Chapter

2

Connections: Looking Back and Ahead in Learning

In this chapter, we describe ways of thinking about the connections in reasoning across different subject areas and grade bands in light of the big idea and the essential understandings described in chapter 1. To do this, we present an example of how ideas about length and area relationships can build from prekindergarten to grade 8. Reasoning in the elementary grades often emerges from examining patterns or relationships that students notice in particular examples. Students may develop conjectures that relate to general reasoning and may attempt to justify or refute their conjectures by building on established, though informal, definitions. We focus on three primary components of the reasoning process: *conjecturing, generalizing,* and *justifying*. In the examples below, we clarify how these processes evolve across the grades by providing examples from three grade bands: prekindergarten–grade 2, grades 3–5, and grades 6–8.

Conjecturing, Generalizing, and Justifying in Pre-K–Grade 2

Consider the situation below, in which students in Mr. Martinez's first-grade classroom are trying to determine which of two equally thick cookies would be larger, cookie A or cookie B. On a whiteboard, Mr. Martinez shows two rectangular shapes that represent cookies (see fig. 2.1) and opens a discussion:

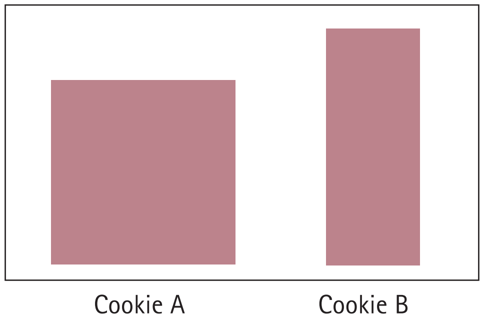

Fig. 2.1. Cookies for comparison by first-grade students

Mr. Martinez: I want you to think about which of these cookies would be larger. Think which one would be "more to eat." Think about this on your own for a minute. [*Pauses for about twenty seconds.*] Now talk to your partner about which cookie would be larger, or more to eat. [*Circulates among the pairs to listen, then calls the class together to discuss.*] Let's hear what you've have been talking about. Michael, share with us what you and your partner have been thinking.

Michael: Cookie A is larger because it's more spread out [*motioning from side to side*].

Tamika: Cookie B is larger because it's bigger [*holding her hands one above the other*].

Micah: What if you divided each cookie into pieces and moved them around to see which was more?

Mr. Martinez: What would that look like, Micah?

Micah: We could lay squares on top of the cookies, like we did last week.

Mr. Martinez: OK, show us how you would do it.

At this point, Micah went forward and laid squares on top of the two diagrams of the cookies, as in fig. 2.2. The students counted the number of square tiles and saw that 16 tiles were needed to cover cookie A, and 10 tiles were needed to cover cookie B. This seemed to convince some students that cookie A was larger, but others still think that cookie B looked larger. The discussion continued.

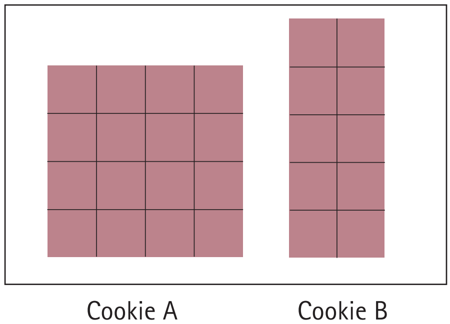

Fig. 2.2. Cookies A and B overlaid with square tiles

Teresa: What if we cut off part of cookie B and moved it to the side of the cookie?

Mr. Martinez: Would cookie B still be the same amount to eat?

Teresa: Yes, because we're just moving the cookie around, not changing how much of it there is. [*Goes to the whiteboard and erases the top piece of cookie B and redraws it on the side, as in fig. 2.3.*]

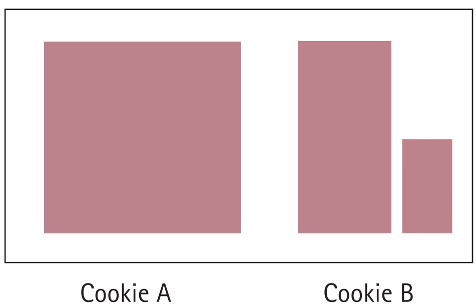

Fig. 2.3. Cookies for comparison by first-grade students, with cookie B "rearranged"

Mr. Martinez: What do you think—which cookie is larger if we use Teresa's method?

Samuel: Cookie A is bigger.

Mr. Martinez: So, if one cookie is taller than another, does that mean that the cookie is larger?

Michael: No, not necessarily. I can show you. [*Goes to the whiteboard and draws two cookies, as shown in fig. 2.4.*] For this one [*pointing to his new pair of cookies*], you can tell that cookie A is a lot bigger, even though cookie B is taller.

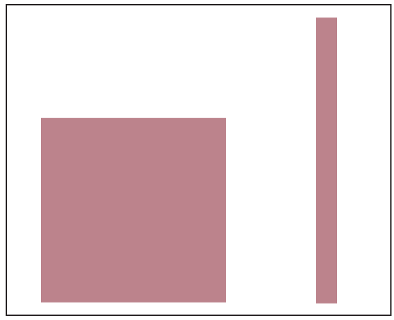

Fig. 2.4. Michael's drawing of two cookies

Mr. Martinez: Could a cookie that is taller be larger? [*Encourages the students to engage in further discussion and to share examples of cookies that are "taller" and also have larger areas.*]

What types of reasoning do students engage in during this lesson? Reflect 2.1 asks you to look closely at the students' reasoning in Mr. Martinez's classroom.

Reflect 2.1

What reasoning occurred in Mr. Martinez's first-grade classroom?

Essential Understanding 1
Conjecturing involves reasoning about mathematical relationships to develop statements that are tentatively thought to be true but are not known to be true. These statements are called conjectures.

Mr. Martinez's students began by *conjecturing* (Essential Understanding 1) about which cookie was larger. They differed in their views about what it means for one cookie to be larger than another cookie. Some attended to the length of one side of a cookie, whereas others seemed to compare the areas of the two cookies informally. The class also engaged in *justifying* (Essential Understanding 6) the idea that a cookie was larger, on the basis of the students' shared understanding of the meaning of comparing the areas of the cookies. Using Micah's idea, which evolved from cutting the cookies into pieces and manipulating the pieces to using squares to measure area as they had previously done, the students investigated how many square tiles covered the cookies as a way to compare the areas, or amounts of cookie. Teresa suggested rearranging the area of cookie B by breaking off one side of the cookie and rearranging it to determine which was larger. Students worked from specific examples and used informal arguments to support their reasoning. Mr. Martinez extended the examples to encourage generalizing (Essential Understanding 2) the relationship between the area of the cookie and the length of one side. Through this discussion, the class recognized that attending to one dimension of the cookies was insufficient for determining which cookie was larger.

In prekindergarten–grade 2, it is common for students to provide arguments using specific examples. Even though a valid mathematical justification for why a generalization is true cannot rely on examples alone (Essential Understanding 9), children may use particular examples to make sense of an idea. Notice that the original conjecture in Mr. Martinez's class was about which cookie was larger, and Micah's method of cutting cookies into equal-sized pieces to compare the areas was the basis of a logical argument that depended on the students' existing understanding that creating a standard square unit is a valid way to measure area. Moreover, Michael's drawing in figure 2.4 relies on the important notion of a counterexample, showing that the idea that a taller cookie will always have more area is false (Essential Understanding 7). Students in prekindergarten–grade 2 may often think in terms of particular examples, but they can still engage in conjecturing, generalizing, and justifying as ways to make sense of general ideas.

At the prekindergarten–grade 2 level, students can make conjectures about mathematical ideas and investigate their conjectures in many different ways. Students should also be able to generalize, even though their generalizations will not be expressed in formal mathematical language. Even a statement such as, "This problem is just like that other one that we did!" is a generalization, in the sense that students are noticing common features across problems (Essential Understanding 2). Justifying is another form of reasoning that students can engage in at the prekindergarten–grade 2 level. Students' justifications at this level may be limited; the justifications may rely inappropriately on examples, or they may simply be a way for students to share their strategies when solving a problem rather than explaining why something always works. However, it is important to help students grow accustomed to creating justifications and learn that justifying is a fundamental part of doing mathematics. Moreover, students' justifications at this level can also contain the elements of more mathematically appropriate arguments; we can see hints of that in the method of using square units to find the areas of the cookies. The more students grow used to creating justifications, the more comfortable they become in explaining their reasoning about general relationships.

Essential Understanding 6
A mathematical justification is a logical argument based on already-understood ideas.

Essential Understanding 2
Generalizing involves identifying commonalities across cases or extending the reasoning beyond the range in which it originated.

Essential Understanding 9
A valid mathematical justification for a general statement is not *an argument based on authority, perception, popular consensus, or examples.*

Essential Understanding 7
A mathematical refutation involves demonstrating that a particular statement is false.

Conjecturing, Generalizing, and Justifying in Grades 3–5

Ms. Timmons's fourth-grade class previously learned how to find the perimeter and area of squares and rectangles. They also examined what happens to the perimeter of a square when its side length doubles. Next, Ms. Timmons asked the students to consider what would happen to the *area* of a square if they doubled the side length:

Ms. Timmons: Based our earlier work, we concluded that when we double the side length of a square, the perimeter doubles. What do you think happens to the area of a square when we double the side length?

Talia: I think it is going to double, since we doubled the side length.

Ms. Timmons: What do others think of Talia's idea? [*Waits while students express agreement with Talia's conjecture and then calls on Selina when she raises her hand.*]

Selina: I think we use the side length of the square to find the area, but I think we should try some squares to see what happens.

In pairs, the students then drew a variety of squares, doubled their side lengths, and found the resulting areas. For instance, Ben and Alejandra drew three squares, with sides of 1 centimeter, 2 centimeters, and 4 centimeters, respectively (see fig. 2.5), and then calculated the area of each square. After the students tested a number of different squares, the teacher called the group back together to share. Ben exclaimed, "We have a picture! We found that for three squares of side length 1 centimeter, 2 centimeters, and 4 centimeters, we got areas of 1 square centimeter, 4 square centimeters, and 16 square centimeters."

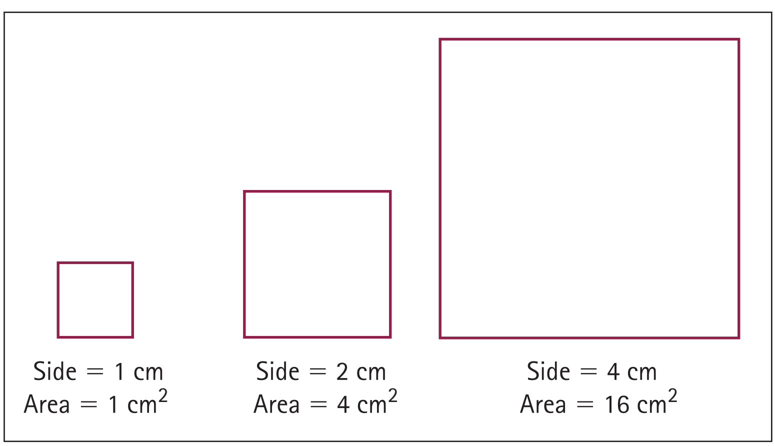

Fig. 2.5. Ben and Alejandra's picture

Ms. Timmons invited Ben to show his drawing and to talk about how the area changed in each square. Ben showed that each square's area was four times the previous square. Ms. Timmons asked the class how Ben and Alejandra's work helped them think about the earlier conjecture, that doubling the side length doubles the area. A discussion ensued:

Petra: We tried two squares with lengths 3 centimeters and 6 centimeters and got areas 9 square centimeters and 36 square centimeters. Nine multiplied by four is 36. I am pretty sure it's going to increase four times, not double.

Ms. Timmons: Are others convinced that the area is increasing four times?

Milo: We tried 5 centimeters and 10 centimeters for the side lengths and got 25 square centimeters and 100 square centimeters.

Ms. Timmons: So, we have tried squares with side lengths of 1 centimeter, 2 centimeters, 4 centimeters; 3 centimeters, 6 centimeters; and 5 centimeters, 10 centimeters. What is our conjecture about the area of a square when the side length is doubled? And how might we *know* this is the case for *all* squares when you double the side length?

Nora: I think our conjecture is that the area increases four times when we double the side length of a square.

Ms. Timmons: So, how do we know that this is going to be the case for all squares? [*Waits while the class remains silent for a minute or so.*]

April: I think it will be, because we already showed it for a lot of squares.

Ms. Timmons: But what if there is one square that we didn't try that won't work?

Nora: Remember when we used those square tiles to find the area? I bet we could do the same thing here!

April: Oh, yeah!

Ms. Timmons: What do you mean?

Nora and her partner, April, used Ben's squares with side lengths 1, 2, and 4. They made the drawing shown in figure 2.6 to show that each time they would need four of the previous squares to find the area of the next.

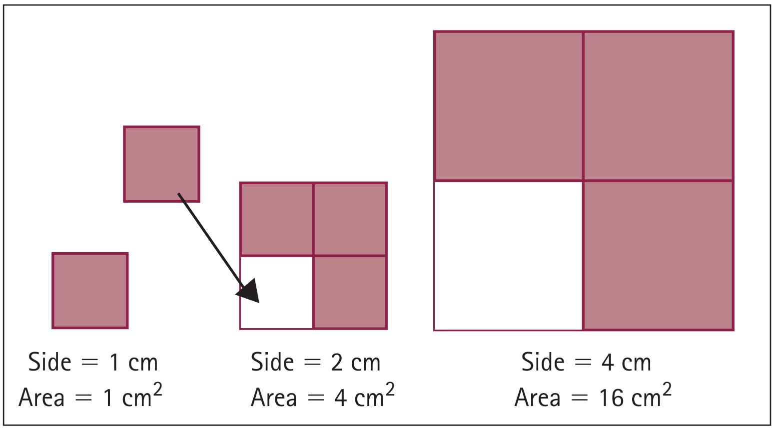

Fig. 2.6. Nora and April's drawing from their work with squares

Milo: I'm not really sure that means it'll work every time, though.

Ms. Timmons: Milo raises a good point. How do we know that the area is four times as much for all squares? Return to your partner work, and think about how we could show that for any square, the area increases by a factor of four when we double the side length.

The students considered a few more examples, trying to create diagrams similar to those that Nora and April had showed the class. Ms. Timmons overheard Selina's reasoning and asked her to share it with the class.

Selina: If you count how the first square fits into the next one, you always get two squares on each side and a fourth square in the hole. It'll always work that way because if you double the side length, the first square fits twice on each side, so that's three squares, and then there's always a hole, and the fourth square goes in the hole.

Essential Understanding 1
Conjecturing involves reasoning about mathematical relationships to develop statements that are tentatively thought to be true but are not known to be true. These statements are called conjectures.

Essential Understanding 6
A mathematical justification is a logical argument based on already-understood ideas.

What reasoning occurred in this fourth-grade classroom? Ms. Timmons asked students to consider the relationship between side length and area. She built on students' previous generalizations, posing a related question that encouraged *conjecturing* (Essential Understanding 1) about the relationship between the side length and area. The vignette demonstrates how elementary students may create conjectures that are not valid and revise them on the basis of examples. In addition, Nora and April engaged in *justifying* (Essential Understanding 6), using a diagram, which Selina relied on to reason more generally. Reflect 2.2 asks you to evaluate Nora and April's drawing and Selina's explanation as mathematical justifications.