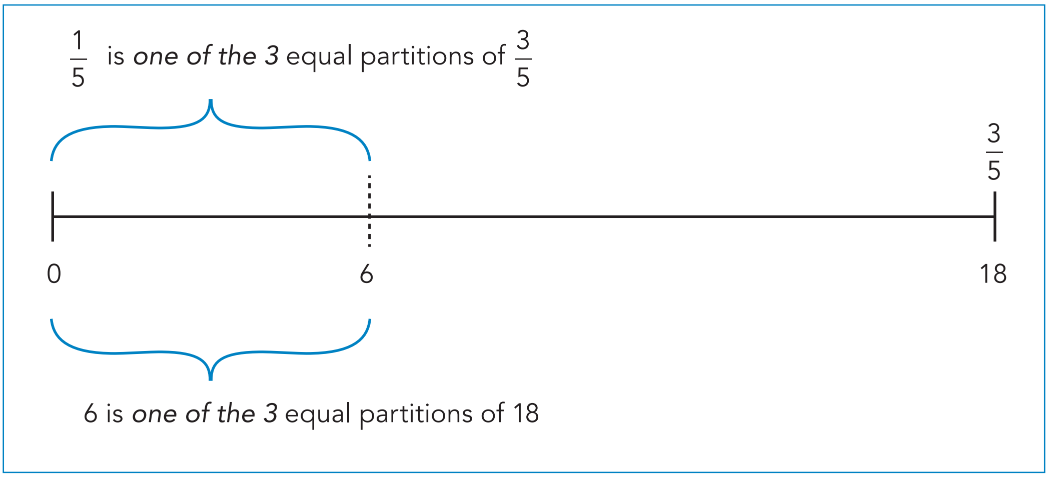

Fig. 1.2. Invariance in the relationship between a quantity of sand going through a timer and one part of an equal partitioning of that quantity of sand, with respect to elapsed time

Specifically, by identifying three equal partitions, each of $^1/_5$, students can determine the invariant relationship of *one to three*. Given that one of the partitions equaling $^1/_5$ is one-third the size of $^3/_5$, students can determine that one-third of 18 is 6. However, to understand the way in which Jamie may have gotten her answer of 30 minutes, they can now use this ratio to state that if one-fifth of the sand took 6 minutes to go through the timer, then five-fifths of the sand in the timer (the whole amount of sand in the timer) will take five times as many minutes—namely, 30 minutes (6 minutes taken 5 times, or 6×5).

The preceding discussions illustrate two of the possible ways in which Jamie might have interpreted the relationships in the problem. Were your predictions of how students might attend to invariance in explaining ways in which Jamie may have solved this problem similar to those in figures 1.1 and 1.2? If so, in what other ways could students attend to invariance? If not, how do your predictions match the students' work shown in figures 1.3–1.5? Use the questions in Reflect 1.2 to guide your examination of the work in these figures.

Reflect 1.2

Figures 1.3, 1.4, and 1.5 show work on problem 1.1 by three seventh-grade students, Marion, Judy, and Gottfried, respectively. How do these students attend to invariance in ratio relationships, as evidenced by their representations and explanations?

What do Marion, Judy, and Gottfried appear to understand about ratios and proportional relationships in the context of explaining Jamie's thinking?

1a. 3/5 = 18 min. × 2 = 6/5 = 36 min.

What I did was I tried to multiply 3/5×2 to get 6/5 and since 3/5=18min. tha multiplied by 2 would be 36min. Then I found that I could divide 18÷3=6 then multiply it by the denominator of 5. Then she got 6minutes for 1/5. She was able to multiply 6×5 to get 30 minutes.

Sand	Minutes
1/5	6 min.
2/5	12 min
3/5	18 min.
4/5	24 min.
5/5 or 1	30 min.

1b.

6+6+6+6+6 = 30 minutes

or

5+5+5+5+5+5 = 30minutes

I did this because porportion would be seperating the 30minutes by the amount of minutes per 1/5.

Fig. 1.3. Marion's response to problem 1.1

By examining students' work—other students' as well as your own—you can discover and explore new ideas and possible strategies for solving problems. To understand students' thinking, you must understand your own thinking—and many possible ways of thinking. Understanding multiple representations allows you to facilitate students' discussions of their own representations and strategies.