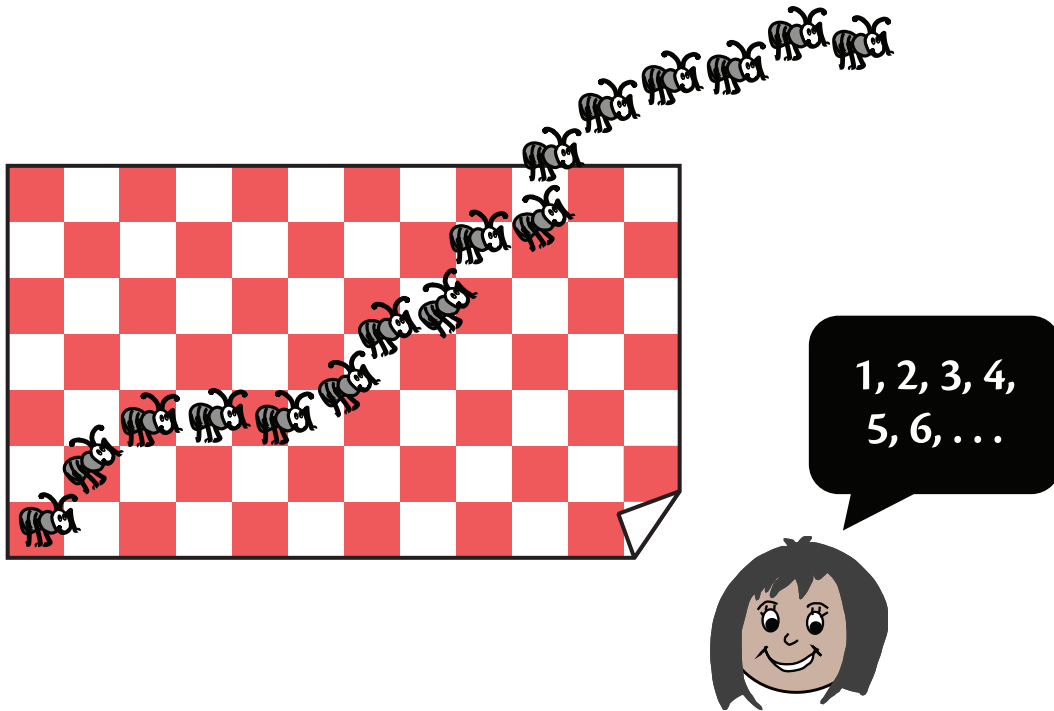


CHAPTER 2

Grades K–2

Counting Up by 1s	CCSS K.CC	6
Counting Back by 1s	CCSS 1.OA	8
Counting Up by 2s	CCSS 1.OA	10
Counting Back by 2s	CCSS 1.OA	12
Counting Up by 5s	CCSS 2.NBT	14
Counting Back by 5s	CCSS 2.NBT	16
Counting Up by 10s	CCSS 2.NBT	18
Counting Back by 10s	CCSS 2.NBT	20
Comparing Numbers by Matching	CCSS K.CC	22
Benchmark Numbers: All About 5	CCSS K.CC	24
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Addition: Commutativity	CCSS 1.OA	36
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Subtraction as Taking Away	CCSS K.OA	42
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Naming Three-Digit Numbers	CCSS 2.NBT	50
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Comparing Sizes of Numbers	CCSS 2.NBT	56
Fractions: Halves	CCSS 2.G	58
Fractions: Quarters	CCSS 2.G	60
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2-D Versus 3-D Shapes	CCSS K.G	68
Comparing 2-D Shapes	CCSS K.G	70
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Composing Shapes	CCSS K.G	74
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COUNTING UP BY 1s



Which ants did Meghan already count?
How high will she go to count all of the ants?

❖ **IN THE EARLIEST GRADES**, counting is fundamental. Counting up by 1s is the first sort of counting students learn about. Students have many opportunities to count in real life and to count using counting books, but the focus should extend beyond the mechanics of counting. Students should be provided with opportunities to consider the *process* of counting.

Questioning students as they count should help them learn some of the fundamental principles of counting: that each number is counted once and only once, that we use a consistent set of number words (in any one language), that the last number spoken tells how many, that the order of objects counted is irrelevant, and that it makes sense to track what has already been counted and what has not. The concept of counting by 1s is first addressed in [Common Core State Standards K.CC.](#)

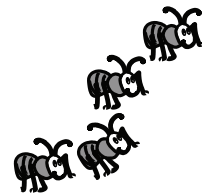
The picture provided here is designed to help students recognize that we want to find a way to distinguish what has already been counted from what must still be counted, and to give students an opportunity to see that we answer the question of

how many there are by counting each item exactly once. It also provides the opportunity for students to use visual estimation: by looking at the ants off the blanket and comparing them to the ants on the blanket, students can see that there are more than 6 ants still on the blanket, so they will need to say more than 6 more numbers. This is a precursor to the notion that a number more than double 6 is more than 12.

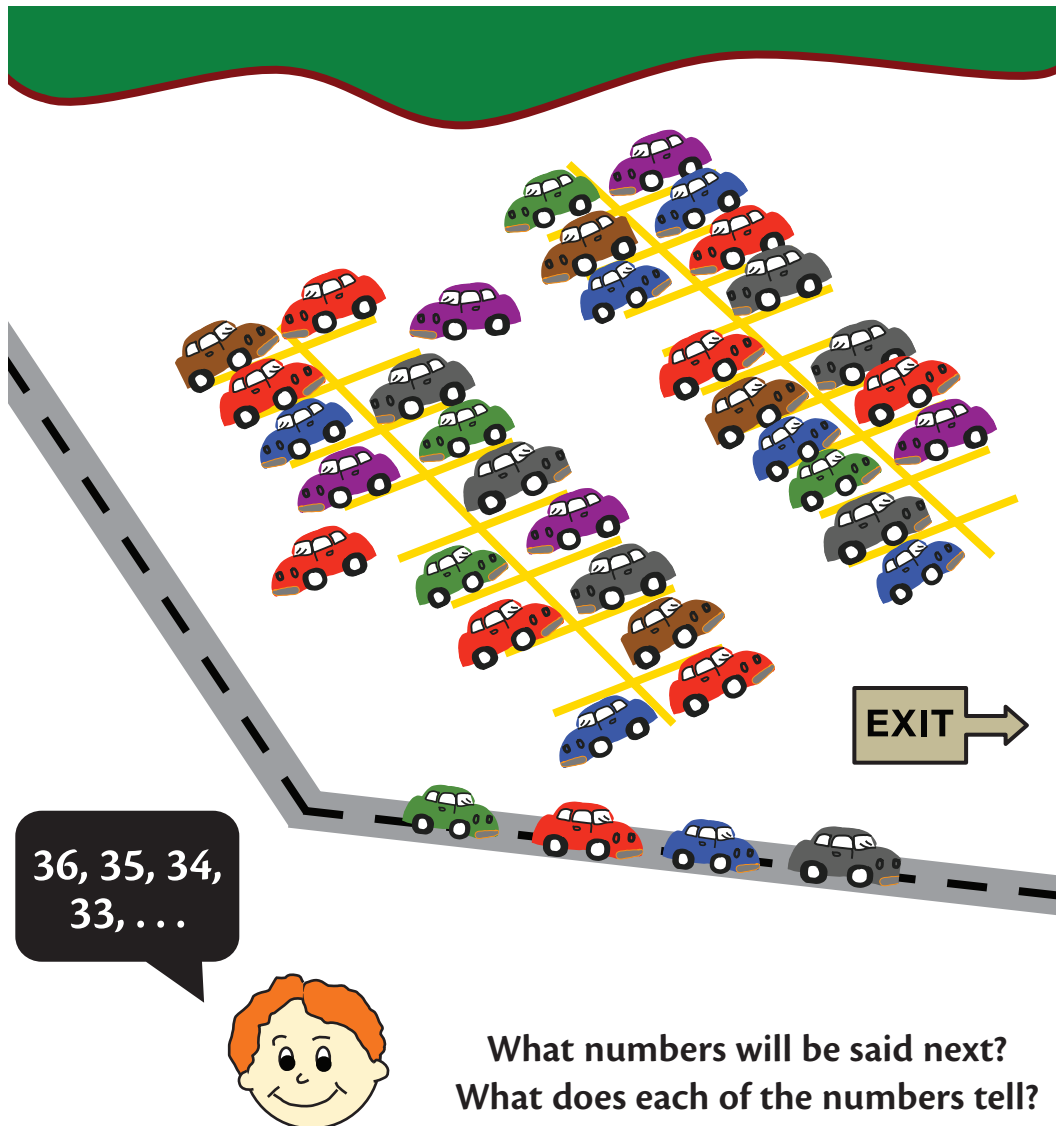
? **QUESTIONS** to supplement the question with the picture and to include in a conversation about the picture include

- *Are you sure which 6 ants Meghan has already counted?* [We want students either to recognize the 6 ants off the blanket that Meghan might have already counted or to realize that Meghan could have counted any 6 of the ants.]
- *How do you know there are more than 6 ants still on the blanket?* [We want students to use visual skills to see that the number off the blanket is less than the number on the blanket.]
- *Which ant will you be touching or looking at when you say 10? 20?* [We want students to show their ability to count and, possibly, their recognition that the count can be done in different ways.]
- *If you had counted differently, would the number of ants be different?* [We want students to realize that a set can be counted in different ways, but the total does not change.]
- *What patterns do you notice when you count?* [We want students to notice patterns as they count, for example, that after 12, there is a set of “teen” numbers.]

◆ **EXTENSION** Ask students to put out a fairly large set of objects and show two different ways to count how many there are.



COUNTING BACK BY 1s



◆ **ALTHOUGH YOUNG STUDENTS** are offered many opportunities to count up, there are fewer occasions when they are expected to count back. Yet not only is counting back practical for its own sake, it also supports later work in subtraction. Counting back is more difficult for students because they have to consider separately both how far back to count and how many are left. For example, to count back 3 from 10, you


have to keep track of how many numbers you say, you have to realize that the number 10 is not counted, and you have to realize that the number left is not the 3 that you were thinking about.

Questioning about counting back should focus not only on how many are removed or how many are left but also on the process of counting. For example, students might notice the pattern of the numbers as they go down or they might notice that the numbers they say relate to what is left, but the *number* of numbers they say relates to the amount that has been removed. The concept of counting back by 1s is first addressed in [Common Core State Standards 1.OA](#).

The picture provided here is designed to highlight the fact that if the first number you say as you are counting back is *thirty-six*, then originally there were 37, and not 36, items; we start counting back once an item has been removed. The picture also is designed to provide practice with counting back and to provide an opportunity to think about when we count back.

 **QUESTIONS** to supplement the question with the picture and to include in a conversation about the picture include

- ***Why might someone count back?*** [We want students to recognize that we count back when we are “getting rid” of things; the numbers we say tell how many items are left as one at a time is removed.]
- ***What does the number of numbers you say tell?*** [We want students to recognize that the number of numbers we say tells how many items are removed.]
- ***How many cars were in the parking lot to start with? How do you know?*** [We want students to realize that the number with which we begin is one more than the first number we say when we say a number each time an item is removed.]
- ***If you continue counting back, will there be any numbers that have a nine in them? Which ones?*** [We want students to notice some of the patterns in the numbers as we count.]
- ***Are there more items or fewer items left as you say more and more numbers as you count back?*** [We want students to become aware that the more numbers we say, the fewer items are left.]

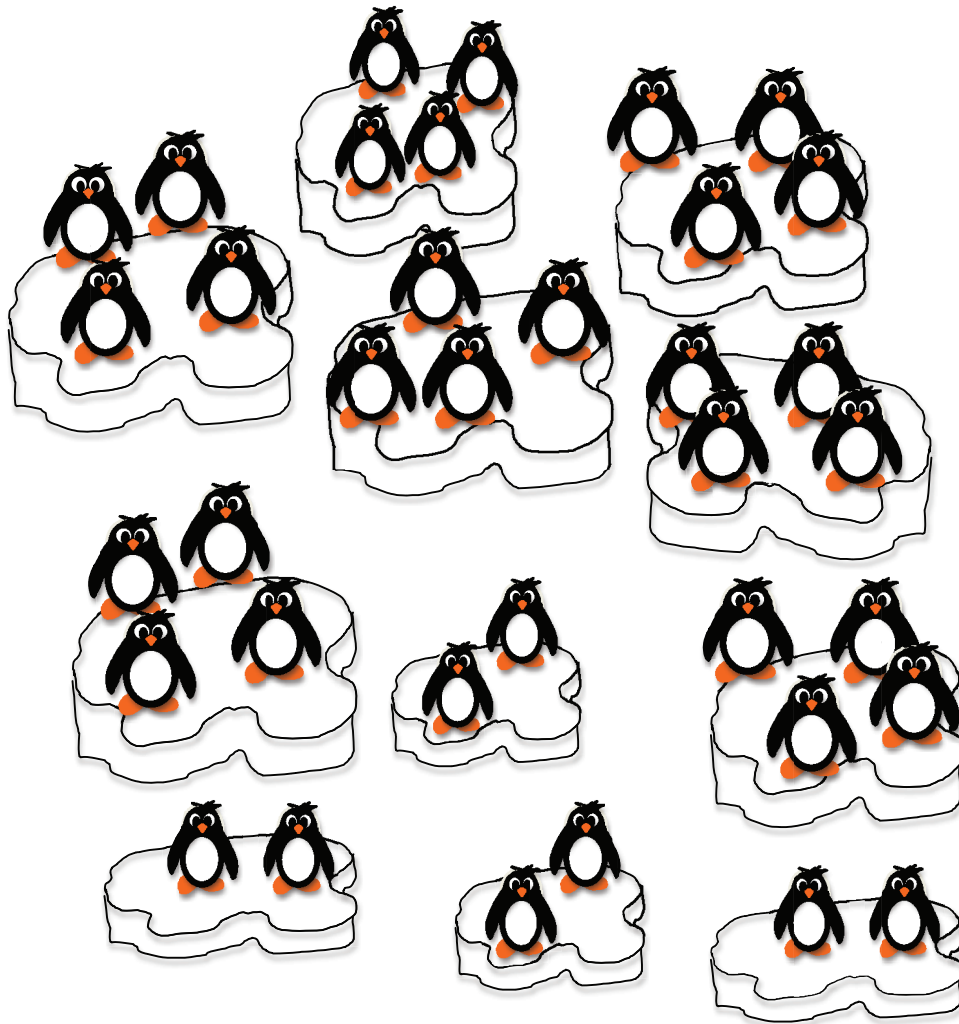
 **EXTENSION** Ask students to count back by 1s from 45 and talk about how that is like counting back from 37 (as in the picture) and how it is different.

CHAPTER 3

Grades 3–5

Multiplication: Equal Groups	CCSS 3.OA	80
Multiplication: Commutativity	CCSS 3.OA	82
Multiplication: The Distributive Principle	CCSS 3.OA	84
Multiplication: 2-Digit by 2-Digit	CCSS 4.NBT	86
Division as Equal Groups or Sharing	CCSS 3.OA	88
Division: Remainders	CCSS 4.OA	90
Rounding Numbers	CCSS 3.NBT	92
Place Value: Multiplying and Dividing by Powers of 10	CCSS 4.NBT	94
Place Value: Renaming Numbers	CCSS 4.NBT	96
Factors: What They Are	CCSS 4.OA	98
Factors Come in Pairs	CCSS 4.OA	100
Fractions: Representing	CCSS 3.NF	102
Fractions: Equivalence	CCSS 3.NF	104
Fractions: Comparing	CCSS 3.NF	106
Fractions: Mixed Number/Improper Fraction Relationship	CCSS 4.NF	108
Fractions: Common Denominators	CCSS 4.NF	110
Adding Fractions	CCSS 5.NF	112
Multiplying Fractions	CCSS 5.NF	114
Fractions: Multiplying as Resizing	CCSS 5.NF	116
Fractions as Division	CCSS 5.NF	118
Decimals: Relating Hundredths to Tenths	CCSS 4.NF	120
Decimals: Equivalence	CCSS 4.NF	122
Decimals: Adding and Subtracting	CCSS 5.NBT	124
Measurement: Time Intervals	CCSS 3.MD	126
Measurement: Area of Rectangles	CCSS 3.MD	128
Perimeter Versus Area	CCSS 3.MD	130
Measurement Conversions	CCSS 4.MD, 5.MD	132
Graphs with Scales	CCSS 3.MD	134
Coordinate Grids	CCSS 5.G	136
Classification of Shapes	CCSS 5.G	138
Parallel and Perpendicular Lines	CCSS 4.G	140
Lines of Symmetry	CCSS 4.G	142
Patterns Versus Non-patterns	CCSS 4.OA	144
Algebraic Thinking: Growing Additively	CCSS 4.OA	146
Algebraic Thinking: Shrinking Additively	CCSS 4.OA	148
Algebraic Thinking: Growing Multiplicatively	CCSS 5.OA	150

MULTIPLICATION: EQUAL GROUPS



Can you write $\square \times \square$ to describe this picture?

❖ **STUDENTS ARE INTRODUCED** to multiplication in this grade band as a way of describing the total count of equal groups. This understanding is critical not only to ensure that students have an understanding of what the operation of multiplication represents but also to promote the use of strategies to decompose and recompose numbers to be able to calculate with them more effectively. These skills are referenced in **Common Core State Standards 3.OA.**

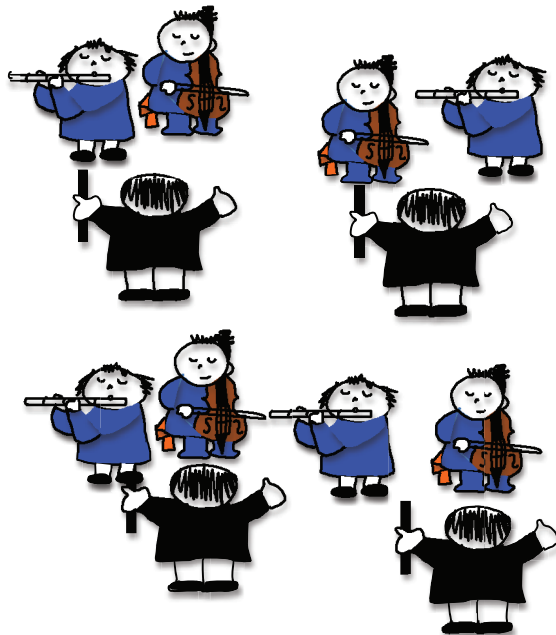
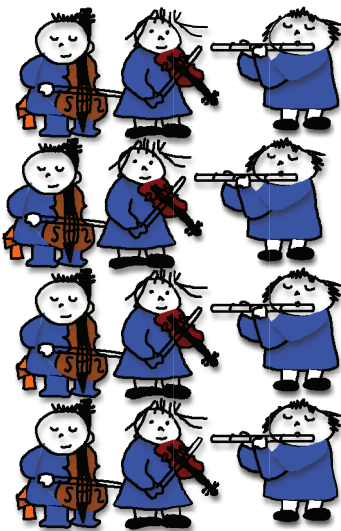
In the picture provided here, most—but not all—groups are equal. However, the groups of 2 can be rearranged to form groups of 4, or the groups of 4 can be broken down into groups of 2. A lively discussion could occur if some students are adamant that the penguin picture does not show multiplication while others recognize the possibility of rearrangement. Notice how much more interesting the question is with groups of 4 and an *even* number of groups of 2 (which can be rearranged into groups of 4) than it would be if there were either all groups of 4 (where there is only one reasonable response) or many groups of 4 and an odd number of groups of 2 (where only groups of 2 would be possible and not groups of either 2 or 4). Questions that are somewhat ambiguous tend to lead to good mathematical conversations.

? **QUESTIONS** to supplement the question with the picture and to include in a conversation about the picture include

- ***When do you use multiplication?*** [We want students to realize that multiplication describes situations involving equal groups.]
- ***Are all the groups of penguins the same size? Does that matter when you are deciding if you can use multiplication?*** [We want students to notice sizes of groups when deciding whether or not to use multiplication.]
- ***Could the penguins be rearranged into equal groups?*** [We want students to see that sometimes rearranging groups can change the way we describe them. For example, $7 + 9$ can be rearranged to $8 + 8$, which is a double, which might help us calculate the sum, but the fact that there was a double was not immediately obvious.]
- ***Is it easier to rearrange the penguins as shown here into equal groups than it would have been if there had been five icebergs with two penguins on them?*** [We want students to see that we could still have created equal groups of 2, but no longer equal groups of 4.]

◆ **EXTENSION** Ask students to create a different picture, using different numbers of items, that does not look like a multiplication situation at first glance but really is.

MULTIPLICATION: COMMUTATIVITY



Which pictures make it easy to see that $3 \times 4 = 4 \times 3$?
Which do not?

❖ **KNOWING THE COMMUTATIVE PRINCIPLE** for multiplication will cut the number of multiplication facts students need to learn almost in half. It will also help them to be more flexible in numerical calculations. Organizing sets in an array makes it easier for students to see a number of principles, including the commutative and distributive principles.

The picture provided here is designed to show that

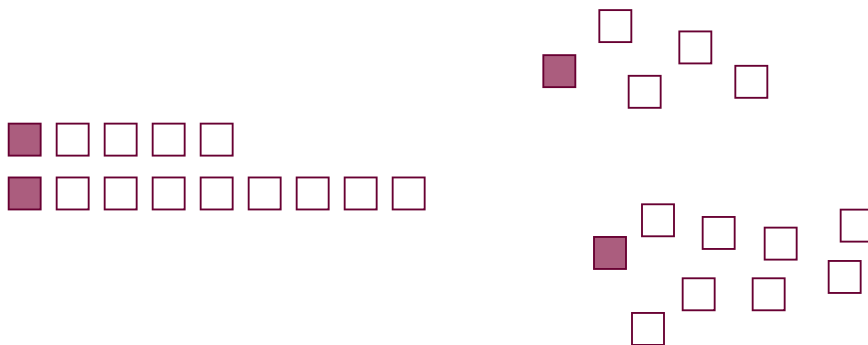
- $a \times b = b \times a$, but it is not always easy to see why
- if a set of equal groups is shown in an array formation, it is easier to see why $a \times b = b \times a$

The value of commutativity is referenced in [Common Core State Standards 3.OA](#).

This picture is designed to contrast three situations: one where an array is used so that commutativity of multiplication is very clear (since 4 rows of 3 is clearly 3 columns of 4); one where it is not too difficult to pull out 3 groups of 4 (the conductors, flute players, and cellists), even though the visual really only shows 4 groups of 3; and one where it is more challenging to find the 3 groups of 4 among the 4 groups of 3. Ideally students should be able to distinguish the three situations by the end of the classroom discussion.

? **QUESTIONS** to supplement the questions with the picture and to include in a conversation about the picture include

- **What does the 4 tell you about each of the three pictures? What does the 3 tell you about each picture?** [We want students to realize that a factor could be the number of groups or the size of a group.]
- **How are the pictures alike?** [We want students to see that there are many ways to represent 4 groups of 3, including the array.]
- **How are the pictures different?** [We want students to see that some visual representations of mathematical ideas make it easier to see principles than do other visual representations. For example, in the diagrams below, the one on the left makes it easy to see why you can add the same amount (the one dark square) to both numbers (4 and 8) without changing the difference, but the picture on the right does not make it as easy.]



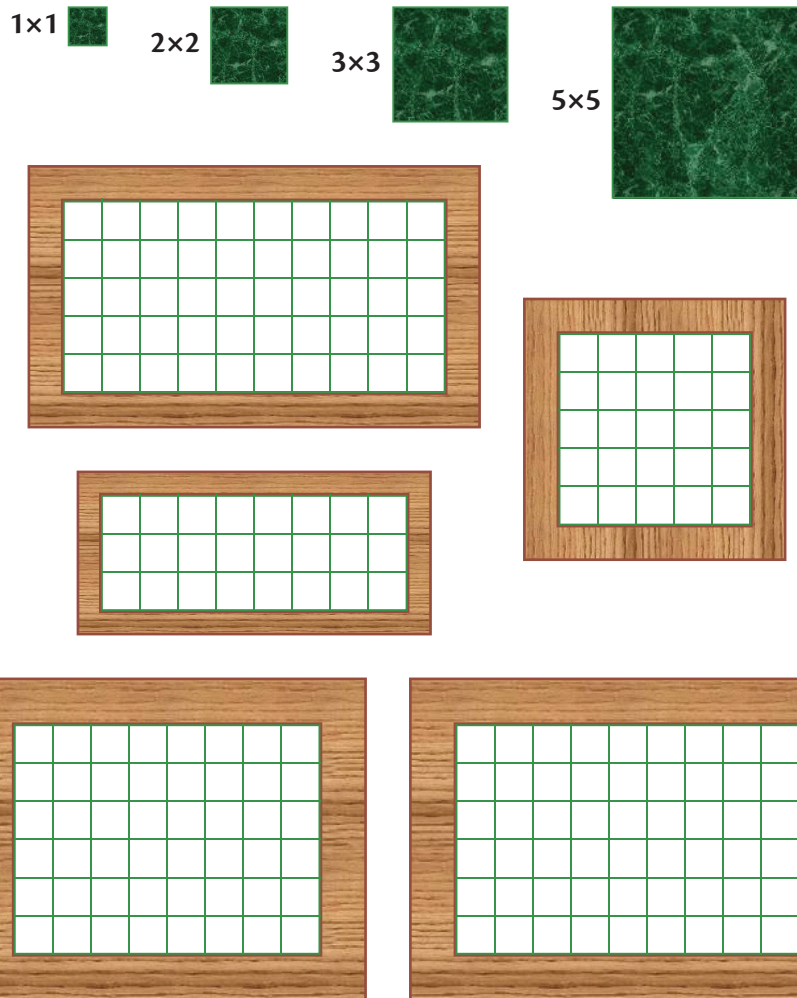
◆ EXTENSION Ask students to create a picture that makes it easy to see why $3 \times 2 = 2 \times 3$. Then ask them to create a picture where it is less obvious.

CHAPTER 4

Grades 6–8

Common Factors	CCSS 6.NS	154
Common Multiples	CCSS 6.NS	156
Square Roots	CCSS 8.EE	158
Fraction Division	CCSS 6.NS	160
Ratios: Multiple Ratios Describe Any Situation	CCSS 6.RP	162
Equivalent Ratios	CCSS 6.RP	164
Equivalent Rates	CCSS 6.RP	166
Solving Rate Problems	CCSS 6.RP	168
Describing Percent	CCSS 6.RP	170
Uses of Integers	CCSS 6.NS	172
The Zero Principle	CCSS 7.NS	174
Subtraction of Integers as a Directed Distance	CCSS 7.NS	176
Multiplication and Division of Integers	CCSS 7.NS	178
Area of a Parallelogram	CCSS 6.G	180
Area of a Triangle	CCSS 6.G	182
The Pythagorean Theorem	CCSS 8.G	184
Pi	CCSS 7.G	186
How Measures Are and Are Not Related	CCSS 7.G	188
Mean	CCSS 6.SP	190
Variability	CCSS 6.SP	192
Sampling	CCSS 7.SP	194
Probability: What It Means	CCSS 7.SP	196
Unpredictability	CCSS 7.SP	198
Rotations, Reflections, and Translations	CCSS 8.G	200
Scale Drawings	CCSS 7.G	202
Dilatations	CCSS 8.G	204
Angles with Parallel Lines	CCSS 8.G	206
Equivalent Expressions	CCSS 6.EE	208
Equation as a Balance	CCSS 6.EE	210
Different Types of Equations	CCSS 6.EE	212
What Is Linear?	CCSS 7.RP, 8.EE	214
Role of the Slope in the Equation of a Line	CCSS 8.EE	216
Systems of Equations	CCSS 8.EE	218
Function Rules	CCSS 8.F	220

COMMON FACTORS



**Which tiles can be used (without cutting)
to perfectly fit each of these rectangle frames?**



TO SIMPLIFY FRACTIONS and to solve certain types of problems, students need to know about common factors. For example, it is useful to know that 3 is a common factor of 6 and 9 when writing the fraction $\frac{6}{9}$ as the simpler equivalent $\frac{2}{3}$. It is also useful to know that 3 is a common factor of 6 and 9 when trying to determine what size tile could be used, without cutting any tiles, to cover an area that is 6 units wide by 9 units long.

Students learn to determine common factors either by guessing and testing or by factoring each number and looking for factors the numbers have in common. Students should be aware that 1 is always a common factor of any two numbers and that any common factor is less than or equal to the lesser of the two numbers being factored. The topic of common factors is addressed in [Common Core State Standards 6.NS](#).

The picture provided here is designed to focus students on the notion that square tiles of different sizes can often exactly fit rectangular spaces, but the tile edge lengths are limited to those that are factors of both the length and the width, if the tiles are not to be cut.

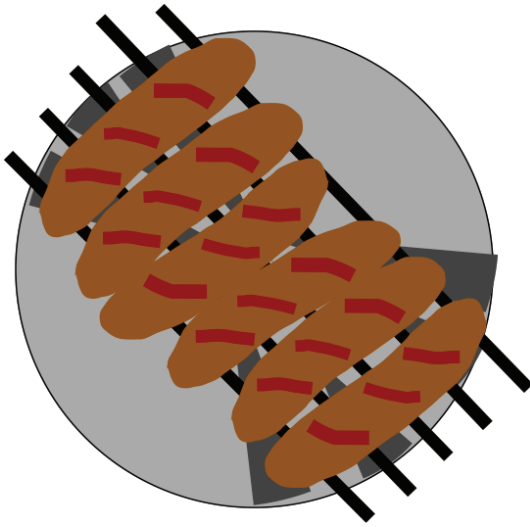
? **QUESTIONS** to supplement the question with the picture and to include in a conversation about the picture include

- **Which tiles can you use for the 6×8 picture? Why?** [We want students to notice that a tile with a linear dimension of 3 or 5 will not work because neither dimension is a factor of both 6 and 8.]
- **Which tiles can you use for the 6×9 picture? Why?** [We want students to notice that a tile with a linear dimension of 1 or 3 will work because either dimension is a factor of both numbers, but a 2×2 tile will not work because 2 is a factor of 6 but not of 9.]
- **In which frames will the 5×5 tile fit? Explain.** [We want students to realize that the only possibility is one where both dimensions are multiples of 5, namely 5×5 or 5×10 in the picture provided.]
- **Which tile always works? Why?** [We want students to recognize that 1 is a factor of every whole number.]
- **In which other sizes of frames, not in the picture, would the 3×3 tile fit?** [We want students to recognize that there is an infinite number of possibilities that can be found by using many different multiples of 3 as lengths and widths.]

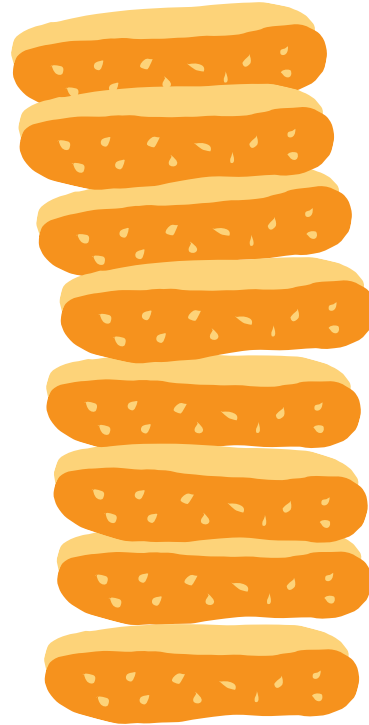
➤ EXTENSION Present the following problem: Classes of two different sizes were each divided into the same number of groups. No students were left out in either class. The group sizes were the same in each class, but different from one class to the next. First ask students how big the classes might have been and how many groups were in each. Then ask how this problem is like the tile problem. The objective is for students to see that this is another common factor problem since both classes had a common group size.

COMMON MULTIPLES

Hot dogs come in
packages of 6



Hot dog buns come in
packages of 8



**How many packages of buns and packages of hot dogs
would you need to buy to have a bun for each dog
and none left over?**

◆ **STUDENTS USE COMMON MULTIPLES** both to determine common denominators for computations with fractions (e.g., adding, subtracting, or possibly dividing fractions) and to solve certain types of problems.

One way to calculate common multiples is to factor each number into primes and use as few of each required prime as possible. In the hot dog/bun example, students might recognize that they need a common multiple of 6 and 8. They might just know that 24 works, or they might factor hot dogs and buns as follows:

$$6 = 3 \times 2$$

$$8 = 2 \times 2 \times 2$$

Thus, the least common multiple is made up of three 2s (since 8 has three 2s, even though 6 has only one) and one 3 (since 6 has one 3, even though 8 has none). We want students to recognize that each common multiple is at least as great as the greater of the two numbers, and that there is an infinite number of common multiples. The topic of common multiples is first addressed in [Common Core State Standards 6.NS](#).

The picture provided here is designed to help students recognize when a common multiple is useful for solving a problem.

? **QUESTIONS** to supplement the question with the picture and to include in a conversation about the picture include

- *Why can't you buy exactly 12 hot dog buns?* [We want students to realize that you can only buy multiples of 8 buns in this situation.]
- *How do you know which numbers of hot dog buns you could buy?*
- *Why can't you buy exactly 16 hot dogs?* [We want students to realize that you can only buy multiples of 6 hot dogs in this situation.]
- *How do you know which numbers of hot dogs you could buy?*
- *Why are there a lot of possible answers to the question with the picture?* [We want students to realize that as soon as you calculate one common multiple, you can multiply it by any whole number at all to get another common multiple.]
- *What do you notice about the numbers of packages of buns and hot dogs?* [We want students to notice that the number of packages of buns is always a multiple of 3 (to ensure that the total number of buns is a multiple of 6) and that the number of packages of hot dogs is always a multiple of 4 (to ensure that the total number of hot dogs is a multiple of 8).]

◆ **EXTENSION** Present the following problem: Ellen works at a shelter every 10th day and Lisa works there every 6th day. If they both work there today, how many days will it be before they work together again? Ask how this problem is related to the problem of the hot dogs and buns. Students should realize that it would have to be the 20th, 30th, 40th, . . . day (multiples of 10) for Ellen to be at the shelter and would have to be the 12th, 18th, 24th, . . . day (multiples of 6) for Lisa to be at the shelter. Clearly, a common multiple is required.