Chapter
1

Number and Numeration: The Big Ideas and Essential Understandings

A discussion of ideas and understandings of whole number begins naturally with quantities (length, area, and discrete quantities) and unit before moving to counting because several fundamental ideas related to number should develop prior to counting in the early grades. Relationships between quantity and unit lay a solid foundation for number. Counting then plays a major role in integrating aspects of number, such as sequence, cardinality, order, and measure. This integration, in turn, supports the more abstract and general ideas of number and our base-ten, place-value number system. Chapter 1 discusses the following five big ideas of number and numeration and related essential understandings:

Big Idea 1. Number is an extension of more basic ideas about relationships between quantities.

> **Essential Understanding 1*a*.** Quantities can be compared without assigning numerical values to them.
>
> **Essential Understanding 1*b*.** Physical objects are not in themselves quantities. All quantitative comparisons involve selecting particular attributes of objects or materials to compare.
>
> **Essential Understanding 1c.** The relation between one quantity and another quantity can be an equality or inequality relation.
>
> **Essential Understanding 1*d*.** Two important properties of equality and order relations are conservation and transitivity.
>
> **Essential Understanding 1e.** The equality relation between two quantities remains unchanged when one or both quantities are

decomposed into parts and when one of the quantities is combined with another quantity to form a larger quantity.

Big Idea 2. The selection of a unit makes it possible to use numbers in comparing quantities.

Essential Understanding 2*a*. Using numbers to describe relationships between or among quantities depends on identifying a unit.

Essential Understanding 2*b*. The size of a unit determines the number of times that it must be iterated to count or measure a quantity.

Essential Understanding 2*c*. Quantities represented by numbers can be decomposed (or composed) into part-whole relationships.

Big Idea 3. Meaningful counting integrates different aspects of number and sets, such as sequence, order, one-to-one correspondence, ordinality, and cardinality.

Essential Understanding 3*a*. The number-word sequence, combined with the order inherent in the natural numbers, can be used as a foundation for counting.

Essential Understanding 3*b*. Counting includes one-to-one correspondence, regardless of the kind of objects in the set and the order in which they are counted.

Essential Understanding 3*c*. Counting includes cardinality and ordinality of sets of objects.

Essential Understanding 3*d*. Counting strategies are based on order and hierarchical inclusion of numbers.

Big Idea 4. Numbers are abstract concepts.

Essential Understanding 4*a*. Patterns in the number-word sequence provide a foundation for the abstract number concept.

Essential Understanding 4*b*. The number sequence is infinite.

Essential Understanding 4*c*. Number symbols are representations of abstract mental objects.

Big Idea 5. A base-ten positional number system is an efficient way to represent numbers in writing.

> **Essential Understanding 5*a*.** Ten different digits can be used and sequenced to express any whole number.
>
> **Essential Understanding 5*b*.** Our base-ten number system allows forming a new place-value unit by grouping ten of the previous place-value units, and this process can be iterated to obtain larger and larger place-value units.
>
> **Essential Understanding 5*c*.** The value of a digit in a written numeral depends on its place, or position, in a number.
>
> **Essential Understanding 5*d*.** Inherent in place value are units of different size.

Big Idea 1

Number is an extension of more basic ideas about relationships between quantities.

Many people assume that getting a good start in learning about number amounts to acquiring such numerical skills as being able to count to 10 or higher in the preschool period and knowing single-digit addition and subtraction facts in kindergarten and first grade. Understanding number is more than that, however.

The concept of number grows out of ideas about relationships between quantities. To understand the mathematical and conceptual foundations of numerical knowledge, it is useful to appreciate the kinds of knowledge that are involved in making nonnumerical comparisons between physical quantities, such as lengths.

Essential Understanding 1*a*

Quantities can be compared without assigning numerical values to them.

It is often possible to evaluate the relationship between two quantities without determining the numerical value of either of them. You can decide which of two lengths is greater, for example, by taking sticks of those lengths and aligning them so that you can determine perceptually which extends farther, as in figure 1.1. If you align the sticks correctly, this is a mathematically sound procedure, although it can lead to errors if you place the sticks so that one stick protrudes farther at one end and the other stick protrudes farther at the other.

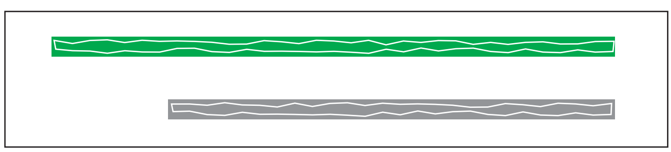

Fig. 1.1. Alignment to compare lengths

Comparing quantities other than lengths without representing them numerically is also possible. For instance, to compare the areas of two two-dimensional shapes, you can superimpose the shapes and possibly determine whether one remains within the boundaries of the other. To compare the masses of two objects, you can place them at equal distances on opposite sides of a fulcrum (or a balance point) to determine whether they balance. And to compare the capacities of two containers, you can fill one container with liquid and then pour that liquid into the other container to determine whether it overflows, just fits, or leaves extra space.

In figure 1.1, length is an attribute of the sticks that allows us to make comparisons without assigning a numerical value. Length, like area, volume, or mass, is not a countable, or *discrete*, quantity, like the number of sticks or the number of objects in a given set. The attributes length, area, volume, and mass are continuous quantities.

Even discrete quantities can be compared without assigning numerical values to them. For instance, if you observe that a child is sitting on each horse on a carousel and that some children are also sitting on benches on the carousel, you can conclude that there are more children than horses, while possibly having very little idea about how many of either there are.

Although comparisons between quantities need not entail any understanding of numbers or numerical symbols, an understanding of number and numerical symbols *does* entail some knowledge of quantities. Because a basic function of numbers is to represent quantities (e.g., "how many?"), an understanding of numbers clearly depends on an understanding of quantity.

Research with very young children who are just learning the words for small numbers, such as *one* and *two*, confirms this dependence (Durkin et al. 1986; Wagner and Walters 1982). Children are attending to quantity when they first use these words as linguistic terms to describe collections of discrete objects—although the collections that very young children describe with them do not necessarily contain the stated number of objects! Only later, as an understanding of number develops, do they combine the number words into a counting sequence.

Essential Understanding 1*b*

Physical objects are not in themselves quantities. All quantitative comparisons involve selecting particular attributes of objects or materials to compare.

Because comparisons between quantities need not entail the use of numerical symbols, they do not presuppose any knowledge of number. They do, however, require some knowledge of several other fundamental mathematical concepts. Consider the situation in Reflect 1.1, for example.

Even an apparently simple quantitative comparison, such as a judgment that one vase is taller than another because it "comes up" farther when the two are placed on the same surface, involves considerable analysis, and the concepts that enter into that analysis are also important in understanding number. Basically, comparisons

Reflect 1.1

Josh said, "The book is bigger than the sheet of paper."

What attribute(s) of the objects might Josh be comparing? In how many different ways could the "size" of the book be compared to the "size" of the paper?

between quantities require an understanding of quantity as distinct from the physical objects themselves. They also entail an understanding of the concepts of equal to, less than, and greater than, which correspond to alternative possible relationships between two quantities. Furthermore, comparisons provide a concrete foundation for the important concept of additive composition (and decomposition).

The vases depicted in figure 1.2 illustrate the idea that quantities must be distinguished from the physical objects in which they are embodied. You might think of the vase on the left as greater than the one on the right if the attributes that you were comparing were heights, but probably not if you were comparing capacities. Clearly, then, it is possible to evaluate which of two objects or sets of physical materials is greater (or whether they are equal) only after you have selected a quantitative attribute on the basis of which to compare them.

Fig. 1.2. Which is greater?

Confusion among different attributes of physical objects contributes to a number of errors commonly observed in young children's reasoning. For instance, consider a child who concludes that a quantity of water has increased when it is poured from a wide container into a narrower one (Piaget 1952). The child has noticed that the water rises to a higher level in the narrower container, and because he is not distinguishing clearly between volume and height, that observation leads him to the conclusion that the quantity of water has increased.

Even the numerical relation that we find between two collections of discrete items depends on how we construe the quantities that we are comparing and what we take as a unit in each case. Consider, for instance, the collections of handbags and slippers in figure 1.3. If you compare individual handbags to *pairs* of slippers, you find that the two collections are equal in number. You can see their equality even without determining how many handbags or pairs of slippers the figure shows, simply by noting that one pair of slippers appears above each handbag. If, however, you compare individual handbags to *individual* slippers, the collections are not equal, since there is not a one-to-one correspondence between their elements.

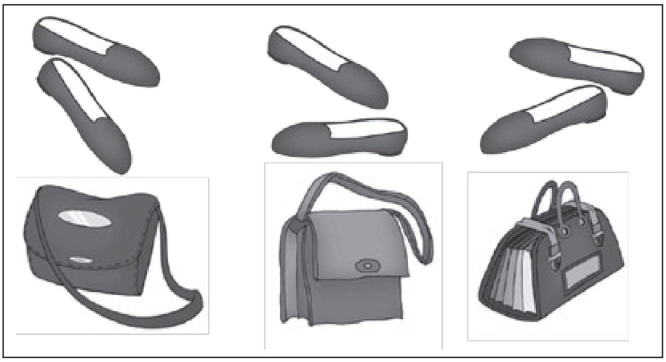

Fig. 1.3. Are the two collections equal in number?

Collections of discrete items are very often compared numerically, usually in a way that treats each discrete item as a separate unit. However, collections invariably have other characteristics as well, such as spatial extent, mass, volume, and total surface area. An important insight into the nature of quantities, including numerical quantities, is that these different dimensions of quantity do not always covary, even though they often do. Thus, in the case of two collections, we cannot necessarily identify which collection has more items by evaluating how much space the collections cover, or by weighing them, although those strategies are effective if we know that all the items in each collection are the same in area or weight. If each of the objects in the collections covers an equal area, we can conclude that there are more items in the collection that covers a larger area altogether, and if every object in the collections has the same weight, we can conclude that there are more items in the collection with the greater aggregate weight.

However, making a comparison of aggregate area or weight is not necessarily a valid way to compare the numbers of items in two collections—that is, their *numerosities*—when the individual items vary in area or weight. For example, if a bag of rocks balances a bag of bricks on a balance scale, it is not clear whether the number of individual bricks is greater than, less than, or equal to the number of individual rocks.

Further, an important understanding of number is the recognition that different quantitative attributes are distinct because transformations that alter one quantitative attribute do not necessarily affect another quantitative attribute and may have no effect on number. For instance, spreading out a row of counters increases the length of the row but does not affect the number of counters in it. Conversely, cutting an item from one collection into two pieces may affect the numerical relation between two collections but not the relation between them in attributes such as mass or total amount. Because the way in which we make nonnumerical comparisons between two quantities depends on the attribute on which we are basing our comparison (e.g., aligning two sticks by their endpoints to compare lengths, putting them on opposite sides of a balance scale to compare weights), carrying out these kinds of comparisons can help to illuminate the distinction between different quantitative attributes.

In addition, the way in which we talk about quantities and the relations between them can have a strong impact on how children think about quantities. In everyday language, we are often not very explicit about the particular quantities that we are comparing. For example, you might say that one vase is “bigger than” another, meaning that its capacity is greater, even when its height is not. Likewise, you might talk about whether one child has “more” of something than another without specifying whether you mean a numerically greater amount or a greater aggregate volume.

The sociocultural perspective on development and learning pioneered by Vygotsky (e.g., 1978) suggests that learning more precise ways of talking about and representing quantities can advance children’s understanding of quantitative relations. Terms that identify specific quantitative dimensions—for example, length, area, volume, and mass—draw children’s attention to a variety of quantitative dimensions on the basis of which particular objects or collections can be compared.

In Reflect 1.1, Josh might be comparing the areas, lengths, or weights of the sheet of paper and the book. From his statement, determining the attribute to which he is attending is not possible. The nonspecific statement does not convey the observed relationship. The precision of the descriptions of relationships between or among quantities can be enhanced or supported by clear identification and articulation of the attribute being compared.

Essential Understanding 1c

The relation between one quantity and another quantity can be an equality or inequality relation.

After we have quantified attributes, we can look at relationships among two or more quantities. We can classify these relationships according to the properties that they have. Reflect 1.2 invites exploration of this idea.

Reflect 1.2

Kara said, "If I know that one quantity is greater than another quantity, I can write four statements."

What statements do you think Kara might write?

Whenever we compare two quantities, we try to decide which of three relationships holds between them: is the first quantity greater than the second, is it less than the second, or are the two equal in magnitude? Thus, comparisons between quantities require an understanding of the relations of *equal to*, *less than*, and *greater than*. These relations are fundamental to an understanding of number as well as to the comparison of unenumerated quantities. Two sets have the same number of objects or elements if there is a one-to-one correspondence between their elements. When two sets differ in the number of objects or elements, one of those sets will have one or more elements remaining when the elements of the two sets have been put into one-to-one correspondence as far as possible; the set with remaining elements is the one with the greater number of objects or elements. The process of constructing a correspondence between the elements of two sets is a means of determining which of the relations—equal to, less than, or greater than—holds between the sets.

In Reflect 1.2, Kara says that she can write four statements if she knows that one quantity is greater than another. Given two unequal quantities, we can say the following:

1. One quantity—say, quantity D—is greater than the other quantity—say, quantity K.
2. Quantity K is less than quantity D.
3. Quantity K is not equal to quantity D.
4. Quantity D is not equal to quantity K.

These four statements are true when we know that two quantities are unequal.

Equality and inequality relations have many important mathematical properties, and these properties apply to relations between nonnumerical quantities as well as to numerical relations. Piaget (e.g., 1952) identified several such properties in his work on children's understanding of quantity. Two particularly important concepts to which he drew attention are the concepts of conservation and transitivity.

Essential Understanding 1*d*

Two important properties of equality and order relations are conservation and transitivity.

Conservation is the idea that the relation between two quantities remains the same when we change irrelevant aspects of the physical objects. For instance, you conserve the number of counters when you spread out a row of counters, and you conserve volume when you pour a liquid into a differently shaped container (assuming no spillage and ignoring any liquid that "sticks" to the first container).

To conclude that two initially equal quantities must still be equal (provided that only quantity-irrelevant aspects of the objects or their arrangement have been changed) when we can no longer compare them directly, we need the concept of *transitivity*. For example, the conclusion that two volumes of liquid are still equal after one has been poured into a different-shaped container rests on two other equality relations: the equality of the two volumes when they were in identical containers, and the equality between a volume of liquid in its initial container and the same volume of liquid after it has been poured into another container. (If we spill some of the liquid while we are pouring it, the second of these equalities no longer holds, and correspondingly, we can no longer conclude that the two quantities are equal after the transformation.) In short, because the quantities were equal initially, and the pouring left the poured quantity the same as it was originally, we know that even though the quantities may look different when they are in different-shaped containers, they are in fact still equal.

Transitivity, which allows comparisons of quantities when direct, physical comparisons are not possible, is useful in many situations. Reflect 1.3 emphasizes the prominent role of transitivity in measurement.

Transitivity in essence consists of comparing each of two or more quantities to an intermediary rather than comparing them directly to each other. Although we usually think of measurement as a process of assigning a numerical value to a quantity, in its simplest form measurement need not involve any use of numbers. Tran's

Reflect 1.3

"I need to compare the areas of these two things, but I can't move them to see how they overlap. They're too big," said Tran.

How can Tran solve this dilemma?

dilemma in Reflect 1.3 points to the need to use an intermediary as a means of measuring both areas and then comparing the outcomes to determine the relationship of the two given areas.

Consider, for example, the problem of comparing the depth of two holes. Bryant and Kopytynska (1976) and Miller (1989) studied young children's responses to this problem. One solution is to insert a stick into one of the holes, mark how far up on the stick the top of that hole comes, and then insert the stick into the other hole. If the mark indicating how far the stick went into the first hole goes inside the second hole, then the second hole is deeper; if the mark remains outside, then the first hole is deeper; and if it falls just at the top edge of the second hole, then the holes are equally deep.

In this solution, the stick (or, more precisely the segment of the stick between the end that is inserted into the holes and the mark made at the edge of the first hole) serves as a measure by means of which two quantitative relations are established: (*a*) the equality between the depth of the first hole and the length of the segment of the stick that was marked off, and (*b*) the equality or inequality relation between that length and the depth of the second hole. Transitive inference combining two relations is the basis for drawing a conclusion about the equality or inequality relation between the depths of the two holes.

Essential Understanding 1e

The equality relation between two quantities remains unchanged when one or both quantities are decomposed into parts and when one of the quantities is combined with another quantity to form a larger quantity.

An important extension of the concept of conservation is the idea that one or both of two quantities can be decomposed into parts or one of the quantities can be combined with another quantity to form a larger quantity without changing the equality relations between the original quantities. In reasoning about the relation between two unequal quantities, we can use this idea by thinking of the larger of the two quantities as composed of (*a*) a quantity that is equal to the smaller quantity and (*b*) a difference quantity. This