

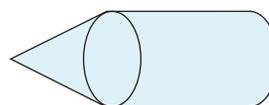
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Two- and Three- Dimensional Shapes, and Area and Volume

Why study the familiar three-dimensional geometrical shapes, namely, prisms, pyramids, cylinders, cones, and spheres? And why study the area of two-dimensional shapes and the surface area and volume of three-dimensional shapes?

In addition to length, area (including surface area) and volume are the geometric ways to describe the size of objects. Area tells us how much material is required to cover a two-dimensional shape or to cover the outer surfaces of a three-dimensional shape. Volume tells us how much material is required to fill a three-dimensional shape.

One reason to study the familiar three-dimensional geometric shapes is that these shapes are relatively simple and we can view many of the physical objects around us as composed of approximate versions of them. The trunks of many trees are approximately in the shape of a cylinder (or perhaps more accurately, a portion of a cone), most rooms are roughly the shape of a prism whose bases are the floor and ceiling, a pile of gravel might be cone-shaped, and a bottle might be roughly a combination of a cylinder and a cone. By studying the surface area and volume of the basic shapes, students develop important tools for describing the size of objects in the world around us.



Essential to understanding volume and surface area of three-dimensional shapes is viewing shapes as composed of other shapes. We usually calculate surface areas and develop formulas for surface areas by viewing the outer surface of a shape as consisting of several pieces joined together. We often calculate volumes and develop formulas for volumes by viewing a shape as decomposed into pieces or layers. Thus the study of area and volume gives students the opportunity to engage in one of the most common and powerful ways of reasoning in mathematics: that of taking apart, analyzing piece by piece, and putting the

analysis together to draw a conclusion. Of prime importance in the study of area and volume is the *reasoning* that leads to the common area and volume formulas. This reasoning connects the geometry of the shape with the algebra of the area or volume formula.

The next section discusses the main ideas leading up to, and in, the fifth-grade focus on area of shapes.

Progression of ideas about, and related to, area of flat two-dimensional shapes

Grade 1	Compose and decompose shapes (e.g., putting two triangles together to make a rhombus; decomposing a hexagon into 6 triangles, 3 rhombuses, or 2 trapezoids; tiling rectangles with squares).
Grade 3	Initially, find the number of objects in a rectangular array by counting one by one. Then recognize the structure of arrays as groups of rows or groups of columns, and use multiplication to find the number of items in a rectangular array. Include the case of rectangles decomposed into arrays of touching squares.
	Investigate, describe, and reason about decomposing, combining, and transforming polygons to make other polygons (include attending to angles and side lengths, and include making composite shapes that are treated as a unit and repeated). In the case of rectangles decomposed as arrays of touching squares, rotate these and view them as made by combining smaller arrays to show to properties of multiplication (e.g., the commutative property, the distributive property).
Grade 4	Understand that the area of a shape (in square units) is the number of unit squares required to cover the shape without gaps or overlaps.
	Fill simple shapes with unit squares, or use graph paper to determine areas, cutting squares in half if necessary so that no gaps or overlaps occur. Realize that shapes that look different can have the same area.
	Distinguish area from perimeter: two shapes can have the same area but different perimeter; two shapes can have the same perimeter but different area. Use graph paper or square tiles and rods or string to see why.
	Understand that areas of rectangles can be found by multiplying two adjacent side lengths because of the way rectangles can be decomposed into arrays of touching squares (restrict to the case of whole-number side length).
	Understand that different units of area (e.g., square yards or square feet, square meters or square centimeters) can be used to describe the area of a given shape. When a larger unit of area is used, the area is a smaller number of those units.

Grade 4	Understand that areas of shapes can be found by decomposing the shapes into (non-overlapping) pieces, finding the area of each piece, and adding. Restrict mainly to shapes that can be decomposed into rectangular pieces. Decomposing rectangles (or rectangular arrays) according to the place value of the side lengths is essential to explaining the multiplication algorithm.
Grade 5	Combine shapes to find an area (e.g., two congruent right triangles combine to make a rectangle, so we see that the area of the triangle is half the area of the rectangle). Decompose shapes, and move the pieces to find an area. Find an area by viewing a shape as a “difference” of two shapes (taking away area).
	Understand that area formulas for triangles and parallelograms can be found and explained by decomposing and composing to make rectangles. Any side can be chosen to be the base for a triangle or parallelogram, and the area formulas are still true. Different reasoning is often required for establishing validity of the area formula, depending on which side is chosen as base. Apply area formulas to solve problems.
	Find areas of complex shapes made from rectangles, triangles, and parallelograms (note that trapezoids and regular polygons can be viewed as such complex shapes).
Grade 6	Extend the area formula for rectangles to the case of fractional or decimal side lengths.
	Know that area formulas are valid even when side lengths are fractions or decimals. Apply area formulas for rectangles, triangles, and parallelograms in these cases.
Grade 7	See that the area formula for circles is plausible by decomposing circles into wedges (sectors) and rearranging the wedges to form a shape that approximates a parallelogram (or rectangle).
Grade 8	Explain why the Pythagorean theorem is true by decomposing a suitable square in different ways and equating the sum of the areas of the pieces.

Area of Polygons

To make sense of area calculations and area formulas, students must first understand what area is. Area is part of the focus of grade 4.

Measures of area, and units of area

What is area? In fourth grade (or before) students should see a need for the concept of area. The need for area is prompted by the desire to compare the sizes of shapes and to describe the size of a shape succinctly to others. To pique students’ interest about area, students can be given two rectangles (which students might view as maps of plots of land), the first of which is wider but shorter than the second, so that which (if either) of the two rectangular plots is bigger is not immediately obvious (for example, see the rectangles in fig. 4.1a). How does one determine which is bigger? How can one describe the size of a rectangle? These questions lead naturally into the study of area.

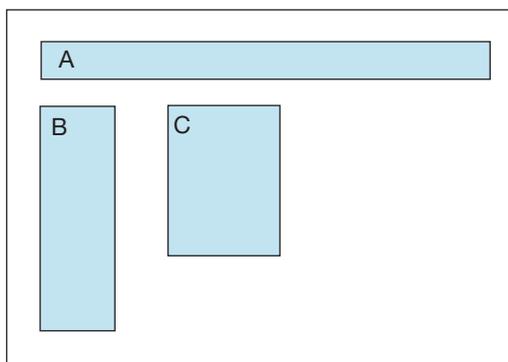


Fig. 4.1a. Which quilt piece covers the most space?

One challenge in learning about area is to understand what area measures. The area of a flat shape is a measure of the amount of space inside the shape; it tells us how much material is needed to cover the shape. As with length, we need standard units of area to be able to communicate clearly. Squares make especially nice units of area because they fit together snugly to make neat rectangular arrays. For this reason, we use squares for our standard units of area. A square centimeter, 1 cm^2 , which is the area of a 1-centimeter-by-1-centimeter square, is a natural first unit of area to use. Alternatively, a square inch, 1 in^2 , could be used initially. The area of a shape, in square units, is the number of 1-unit-by-1-unit squares (“unit squares”) needed to cover the shape without gaps or overlaps.

The students in Ms. C’s class discovered the usefulness of a unit square for comparing areas when they were asked which of three quilt pieces, like those shown in figure 4.1, covers the most space (Lehrer et al. 1998). To make the rectangles in figure 4.1a, use the following dimensions: A is 1 by 12, B is 2 by 6, and C is 3 by 4 (but do not give students these dimensions).

One student showed that pieces A and C cover the same amount of space by folding C into four equal horizontal parts. Another student folded C into three equal vertical parts to reach the same conclusion (see fig. 4.1b). When both sets of folds were made on C, the students were able to use the resulting squares as a common unit to show that all three quilt pieces cover the same amount of space.

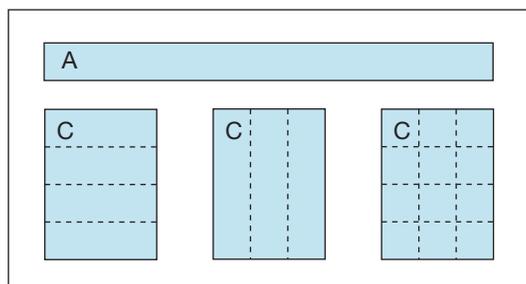


Fig.4.1b. Different ways to compare quilt pieces A and C

Initial area work should involve only whole numbers of units, but students should eventually work with shapes that need to be cut apart and rearranged to make a whole number of unit squares. Initial examples can include triangles of area 1 cm^2 , such as a right triangle with 1 cm and 2 cm legs.

By drawing a variety of shapes on graph paper or by making shapes out of tiles (perhaps in a context, such as designing a floor plan for a guinea pig cage), students should come to understand that many different shapes have the same area.

Area versus perimeter

One challenge for students is to understand the distinction between length and area. In fourth grade, students should have the opportunity to distinguish between the area and the perimeter of a shape and to realize that neither one determines the other.

For example, by using a fixed number of rods of a fixed length to make the outline of various different shapes, or by considering or making drawings on graph paper, students could see that two shapes can have the same perimeter but different areas. Because the lengths of two adjacent sides of a rectangle add to half the perimeter of the rectangle, to find rectangles of a given, fixed perimeter, say, 12 feet, find lengths that add to 6 feet. So a 5-foot-by-1-foot rectangle, a 4-foot-by-2-foot rectangle, and a 3-foot-by-3-foot rectangle (and even a $4\frac{1}{2}$ -foot-by- $1\frac{1}{2}$ -foot rectangle) all have the same perimeter, 12 feet, but have different areas.

Students could also arrange square tiles in different configurations or use drawings on graph paper to show that two shapes can have the same area but different perimeter. Given a fixed (whole) number of square-inch tiles, say, twenty-four tiles, finding all the (“filled in”) rectangles that can be made with exactly that many tiles is the same as finding the pairs of factors for 24:

$$24 = 1 \times 24, \quad 24 = 2 \times 12, \quad 24 = 3 \times 8, \quad 24 = 4 \times 6.$$

Note that if students take orientation into account, they may think of a pair of factors as giving rise to two rectangles, such as a 3-by-8 rectangle and an 8-by-3 rectangle. These two rectangles are congruent because one can be rotated to match the other, so in an abstract setting, a factor pair produces a single rectangle. But in many practical situations, such as when a rectangular patio is attached to the side of a house, the different orientations can be distinct.

Students who have already studied areas of rectangles might also solve the next Puppy Run and Farmer’s Gardens problems.

Puppy Run

You want to make a run for your new puppy. You have 36 feet of fencing. The floor of the puppy run will be in the shape of a rectangle, and the fence will go all the way around the rectangle.

1. Find the lengths of the sides of at least 3 different puppy runs that you could make using all 36 feet of fencing. (Answers vary: Adjacent sides add to 18 feet.)
2. Find the areas of the floors of your puppy runs in part 1. Are all the areas the same? (Answer: The areas are different.)
3. What do you think is the largest area you can make with this amount of fencing? Explain your answer. (Answer: the largest area that can be made occurs when the pen is a 9-foot-by-9-foot square of area 81 square feet. Some students may notice that the more “squarelike” the rectangle is, the greater the area, whereas the greater the difference between lengths of adjacent sides, the smaller the area.)



Areas of rectangles

Farmer's Gardens



A farmer has three gardens. Each garden has a perimeter of 36 feet. The first garden has an area of 81 square feet. The second garden has an area of 72 square feet. The third garden has an area of 45 square feet. What might the dimensions of each garden be? (Answer: If the gardens are rectangular, then two adjacent sides must add to 18 feet (half of 36), so students can look for numbers that add to 18. A 9-foot-by-9-foot garden has area 81 square feet. A 6-foot-by-12-foot-garden has area 72 square feet. A 15-foot-by-3-foot garden has area 45 square feet.)

The multiplication formula for areas of rectangles

Why can we multiply the length and width of a rectangle to find its area? A rectangle whose length and width are whole numbers of units can be decomposed into an array of unit squares. By viewing these arrays as groups of rows or groups of columns, as in figure 4.2, students should come to see that they can use multiplication, rather than count one by one, to determine the total number of unit squares covering the rectangle in an efficient way. Thus students should see that the length-times-width formula for areas of rectangles makes sense by viewing rectangles whose sides are whole numbers of units as arrays of unit squares that can be decomposed into equal groups by the rows or the columns of the array. For consistency with other area formulas (for triangles and parallelograms), instead of referring to the side lengths of a rectangle as “length” and “width,” we can instead use “base” and “height.”

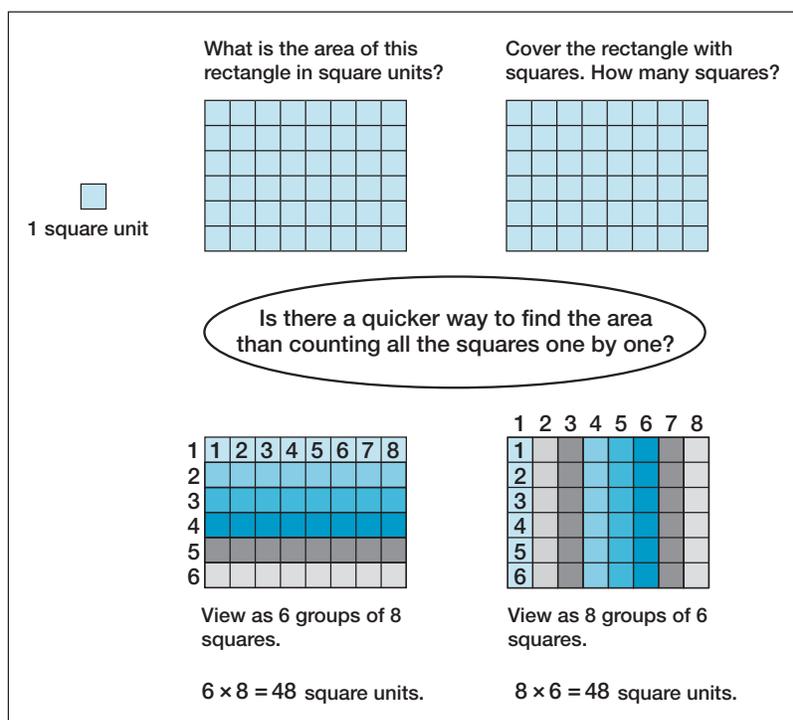


Fig. 4.2. Decomposing a rectangle into groups of squares to see why we multiply

For some students, a challenge in learning the area formula for rectangles is to see that rectangles of whole-number side length can be decomposed into neat arrays of unit squares and that these arrays can be decomposed into equal groups. Such students may need to work with actual squares or make drawings of rows and columns on square grid paper. They need to build rectangles and create mental images of the interior space of rectangles as composed of arrays of squares. Being able to view rectangles as decomposed into groups of squares is essential to understanding the area formula for rectangle.

Rectangle Area Problems

1. A rectangular patio is 8 feet wide and has area 120 square feet. How long is the patio?
(Answer: 15 feet)
2. A rectangular garden is 25 meters wide and is surrounded by 120 meters of fencing. What is the area of the garden?
(Answer: The length of the garden is 35 meters because the length plus width is half of the perimeter. So the area is 25×35 , or 875, square meters.)
3. A rectangle is 40 millimeters wide and 7 centimeters long. What is the area of the rectangle? Why can we not multiply 7×40 to find the area?
(Answer: 28 square centimeters, or 2800 square millimeters. We have to use the same units for the length and the width when we apply the area formula.)

Finding areas of regions composed of several rectangles

In preparation for finding areas of triangles and parallelograms in grade 5 by decomposing and combining shapes, fourth-grade students should become flexible in decomposing shapes into rectangular pieces to find the area. L-shaped regions, such as the one at the top of figure 4.3, are good starting points because students may find several different ways to determine the area, such as the four methods shown in the figure. Although method 4 does not work for all L-shaped regions, it does work in this instance, and it previews a method for determining areas of triangles. Asking students to associate strategies with the corresponding numerical expressions for the area (shown below each strategy in fig. 4.3) forges a connection with algebra.

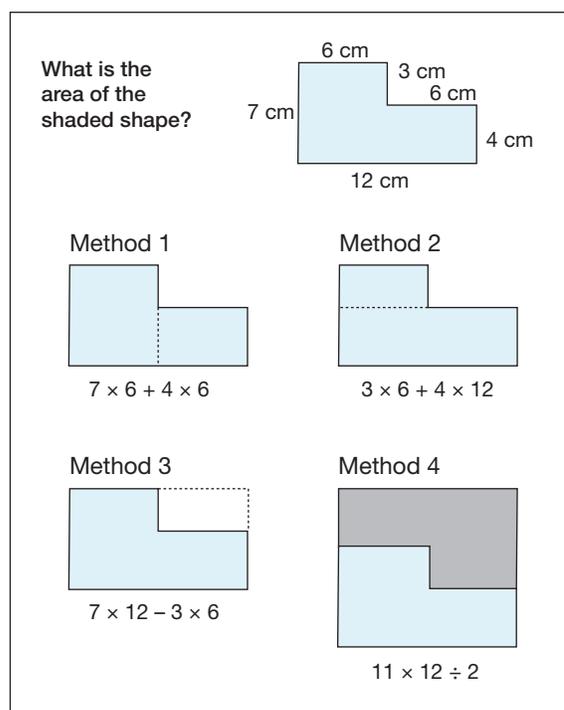


Fig. 4.3. Finding the area of an L-shaped region in several ways

L-shaped Polygons of Area 84

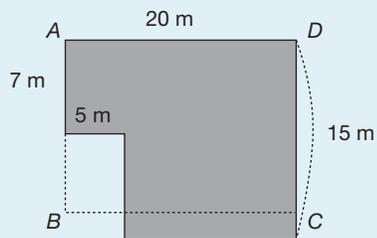
The area of a mystery “L shaped” polygon is 84 square units. What are the lengths of the sides of this mystery polygon? Can we tell for sure, or is more than one L-shaped polygon of area 84 square units possible?

(Answer: Many different L-shaped polygons have area 84 square units.)

Changing the Shape but Keeping the Area the Same

This is a good, challenging multistep problem about area.

The shaded shape is a diagram of a lot. If we want to make the lot into the rectangle $ABCD$ of the same area, how long should CD be?



(Answer: 13 meters because the area of the shape is 260 square meters and $260 \div 20 = 13$)

Other units of area

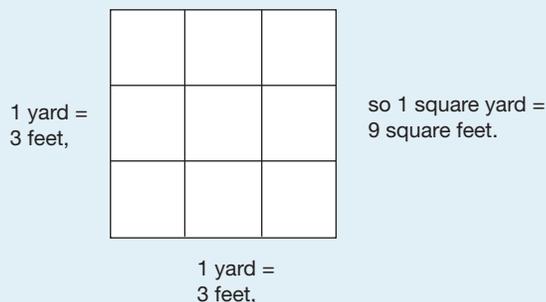
When students first start studying area in fourth grade (or before), they will use square inches or square centimeters as units. Wanting to describe the area of a room or a field presents the need for other units of area. Square meters and square feet are natural choices of units for describing areas of rooms, fields, and the like. Square kilometers and square miles are natural choices for describing areas of cities, states, and countries.

Students should understand that the area of a region can be given in square meters or in square centimeters but that the number of square meters will be less than the number of square centimeters (and similarly for square feet and square inches). More specifically, students should understand why 100×100 square centimeters are in a square meter and why 12×12 square inches are in a square foot.

When considering area in real-world situations, problems involving estimation are especially appropriate.

Area of a Classroom in Square Feet and Square Yards

1. Tell students: A fourth- (or fifth-) grade class at another school says that their classroom has area approximately 70 square yards. Is our classroom larger, smaller, or about the same? First, make an estimate of the area of our classroom. Then make a plan to determine the area of our classroom.
2. Discuss students' estimates and plans, then ask them to carry out one (or several) of the plans. After students have carried out the plans, have a class discussion in which students compare their answers with their estimates and with the estimate of the other fourth- (or fifth-) grade class.
3. Ask students: What if the other fourth- (or fifth-) grade class reported the area of their classroom in square feet instead of as 70 square yards. Would the number of square feet be greater than or fewer than 70? How do you know?
4. Ask students: Is there a way we can find out the area of the other classroom in square feet, given that it is 70 square yards? (Students should determine that since a 1-yard-by-1-yard square is 3-foot-by-3-foot, 1 square yard is 9 square feet, so 70 square yards is 70×9 , or 630, square feet.)



5. As an optional follow-up, take students on a “field trip” through the school building. Have students estimate various areas. Take measurements so that the areas can be determined at a later time. Back in the classroom, have students determine the areas using the measurements, then compare their answers with their estimates.

Areas of parallelograms, triangles, and other polygons

Since students study many polygons other than combinations of rectangles, a natural extension is to ask about finding areas of familiar polygons. Finding areas of many of these familiar polygons is a focus in grade 5. A central idea is that we can often find the area of a polygon by relating it to another polygon whose area we already know how to find. In particular, parallelograms and triangles can be related to rectangles. Triangles can also be related to parallelograms.

Areas of parallelograms

To help students reason about areas of parallelograms and develop and understand an area formula for parallelograms, students could be given parallelograms on centimeter graph paper and asked to relate them to rectangles to find the area. Students might come up with several different methods, including those indicated in figure 4.4. In each of these cases, the area of the parallelogram is the same as the area of the related rectangle because pieces of the parallelogram have just been rearranged, but no area has been