Chapter
1

Addition and Subtraction: The Big Ideas and Essential Understandings

Young children begin learning mathematics before they enter school. They learn to count, and they can solve simple problems by counting. In the primary grades, mathematics instruction focuses on the development of number sense, understanding of numerical operations, and fluency in performing computations. *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics [NCTM] 2000) describes the development of these skills and concepts, asserting that by the end of grade 2, students should—

- understand numbers, ways of representing numbers, relationships among numbers, and number systems;
- understand meanings of operations and how they relate to one another;
- compute fluently and make reasonable estimates. (NCTM 2000, p. 78)

Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics: A Quest for Coherence (NCTM 2006) continues to emphasize the importance of developing both conceptual understanding and procedural understanding of addition and subtraction. Building on *Principles and Standards, Curriculum Focal Points* recommends that instruction focus on developing this understanding throughout the early grades. Activities in kindergarten should center on joining and separating sets:

> Children use numbers, including written numerals, to represent quantities and to solve quantitative problems, such as . . . modeling simple joining and separating situations with objects. They choose, combine, and apply effective strategies for answering quantitative questions, including quickly recognizing the number in a small set, counting and producing sets of given sizes, counting the number in combined sets, and counting backward. (NCTM 2006, p. 12)

In grade 1, instruction should focus on developing students' understanding of addition and subtraction as well as related facts and strategies associated with these operations:

> Children develop strategies for adding and subtracting whole numbers on the basis of their earlier work with small numbers. They use a variety of models, including discrete objects, length-based models (e.g., lengths of connecting cubes), and number lines, to model "part-whole," "adding to," "taking away from," and "comparing" situations to develop an understanding of the meanings of addition and subtraction and strategies to solve such arithmetic problems. Children understand the connections between counting and the operations of addition and subtraction (e.g., adding two is the same as "counting on" two). They use properties of addition (commutativity and associativity) to add whole numbers, and they create and use increasingly sophisticated strategies based on these properties (e.g., "making tens") to solve addition and subtraction problems involving basic facts. By comparing a variety of solution strategies, children relate addition and subtraction as inverse operations. (NCTM 2006, p. 13)

In grade 2, the instructional focus should shift to helping students develop quick recall of addition and related subtraction facts, as well as fluency with multi-digit addition and subtraction:

> Children use their understanding of addition to develop quick recall of basic addition facts and related subtraction facts. They solve arithmetic problems by applying their understanding of models of addition and subtraction (such as combining or separating sets or using number lines), relationships and properties of number (such as place value), and properties of addition (commutativity and associativity). Children develop, discuss, and use efficient, accurate, and generalizable methods to add and subtract multidigit whole numbers. They select and apply appropriate methods to estimate sums and differences or calculate them mentally, depending on the context and numbers involved. They develop fluency with efficient procedures, including standard algorithms, for adding and subtracting whole numbers, understand why the procedures work (on the basis of place value and properties of operations), and use them to solve problems. (NCTM 2006, p. 14)

This chapter discusses in detail the mathematical concepts that these publications outline. In addition, it relates various mathematical processes to these concepts by exploring—

- situations for which addition and subtraction can be used to solve problems;

- ways to represent addition and subtraction;
- ways to reason with addition and subtraction; and
- connections and relationships among these and other mathematical topics.

It also examines numerical relationships that arise from studying multiple representations and the reasoning required for the meaningful use and understanding of computational algorithms, written and mental, standard and nonstandard. The representations have been chosen primarily for their usefulness in illustrating the mathematical concepts. Most of the early examples use counters, since these constitute the most elementary representation, but later discussions involve the use of other representations for addition and subtraction, such as the number line, a hundreds chart, and base-ten place-value blocks.

"Unpacking" ideas related to addition and subtraction is a critical step in establishing deeper understanding. To someone without training as a teacher, these ideas might appear to be simple to teach. But those who teach young students are aware of the subtleties and complexities of the ideas themselves and the challenges of presenting them clearly and coherently in the classroom. Teachers of young students also have an idea of the overarching importance of addition and its inverse operation, subtraction:

> **Overarching idea:** Addition and its inversely related operation, subtraction, are powerful foundational concepts in mathematics, with applications to many problem situations and connections to many other topics. Addition determines the whole in terms of the parts, and subtraction determines a missing part.

This overarching idea anchors teachers' understanding and their instruction. It incorporates two big ideas about addition and subtraction that are crucial to understand. The first relates to when to use each operation, and the second deals with how to get answers efficiently and accurately. Each of these two big ideas involves several smaller, more specific essential understandings.

These big ideas and essential understandings are identified here as a group to give you a quick overview and for your convenience in referring back to them later. Read through them now, but do not think that you must absorb them fully at this point. The chapter will discuss each one in turn in detail.

Big Idea 1. Addition and subtraction are used to represent and solve many different kinds of problems.

Essential Understanding 1*a*. Addition and subtraction of whole numbers are based on sequential counting with whole numbers.

Essential Understanding 1*b*. Subtraction has an inverse relationship with addition.

Essential Understanding 1*c*. Many different problem situations can be represented by part-part-whole relationships and addition or subtraction.

Essential Understanding 1*d*. Part-part-whole relationships can be expressed by using number sentences like $a + b = c$ or $c - b = a$, where a and b are the parts and c is the whole.

Essential Understanding 1*e*. The context of a problem situation and its interpretation can lead to different representations.

Big Idea 2. The mathematical foundations for understanding computational procedures for addition and subtraction of whole numbers are the properties of addition and place value.

Essential Understanding 2*a*. The commutative and associative properties for addition of whole numbers allow computations to be performed flexibly.

Essential Understanding 2*b*. Subtraction is not commutative or associative for whole numbers.

Essential Understanding 2*c*. Place-value concepts provide a convenient way to compose and decompose numbers to facilitate addition and subtraction computations.

Essential Understanding 2*d*. Properties of addition are central in justifying the correctness of computational algorithms.

Representing and Solving Problems: Big Idea 1

Big Idea 1. *Addition and subtraction are used to represent and solve many different kinds of problems.*

Many different types of problems can be represented by addition or subtraction. It is important to learn how to recognize these situations and represent them symbolically, building on counting with whole numbers. By understanding these situations and their representations well, teachers can provide students with many different examples of addition and subtraction problems. The discussion below of Big Idea 1 presents and examines fifteen examples that illustrate situations that can be represented by addition or subtraction.

Building on sequential counting

Essential Understanding 1a. *Addition and subtraction of whole numbers are based on sequential counting with whole numbers.*

Situations that can be represented by addition or subtraction can be considered as basic applications of counting forward or back. Even very young children can solve simple addition and subtraction story problems by counting concrete objects (e.g., Starkey and Gelman 1982; Carpenter and Moser 1983). They establish a one-to-one correspondence by moving, touching, or pointing to each object that they are counting as they say the corresponding number words. The following two examples demonstrate how counting relates to addition and subtraction situations.

For an extended discussion of counting strategies and number ideas, see *Developing Essential Understanding of Number and Numeration for Teaching Mathematics in Prekindergarten–Grade 2* (Dougherty et al. 2010).

Example 1 lends itself to a number of simple counting strategies:

> **Example 1:** Max has 2 apples. He picks 5 more. How many apples does Max have now?

This problem can be represented with concrete objects by first placing 2 counters (the quantity that Max starts with) and then placing a second group of 5 more counters. Figure 1.1 illustrates the two groups of counters.

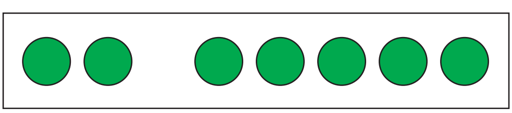

Fig. 1.1. Counters representing 2 and 5

A variety of counting strategies might be used to find the total number of counters:

- *Count all:* Count each of the counters: 1, 2 [pause] 3, 4, 5, 6, 7.
- *Count on from the first number:* A more efficient way to find the total is to count on, beginning with the first quantity given in the problem (in this case, 2): 2 [pause], 3, 4, 5, 6, 7.
- *Count on from the larger number:* A still more efficient way to find the total is to count on, beginning with the larger number (5, in this case) and counting on the smaller number (2): 5, [pause] 6, 7.

Reflect 1.1 explores possible ways to use counters with these strategies.

Reflect 1.1

What counters might a child point to as she uses each of the counting strategies shown above?

Does a child need all of the counters for counting on?

Each of the "counting on" strategies is more efficient when problem solvers recognize the first number that they use without counting. This process reduces the difficulty of many tasks and is frequently useful in playing games, counting coins, or other simple everyday tasks. Recognizing patterns on number cubes and dominoes, such as in figure 1.2 is particularly helpful.

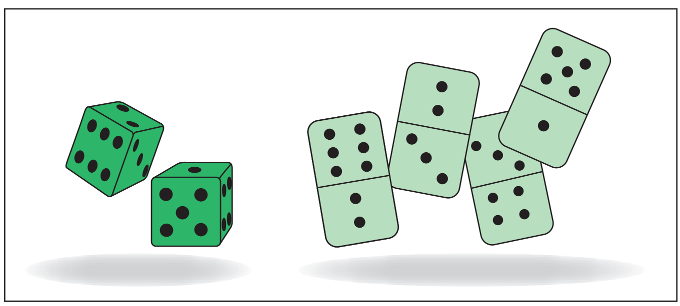

Fig. 1.2. Number cubes and dominoes

Relating numbers to the benchmark quantities 5 and 10 helps students see the relative sizes of numbers and can therefore support their transition from counting to later work in addition and subtraction. In the five-frame on the left in figure 1.3, we not only recognize the three counters without counting, but we also note without counting that there are two empty spaces, so 3 is 2 less than 5, or $3 + 2 = 5$, or $5 - 2 = 3$. In the ten-frame on the right, we see that 6 is 1 more than 5, or $6 = 5 + 1$, and that 6 is 4 less than 10, or $6 + 4 = 10$, or $6 = 10 - 4$.

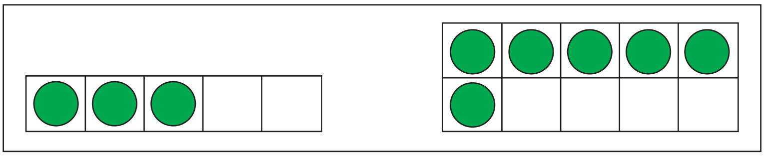

Fig. 1.3. Five-frame and ten-frame representing 3 and 6, respectively

Working with dot patterns, whether on a ten-frame or a number cube, can help students recognize the number of objects without counting. Some representations are more useful for building recognition of multiples—especially doubles. The arrangement of six dots on a number cube is similar to the arrangement of counters on the ten-frame shown in figure 1.4. This arrangement leads to thinking of 6 as two rows of 3, or $6 = 3 + 3$. Reflect 1.2 explores extending this thinking to other arrangements of dots.

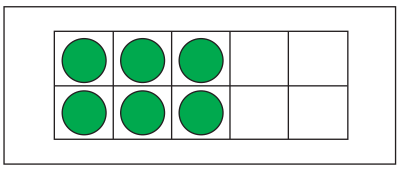

Fig. 1.4. Another way to show 6

Reflect 1.2

What number relationships might students perceive from the standard arrangements of dots on a number cube?

Example 2 lends itself to a different counting strategy:

Example 2: Sari has 5 apples. Three are red. The rest are yellow. How many of Sari's apples are yellow?

One way of using counting to solve this problem is to lay out 5 counters, separate (perhaps by circling) the 3 that represent red apples, and then count the remaining counters. Figure 1.5 depicts this situation.

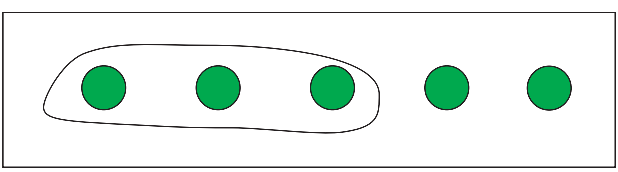

Fig. 1.5. Counters representing 5 with 3 as one part

We might also solve this problem by "counting on." We could lay out 3 counters for the 3 red apples and then count on until we had counters for 5 apples, as shown in figure 1.6.

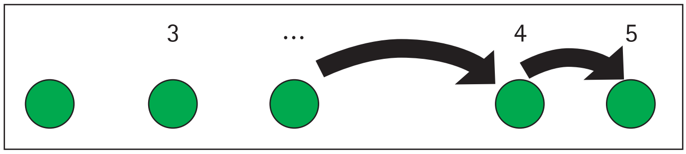

Fig. 1.6. Counting on to represent 5 with 3 as one part

Alternatively, we might represent the problem by "counting back." We could start with 5 counters and then count back 3 for the 3 red apples, as illustrated in figure 1.7. Consider the question in Reflect 1.3 to compare "counting on" and "counting back."

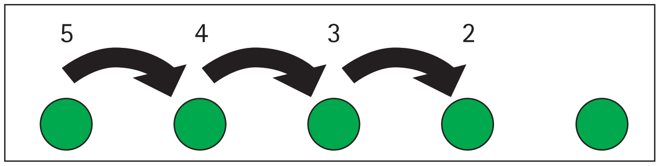

Fig. 1.7. "Counting back" to represent 5 with 3 as one part

Reflect 1.3

Why is "counting back" so much more difficult than "counting on"?

The inverse relationship of addition and subtraction

Essential Understanding 1*b*. *Subtraction has an inverse relationship with addition.*

The chart in figure 1.8 shows the input and the output for the algebraic rule "add 2." The output number is always two more than the input number.

Rule: Add 2	
Input	Output
1	3
5	7
8	10
11	13

Fig. 1.8. Input/output table

The relationship in figure 1.8 is a *function* and can be represented by a function machine, as shown in figure 1.9. We can reverse the action of adding 2 by subtracting 2.

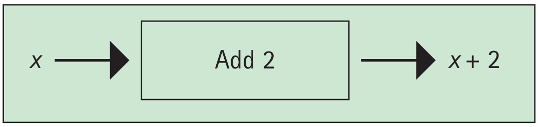

Fig. 1.9. A function machine for "add 2"

Example 2, about Sari's apples, showed a problem situation that some people would represent by addition, while others would use subtraction. The "counting on" subtraction strategy described above is grounded in the fact that $5 - 3 = \square$ is equivalent to $5 = 3 + \square$. The result of subtracting b from a, $a - b$, is formally defined as the number y where $a = b + y$. This definition builds logically on what students already know about addition, demonstrating why some problems can be solved by either operation.

For a discussion of the inverse relationship between multiplication and division, see *Developing Essential Understanding of Multiplication and Division for Teaching Mathematics in Grades 3–5* (Otto et al. 2011).

Understanding the relationship between addition and subtraction reduces the number of facts that students must "know" by giving them a consistent, reliable strategy for subtraction: use the related addition fact. These related facts then form fact families. The third column in figure 1.10 shows a more formal algebraic description of a fact family.

$2 + 3 = 5$	$4 + 4 = 8$	$b + y = a$
$3 + 2 = 5$		$y + b = a$
$5 - 2 = 3$	$8 - 4 = 4$	$a - b = y$
$5 - 3 = 2$		$a - y = b$

Fig. 1.10. Examples of fact families

The language describing subtraction is often very difficult for students to comprehend and use correctly. We may read the expression $5 - 3$ in many ways. Thinking about this expression in terms of parts and wholes may be helpful, since 5 (the *minuend*) is the whole, and 3 (the *subtrahend*) is a part. "Five minus 3" is the way that many adults would read the expression. Students might read it as "5 take away 3," but they might also say it as "3 taken away from 5." The same expression might be read either as "5 subtract 3" or as "3 subtracted from 5." It also might be read as "5 less 3" or as "3 less than 5." Note that in these phrases, the order of the numbers shifts, and some expressions include a preposition (*from* or *than*). It is very difficult for children to distinguish differences among the meanings of these phrases, and this confusion leads them to make frequent reversal errors. Chapter 3 describes strategies for helping students make sense of actions in word problems and the language of addition and subtraction.

For a discussion of using appropriate terminology and representing word problems with multiplication and division, see *Developing Essential Understanding of Multiplication and Division for Teaching Mathematics in Grades 3–5* (Otto et al. 2011).