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Mathematical Reasoning and Sense Making in Grades 6-8[1]

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Reasoning and sense making are the foundation of mathematical competence and proficiency, and their absence from the curriculum leads to failure and disengagement in mathematics instruction. Thus, developing students' reasoning and sense-making capabilities should be the primary goal of mathematics instruction. In order to achieve this goal, all mathematics classes should provide ongoing opportunities for students to implement these processes.

What are mathematical reasoning and sense making? *Reasoning* is the process of manipulating and analyzing objects, representations, diagrams, symbols, or statements to draw conclusions based on evidence or assumptions. *Sense making* is the process of understanding ideas and concepts in order to correctly identify, describe, explain, and apply them. Genuine sense making makes mathematical ideas "feel" clear, logical, valid, or obvious. The moment of sense making is often signaled by exclamations such as "Aha!" "I get it!" or "Oh, I see!"

Why Focus on Reasoning and Sense Making?

Reasoning and sense making are critical in mathematics learning because students who genuinely make sense of mathematical ideas can apply them in problem solving and unfamiliar situations and can use them as a foundation for future learning. Even with mathematical skills, "[i]n order to learn skills so that they are remembered, can be applied when they are needed, and can be adjusted to solve new problems, they must be learned with understanding [i.e., they must make sense]" (Hiebert et al. 1997, p. 6).

Sense making is also important because it is an intellectually satisfying experience, and not making sense is frustrating (Hiebert et al. 1997). Students who achieve genuine understanding and sense making of mathematics are likely to stay engaged in learning it. Students who fail to understand and make sense of mathematical ideas and instead resort to rote learning will eventually experience continued failure and withdraw from mathematics learning.

Understanding Students' Thinking

An abundance of research describing how students learn mathematics indicates that effective mathematics instruction is based on the following three principles (Battista 2001; Bransford, Brown, and Cocking 1999; De Corte, Greer, and Verschaffel 1996; Greeno, Collins, and Resnick 1996; Hiebert and Carpenter 1992; Lester 1994; NRC 1989; Prawat 1999; Romberg 1992; Schoenfeld 1994; Steffe and Kieren 1994):

1. To genuinely understand mathematical ideas, students must construct these ideas for themselves as they intentionally try to make sense of situations; the meanings they construct for new mathematical ideas that they encounter is determined by their preexisting knowledge and reasoning and by their commitment to making personal sense of those ideas.
2. To be effective, mathematics teaching must carefully guide and support students as they attempt to construct personally meaningful mathematical ideas in the context of problem solving, inquiry, and student discussion of multiple problem-solving strategies. This sense-making and discussion approach to teaching can increase equitable student access to powerful mathematical ideas as long as it regularly uses embedded formative assessment to determine the amount of guidance each student needs. (Some students construct ideas quite well with little guidance other than well-chosen sequences of problems; other students need more direct guidance, sometimes in the form of explicit description.)
3. To effectively guide and support students in constructing the meaning of mathematical ideas, instruction must be derived from research-based descriptions of how students develop reasoning about particular mathematical topics (such as those given in research-based learning progressions).

Consistent with this view on learning and teaching, professional recommendations and research suggest that mathematics teachers should possess extensive research-based knowledge of students' mathematical thinking (An, Kulm, and Wu 2004; Carpenter and Fennema 1991; Clarke and Clarke 2004; Fennema and Franke 1992; Saxe et al. 2001; Schifter 1998; Tirosh 2000). Teachers should "be aware of learners' prior knowledge about particular topics and how that knowledge is organized and structured" (Borko and Putnam 1995, p. 42). And because numerous researchers have found that students' development of understanding of particular mathematical ideas can be characterized in terms of developmental sequences or *learning progressions* (e.g., Battista and Clements 1996; Battista et al. 1998; Cobb and Wheatley 1988; Steffe 1992; van Hiele 1986), teachers must understand these learning progressions. They must understand "the general stages that students pass through in acquiring the concepts and procedures in the domain, the processes that are used to solve different problems at each stage, and the nature of the knowledge that underlies these processes" (Carpenter and Fennema 1991, p. 11). Research clearly shows that teacher use of such knowledge improves students' learning (Fennema and Franke 1992; Fennema et al. 1996). "There is a good deal of evidence that learning is enhanced when teachers pay attention to the knowledge and beliefs that learners bring to a learning task, use this knowledge as a starting point for new instruction, and monitor students' changing conceptions as instruction proceeds" (Bransford et al. 1999, p. 11).

Beyond understanding the development of students' mathematical reasoning, it is important to recognize that to be truly successful in learning mathematics, students must stay engaged in making personal sense of mathematical ideas. To stay engaged in mathematical sense making, students must be successful in solving *challenging but doable* problems. Such problems strike a delicate balance between involving students in the hard work of careful mathematical reasoning and having students succeed in problem solving, sense making, and learning. Keeping students successfully engaged in mathematical sense making requires teachers to understand each student's mathematical thinking well enough to continuously engage him or her in *successful* mathematical sense making. Furthermore, to pursue mathematical sense making during instruction, students must *believe*—based on their past experiences—that they are capable of making sense of mathematics. They must also believe that they are supposed to make sense of all the mathematical ideas discussed in their mathematics classes.

Finally, as part of the focus on reasoning and sense making in mathematics learning, students must adopt an inquiry disposition. Indeed, students learn more effectively when they adopt an active, questioning, inquiring frame of mind; such an inquiry disposition seems to be a natural characteristic of the mind's overall sense-making function (Ellis 1995; Feldman and Kalmar 1996).

Reaching All Students

The foregoing description of principled, student-reactive teaching not only helps all students maximize their learning but also benefits those who are struggling (Villasenor and Kepner 1993). In fact, this type of teaching supports all three tiers of Response to Intervention (RTI) instruction. For Tier 1, high-quality classroom instruction for all students, research-based instructional materials include extensive descriptions of the development of students' learning of particular mathematical topics. Research shows that teachers who understand such information about student learning teach in ways that produce greater student achievement. For Tier 2, research-based instruction enables teachers to better understand and monitor each student's mathematics learning through observation, embedded assessment, questioning, informal assessment during small-group work, and formative assessment. They can then choose instructional activities that meet their students' learning needs: whole-class tasks that benefit students at all levels or different tasks for small groups of students at the same level. For Tier 3, research-based assessments and learning progressions support student-specific instruction for struggling students so that they receive the long-term individualized instruction sequences they need.

Because extensive formative assessment is embedded in this type of teaching, support for its effectiveness also comes from research on the use of formative assessment, which indicates that formative assessment helps all students—and perhaps particularly those who are struggling—to produce significant learning gains, often reducing the learning gap between struggling students and their peers.

What Does Sense Making Look Like during Learning and Teaching?

The following two examples, which illustrate the development of students' reasoning and sense making about particular mathematical ideas, will allow us to examine obstacles to sense making, variations in student sense making, and how teaching can support sense making at various levels of sophistication.

Making Sense of Division of Fractions

To illustrate the nature of mathematical sense making, reasoning, and understanding, consider two different ways that students might reason about and make sense of the problem "What is $2^1/_2$ divided by $^1/_4$?" (Battista 1999). Many students solve this problem by using the "invert-and-multiply" procedure they memorize and almost never understand:

$$2\frac{1}{2}+\frac{1}{4}=\frac{5}{2}\square\frac{4}{1}$$

They do not make conceptual sense of this procedure, and the only way they can justify it's validity is by saying something like "That's the way my teacher taught me."

In contrast, students who have made sense of and understand division of fractions do not need a symbolic procedure to compute an answer to this problem. They can think about the symbolic problem physically as one that requires finding the number of pieces of size one-fourth that fit in a quantity of size two and one-half (see fig. 1.1). They reason that because there are 4 fourths in each 1 and 2 fourths in $^1/_2$, there are 10 fourths in $2^1/_2$.

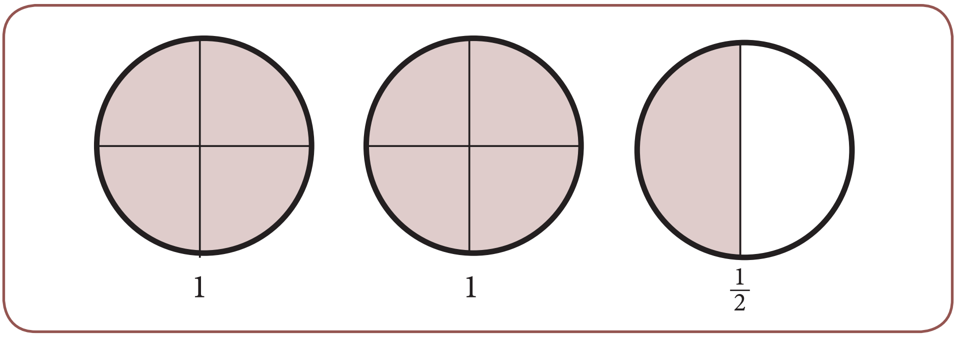


Fig. 1.1

Furthermore, having this mental model–based intuitive understanding of division of fractions can help students start to make personal sense of the symbolic algorithm. In the problem $2^1/_2 \div {}^1/_4$, why do we change division by $^1/_4$ to multiplication by 4? The reason is that because there are 4 fourths in each whole, we must multiply the number of wholes in the dividend (including fractional parts) by 4 to determine how many fourths are in the dividend $2^1/_2$. Expressing this reasoning in equation form would look like the following:

$$2\frac{1}{2} \div \frac{1}{4} = 2\frac{1}{2} \times 4 = (2 \times 4) + (\frac{1}{2} \times 4) = 8 + 2 = 10$$

As another example, what is 10 divided by $^1/_4$? Because there are 4 fourths in each 1, and there are 10 ones in 10, there are 10 times 4 fourths in 10. So the answer is found by multiplying the dividend 10 by 4; that is, $10 \div {}^1/_4 = 10 \times 4$. To have students continue this reasoning, we can ask them to describe how to find the quotients for problems like $12 \div {}^1/_5$, $8^1/_2 \div {}^1/_2$, and $7^1/_3 \div {}^1/_6$, and to describe in words why their solution procedures work.

Reasoning and Sense Making About Geometric Properties of Shapes

To illustrate the ideas just described, we examine students' sense making and reasoning about one particular topic—using geometric properties of shapes and transformations to find missing side lengths in polygons. We look at the different ways that students make sense of and reason about this topic, and we examine how instruction can encourage and support students' increasingly more sophisticated reasoning about it. The key to helping students make sense of a formal mathematical idea is first determining empirically how they currently are making sense of the idea, second hypothesizing how their understanding of the idea might progress, and finally choosing problems and representations that can potentially help them progress to more sophisticated levels of reasoning. Because of the wide variety of strategies students use when solving these problems, the problems offer an excellent context for students to implement mathematical practice 3: construct viable arguments and critique the reasoning of others.

Types of Student Reasoning

Consider the set of problems shown in figure 1.2. Students use two fundamentally different types of reasoning and sense making on these problems: measurement and non-measurement reasoning. And, as illustrated, their uses of these types of reasoning have varying levels of sophistication.

Non-measurement Reasoning

Non-measurement reasoning does not use numbers. Students reason by using visualization, transformations, and properties of shapes. The following examples illustrate several levels of sophistication in this type of reasoning that differ in both abstractness and validity.

Some students, such as those discussing problem 2 in figure 1.2, use vague holistic visual reasoning.

Jackson: Shape A looks bigger.

Rafael: Shape B looks longer because it has more turns.

Feliciti: Shape A has more room inside, so it has to have more length around.

Other students visualize moving parts of shapes around in ways that do not lead to valid reasoning.

Angelo: In problem 2 the bottom middle of B can be moved down to make A [see fig. 1.3], so they [the perimeters] are the same.

Instructional Support for Students Who Use Vague Non-Measurement Reasoning

One way to help students like Jackson, Rafael, Feliciti, and Angelo is to give them large drawings of the shapes in problems 1–5 (with dimensions, say, in inches) and sets of appropriately cut unit straws that they can place on top of the drawings (fig. 1.4). For example, in figure 1.4a, students can show that both shapes A and B can be made from the same set of straws, so they have the same perimeter. For students who need more support in making sense of this reasoning, we might use two identical sets of straws like those shown in figure 1.4a and put them in straight lines next to each other to see that the total lengths of each set are equal. Such an activity can help them make sense of the reasoning that if one shape can be made from the other shape by rearranging its sides, then the two shapes have equal perimeters. Also, some students might argue that shapes A and B have the same length because they both can be made from 1 8-rod, 1 6-rod, and 2 3-rods, *In the shapes below, angles that look like right angles are right angles. In each problem, which shape has the greater perimeter, or do they have the same perimeter? (The perimeter of a shape is the distance traveled as you trace it.) Describe your reasoning in writing.*

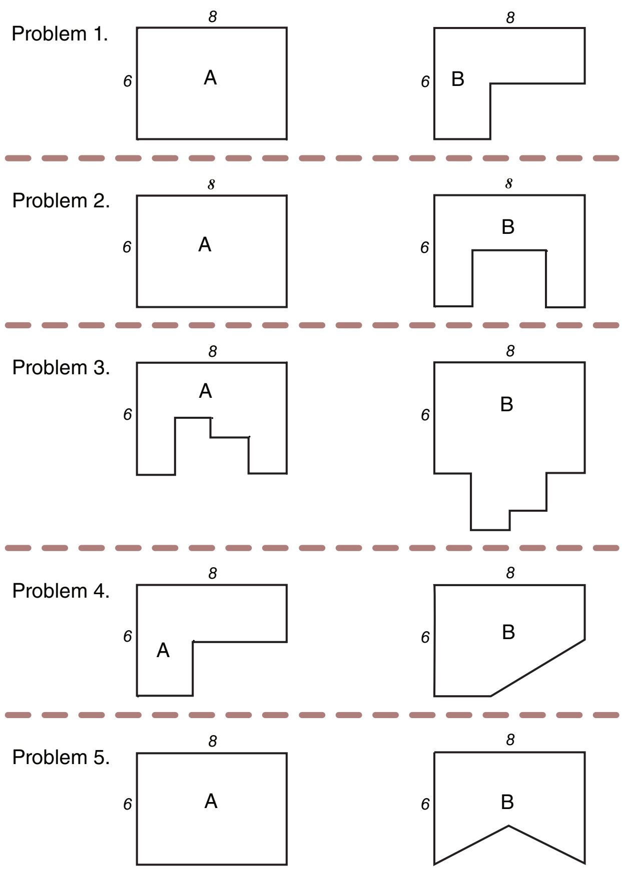


Fig. 1.2. Initial problem set

1 2-rod, and 1 4-rod. In figure 1.4b, we ask students to use the colored segments to decide which shape has the larger perimeter. They can find that all the segments are needed to cover shape B but that shape A can be covered without using the brown segments; therefore shape B has the greater perimeter. (Use two identical sets and line them up for students who do not believe this.) Decomposing shapes into parts and rearranging those parts is a powerful form of geometric reasoning that is useful in measuring not only length but also area and volume.

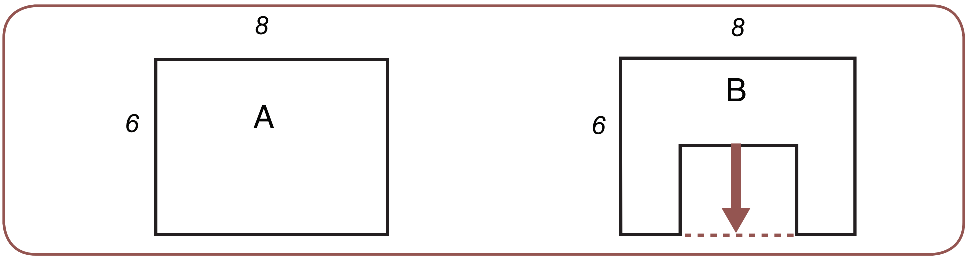


Fig. 1.3. An example of incorrect movement-based reasoning

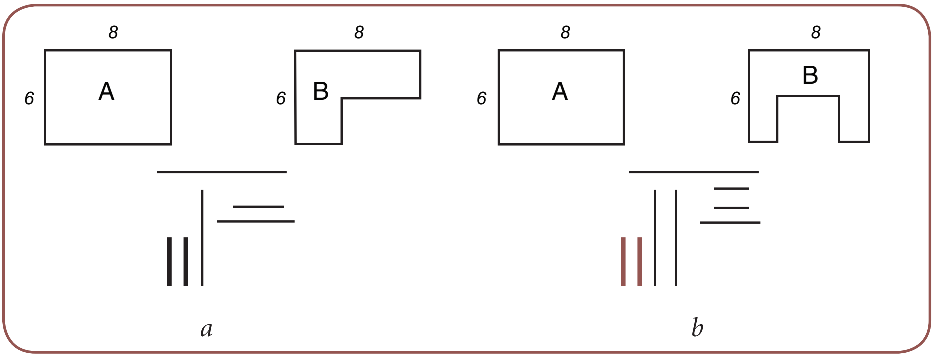


Fig. 1.4. Using straws to compare perimeters of varying Shapes A and B

Other students use more sophisticated movement-based visual reasoning that, although informal, is consistent with geometric properties of figures and transformations. For example (see fig 1.5), for problem 2 Chen reasons that the three bottom horizontal segments of B can be moved to make the top side, while Gia reasons that the middle horizontal segment of B can be moved to fit exactly on the bottom of B. Both students use this reasoning to conclude that the sides of shape A can be made from the sides of shape B but with segments *d* and *d* left over; so B has the greater perimeter. Implicit in the movement-based arguments of these two students is that these movements preserve length.

Chen: [*Motioning toward B as shown in fig. 1.5*] I can move these 3 lines [horizontal segments] on B up to equal the top side of B. So if we take these 3 lines, the top, and both sides of B, that's the same as A. But B also has these 2 lines [segments marked *d* on B]. So B is bigger.

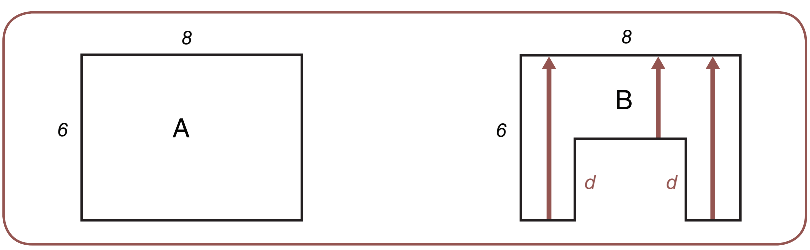


Fig. 1.5. An example of sophisticated movement-based visual reasoning

Gia: Slide the bottom middle segment all the way down [*see shape B in fig. 1.6*]. That makes rectangle A. But B has these extra lines [*draws squiggles*]. So B has the bigger perimeter.

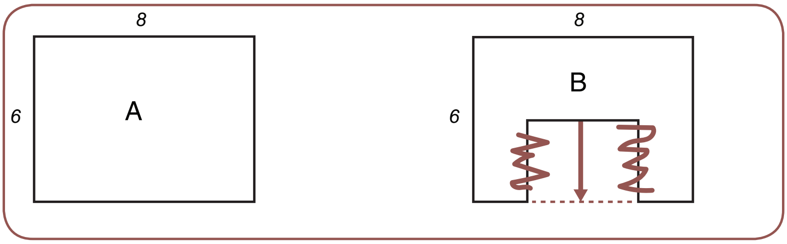


Fig. 1.6. A second example of correct movement-based reasoning

Formalizing Non-Measurement Reasoning Using Isometries

One way to help students formalize their intuitive movement-based reasoning is to specify motions explicitly with isometries. In problem 2, for instance, if we translate segment *xy* through vector *xz*, we will get rectangle A but with extra segments (*d* and *d*; see fig. 1.7). So B has the greater perimeter. This is the reasoning that Gia used, but instead of referring to her visualization as a *translation*, she used the informal term *slide*, and she did not specify the translation vector. If Gia had studied isometries, asking her whether she knew another name for a slide and whether she could define the *translation vector* would have encouraged her to move to a more abstract and sophisticated level of reasoning.

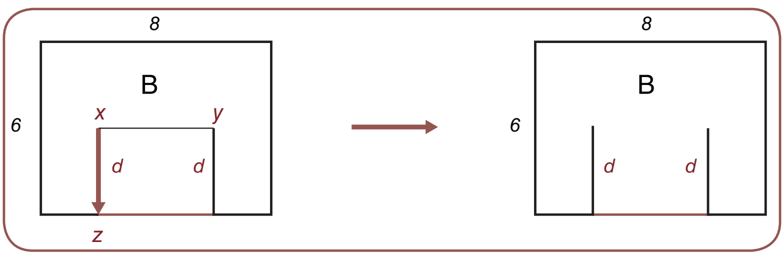


Fig. 1.7. Using isometries to formalize intuitive movement-based reasoning

While working on problem 3, many students reason that shape B can be transformed into shape A by reflecting (many students say *flipping*) the bottom section of the shape.

Ellie: [*Draws the dashed brown segments in figure 1.8*] The perimeters are equal because if you reflect this part [*points to indented 5-piece section of shape A*] about the dotted line, you get this part [*points to the congruent extended 5-piece section of shape B*].

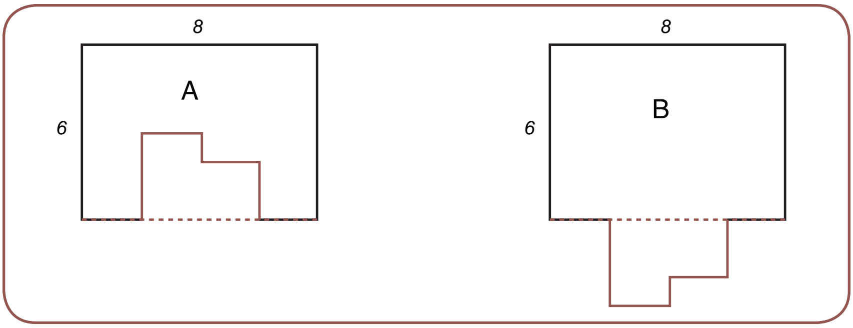


Fig. 1.8. Some students "flip" the bottom of shape B to transform it to Shape A.

Distance-Preserving and Nonpreserving Transformations

Some students use movement-based reasoning that does not preserve lengths and therefore invalidates their conclusions. Consider Deshawn's argument for why shapes A and B in problem 5 have the same perimeter.

Deshawn: [*Referring to problem 5*] If you slide these two lines down (see fig. 1.9a), you get shape A. So A and B have the same perimeter.

In this case, Deshawn used a transformation *that did not preserve the lengths* of the line segments. We might use Dynamic Geometry or physical materials to help Deshawn see that to move segments XP and YP so that they are horizontal—in a way that preserves their lengths—we need to rotate them as radii of circles, not shrink them. As shown in figure 1.9b, if we rotate circle radii XP and YP to get XQ and YR so that the segments are horizontal, they will overlap. So XPY is longer than XY, which means that the perimeter of shape B is greater than that of shape A. To help Deshawn build on his informal transformation-based reasoning, it is important to discuss valid transformation-based reasoning (instead of referring, for instance, to the triangle equality theorem discussed below).

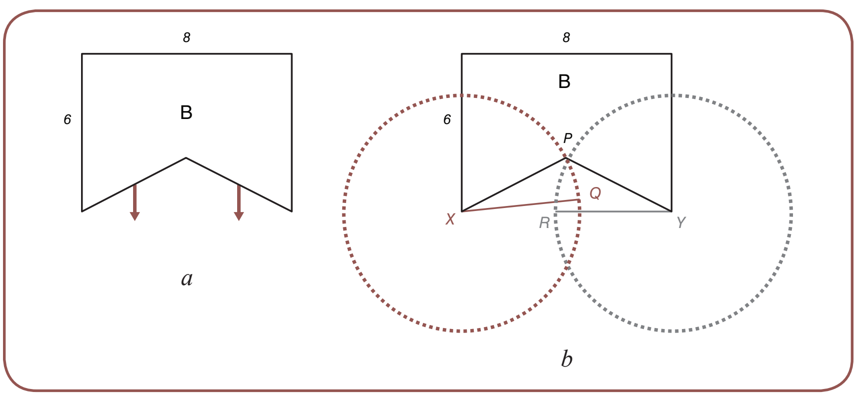


Fig. 1.9. Using a transformation that does not preserve segment lengths

Making Non-measurement Reasoning More Rigorous by Using Properties of Rectangles

Another way to formalize and add precision to reasoning about these problems is to construct arguments that explicitly reference properties of rectangles.

Suchi: [*Working on problem 2*] Because all the angles are right angles (see fig. 1.10), shape A and the outside of shape B are congruent rectangles. I know that opposite sides are equal in rectangles, so I know the bottoms are 8 like the tops, and the right sides equal 6 like the left side [*labels these lengths*]. Also, right here [*draws the dashed segment in B and labels points P, Q, R, S*], PQRS is a rectangle. So QR equals PS. [*Labels points M and N.*] If we add MP, QR, and SN, they equal the bottom of B, which is 8. But B also has QP and RS in its length, so B has the bigger perimeter.

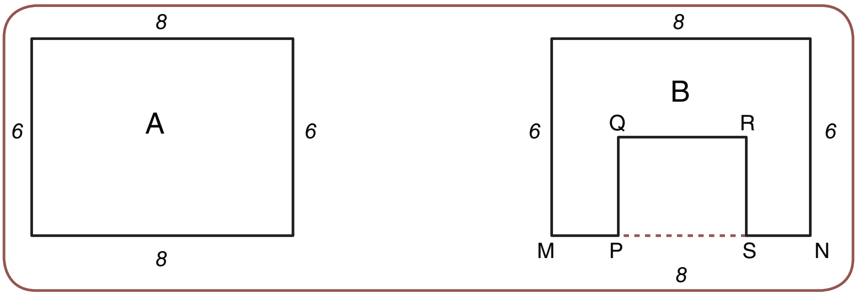


Fig. 1.10. Using the properties of rectangles for non-measurement reasoning

Measurement Reasoning

Measurement reasoning uses numbers. In this second type of reasoning about the perimeters of shapes, students count, estimate, and use properties of shapes to explicitly draw conclusions about shapes' *numerical measurements*. As students use increasingly sophisticated measurement strategies, they are developing fluency in mathematical practice 2a: make sense of quantities and their relationships. As the following examples illustrate, there are many levels of sophistication in this type of reasoning.

At perhaps the lowest level of such reasoning, many students use a numerical procedure that has little connection to the required measurement concepts. For example, in any of the problems 1–5, a student might multiply 6 times 8 for each shape and conclude that the shapes have the same perimeter. Other students use counting, even though in this case counting is too imprecise. For instance, one student used finger taps to find the perimeters of both figures and concluded that B is bigger (see fig. 1.11).

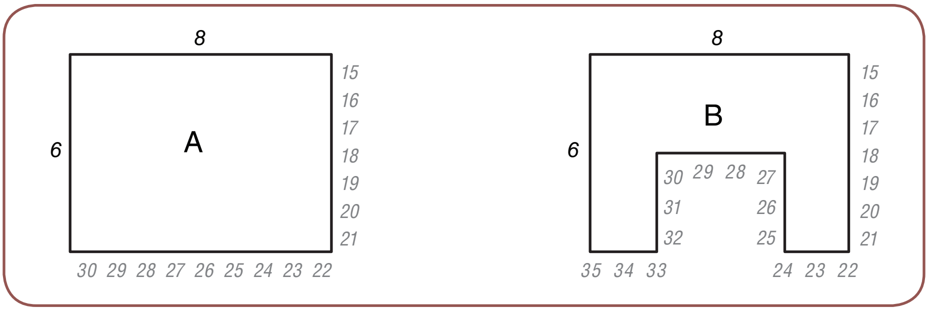


Fig. 1.11. A low level of reasoning, using finger taps to measure perimeters

When reasoning numerically about these problems, many students do not need to count, but they do need to label missing side lengths with numbers. They cannot reason abstractly and logically with unlabeled lengths; they need sides labeled with numbers. To get these numbers, many students incorrectly estimate the lengths of the bottom segments of B in a way that violates the properties of rectangles. In the example shown in figure 1.12, the student did not understand that the sum of the length estimates for the bottom horizontal segments of B [2 + 5 + 2] should equal the length of the top of B, which is 8. Note that this student would benefit from hearing Suchi's rectangle-based reasoning.

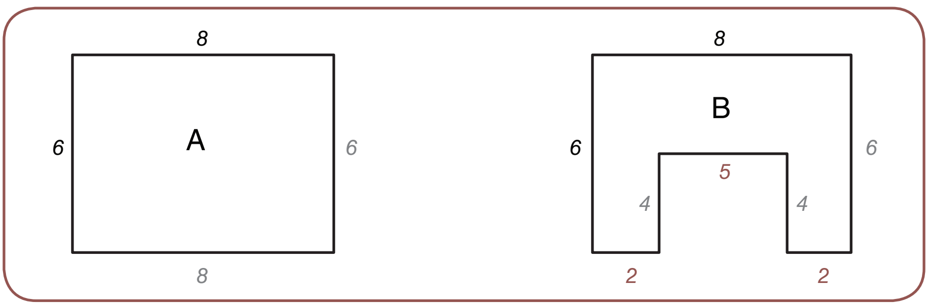


Fig. 1.12. Not understanding that the sum of the bottom lengths of shape B must equal the top length

Other students who need numbers to reason about the problems know that the lengths of the three bottom horizontal segments of B must sum to 8 because they must equal the length of the top of B (again, using reasoning similar to Suchi's). Their estimates for these missing lengths incorporate appropriate geometric properties of the shapes, like Andre's estimate, which follows.

Andre: Well I know that these 3 lines [bottom horizontal segments in B (see fig. 1.13)] have to equal 8. So I'll say they are 2, 4, and 2 [*labels sides on B*]. And these 2 lines [bottom vertical segments on B], look like they are half of 6, so I'll make them 3 and 3 [*labels sides on B]*. So, for B, I have 6 + 8 + 6 + 2 + 4 + 2 + 3 + 3, which equals 34, and A is 8 + 8 + 6 + 6, which is 28. So B has the larger perimeter.

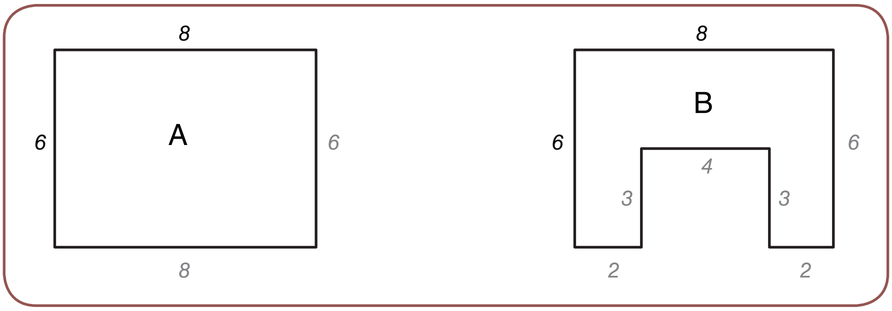


Fig. 1.13. Correctly using the properties of rectangles to help with reasoning

Instructional Support for Students' Numerical Reasoning Using Drawings on Graph Paper

At first, many students will not be able to apply properties of rectangles intuitively to infer lengths of segments, as did Andre. These students need to see the actual unit-lengths that make up the lengths, and they need to see empirically, with the help of many examples, that opposite sides of rectangles have the same length. Teachers can help such students by presenting the problems on graph paper. First, have students attempt to solve a problem as presented in the initial problem set. Then, have the students who need help redraw the figures on graph paper. Always have students first predict an answer without graph paper; then check their answer with graph paper. Comparing their nongrid and grid answers will help them abstract the structures of the shapes, which will eventually enable them to obtain answers without graph paper.

Teacher: [*Showing the alternate form of problem 2 on a document projector* (fig. 1.14)] Let's check our reasoning on problem 2 by drawing the shapes on graph paper and writing in the missing lengths. We can count unit-lengths to find the lengths of each side.

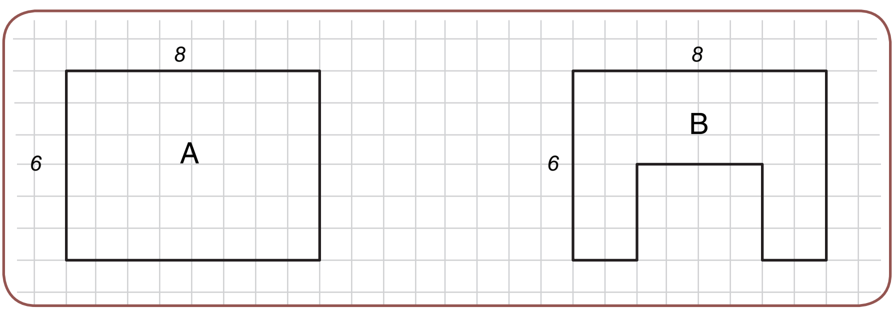


Fig. 1.14. Shapes A and B drawn on graph paper

Teacher: Can somebody come up to the projector and write the lengths of all the unlabeled sides? [*Emma labels the side lengths correctly as shown in figure 1.15.*]

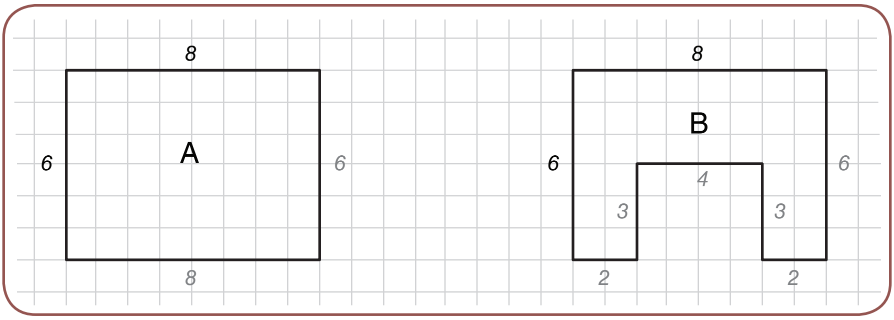


Fig. 1.15. Correct labeling of shape B

Teacher:	So what are the perimeters of shapes A and B?
Emma:	I got the perimeter of A as 8 + 8 + 6 + 6 = 28.
Haseen:	I got the perimeter of B as 8 + 6 + 6 + 2 + 4 + 2 + 3 + 3 = 34. So B is longer than A.
Teacher:	What do you notice about the lengths of the three bottom sides of B: 2, 4, and 2?
Haseen:	They add up to 8, the same as the 8 for the top of B.
Teacher:	Why is that?
Andre:	It's like I said before. The outside of B is a rectangle just like A [*traces around B as shown in figure 1.16*]. The bottom side is 8 just like on A. Eight equals 2 plus 4 plus 2 [*circling 2, 4, and 2*].

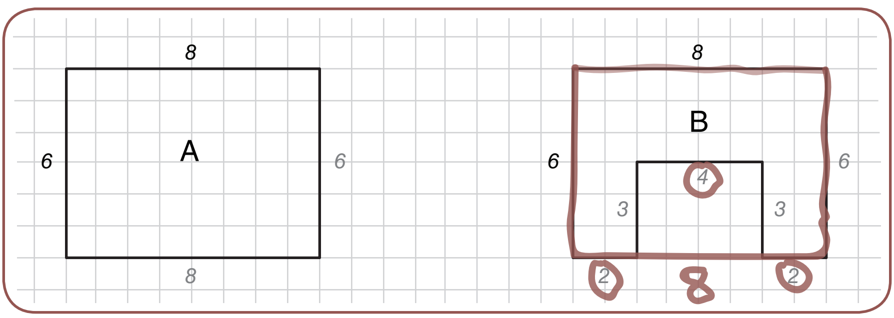


Fig. 1.16. Correctly using the properties of rectangles to reason about perimeter

Note how the teacher not only had the students solve the problem on the grid but also had them reflect on their measurements in a way that promoted using properties of rectangles in their reasoning.

Making Measurement Reasoning More Rigorous

As students move into seventh and eighth grade, they should gradually increase the sophistication of their arguments and justifications. Indeed, MP3 includes constructing viable arguments; understanding and using assumptions, definitions, and previously established results; building logical progressions of statements to explore the truth of conjectures; analyzing situations; justifying conclusions; and distinguishing correct logic or reasoning from that which is flawed. As students work on the missing-length tasks, we must help them advance gradually to formalizing the arguments previously presented. Connecting their more intuitive arguments to the more formal arguments that follow is a critical form of sense making and reasoning for students.

Explicit Use of Properties of Rectangles to Reason About Length Measurements

In the following example, a student uses more sophisticated reasoning by explicitly referring to and using properties of rectangles.

Sophia: Because all the angles are right angles, shape A is a rectangle, and so is the outside of shape B [*draws the dashed segment in B* (fig. 1.17)]. Because opposite sides in rectangles are equal, the right sides of A and B are both 6 [*writes 6 next to both sides*].

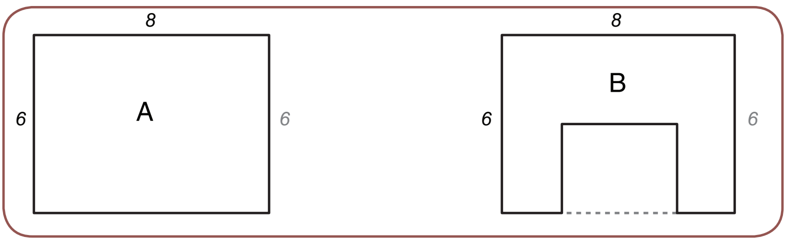


Fig. 1.17. Sophia's first diagram

Sophia: We can also divide B into three rectangles [*draws dashed line segments on B* (see fig. 1.18)]. Because rectangles have opposite sides equal, this side equals this side [*pointing to* a *and* a], this side equals this side [*pointing to* b *and* b], and this side equals this side [*pointing to c and c*]. So the total length of the three bottom segments on B equals the length of the top of B, 8 [*writes 8 under the bottoms of A and B*]. The perimeter of A is 8 + 8 + 6 + 6 = 28. The perimeter of B is 8 + 8 + 6 + 6 plus these two segments [*points to* d *and* d]. So B's perimeter has to be bigger than A's.

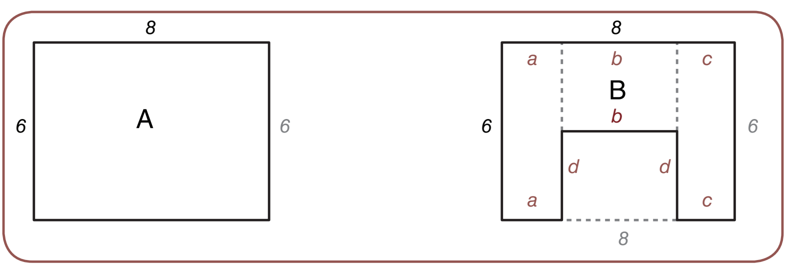


Fig. 1.18. Sophia's second diagram

Using Algebra

Students who have studied algebra often use it to express relationships between and among the sides of shapes to determine their perimeters. This is another step up in levels of sophistication of measurement reasoning and promotes SMP 2.

David: [*Labeling sides on A* (fig. 1.19)] Because this is a rectangle, I know these opposite sides are 6 and 8. On B, this side [right] is 6, and I'll label the other sides with letters because I don't know what their lengths are.

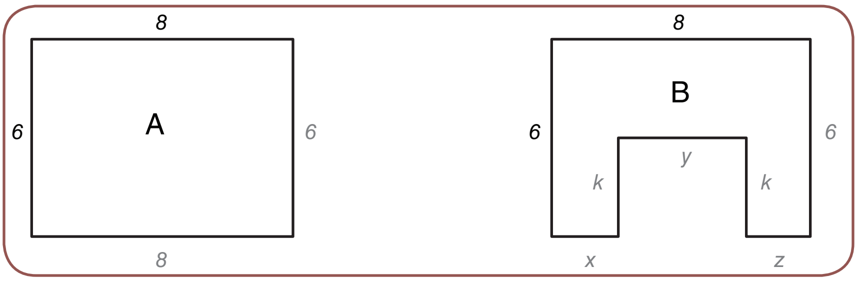


Fig. 1.19. David's diagram

David: OK. Because the outside of B is really a rectangle and this indented part is a rectangle, I know $x + y + z = 8$ [*writes*].

$$x + y + z = 8$$

David: And the other two sides are equal, so I'll call them k. So I can write what the perimeter of B is [*writes*].

$$\text{Perimeter of B} = 6 + 8 + 6 + (x + y + z) + k + k$$

David: I know that $x + y + z = 8$. So I can substitute 8 for it, and I get 28 plus $2k$.

$$\text{Perimeter of B} = 6 + 8 + 6 + (8) + 2k = 28 + 2k$$

David: The perimeter of A is 28. So I can substitute again.

$$\text{Perimeter of B} = \text{Perimeter of A} + 2k$$

David: So the perimeter of shape B is $2k$ greater than the perimeter of shape A.

Using the Triangle Inequality and the Pythagorean theorem

Middle school students should have already discovered the triangle inequality: the sum of the lengths of any two sides of a triangle is greater than the length of the third side. This principle can be used in reasoning about problems 4 and 5.

Cammi: [*Working on problem 4*] B has less perimeter. Because if we draw these two sides from A on B [*draws dotted segments* (fig. 1.20)], we get a right triangle. And the length of the hypotenuse is always less than when you add the two other sides together.

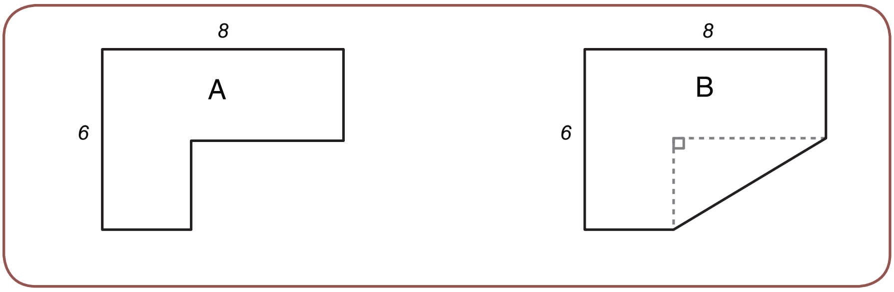


Fig. 1.20. Cammi's diagram

Keep in mind, however, that as previously discussed, many students will not be able to make sense of this argument unless they can reason about actual numbers.

Teacher: Does that make sense to everyone?

Olivia: Sort of. But I'm not sure I really get it.

Teacher: How can we check out this reasoning to be sure it is correct?

Tia: It helps me when we draw the problems on graph paper.

Teacher: Ok, let's do that. [*The teacher, who has anticipated this student's need, has a copy of problem 4 drawn on graph paper, which she shows on a document projector.*] What should we do next?

Tia: Label the side lengths [*goes to the projector table and labels the lengths as shown in figure 1.21*]. You can just count on sides.

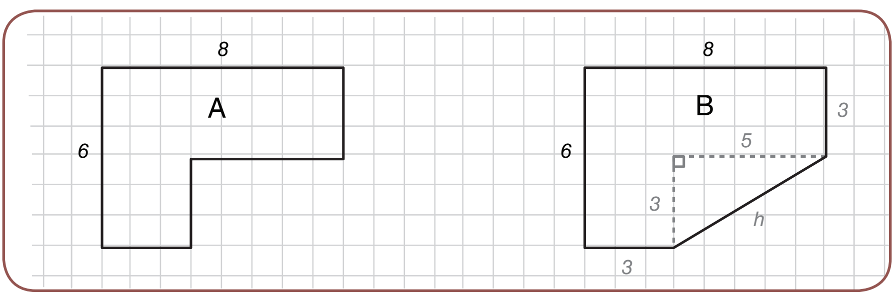


Fig. 1.21. Problem 4 drawn on graph paper

Teacher: Now what…how do we find this tilted side?

Cammi: Use the Pythagorean theorem [writes on projected sheet].

$$h=\sqrt{3^2+5^2}=\sqrt{34}=5\ 83$$

Cammi: And 5.83 is less than 3 + 5 = 8.

Teacher: So what is the perimeter of B?

Tia: 5.83 + 6 + 8 + 3 + 3 = 5.83 + 20 = 25.83.

Teacher: And what is the perimeter of A?

Tia: 6 + 8 + 6 + 8 = 28. So Andre was right; shape B has lower perimeter.

A similar kind of triangle-inequality reasoning can be used to conclude that shape B has a perimeter greater than that of shape A in problem 5 (see William's reasoning below). Of course, students might make this problem a little more concrete by using the Pythagorean theorem to find the actual lengths of *a* and *b* (see fig. 1.22).

$$a = \sqrt{2^2 + 4^2} = \sqrt{20} = 4.47 = b$$

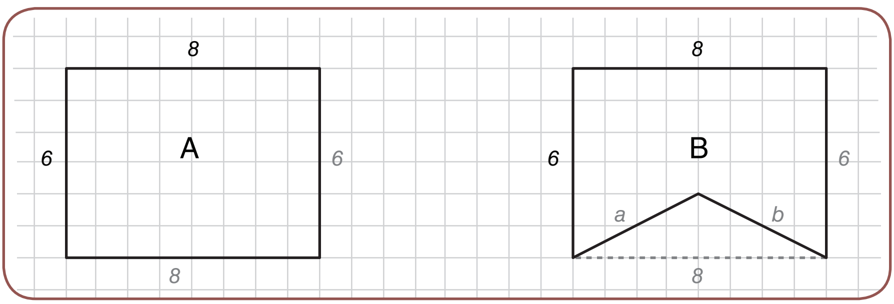


Fig. 1.22. Using the Pythagorean theorem to find the actual lengths of *a* and *b*

William: [*writes*]

Perimeter of A = 6 + 6 + 8 + 8 = 20 + 8

Perimeter of B = 6 + 6 + 8 + (*a* + *b*) = 20 + (*a* + *b*)

$a + b > 8$

Perimeter of B > Perimeter of A

Teacher: Can we use the Pythagorean theorem to actually find *a* + *b*?

Mulan: Yeah, I got that both *a* and *b* equal 4.47. Add *a* and *b*, and you get 8.94, which is more than 8.

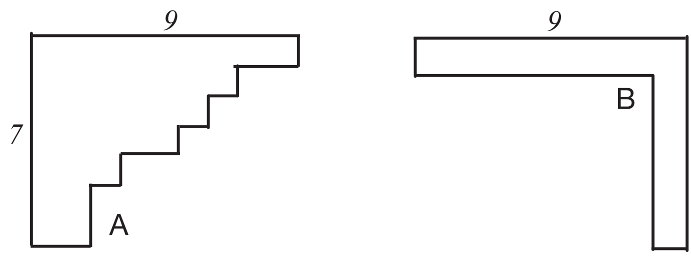
Teacher: What's the perimeter of B?

Mulan: I got 28.94. That's more than 28.

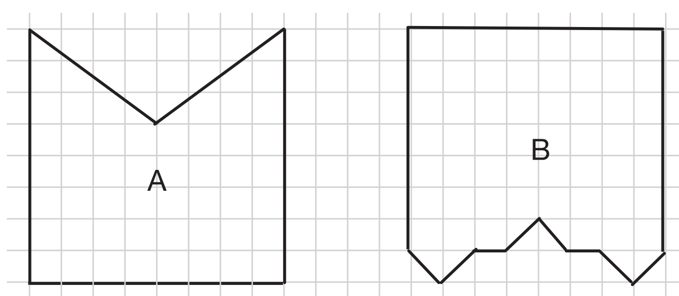
Helping Students Extend and Generalize Their Reasoning

Part of deep mathematical sense making is understanding ideas well enough to apply them appropriately to problems that are different from the problems in which the ideas were developed. Problem 6 (below) looks a bit different from problems 1 to 3 (see fig. 1.2), but can be solved using both measurement and non-measurement reasoning as previously described for these problems. In problem 7 (below), students must actually find the lengths of all the segments by using a grid to find the lengths of vertical and horizontal segments and the Pythagorean theorem to find the lengths of oblique segments (A has the greater perimeter). Finally, students should be able to clearly explain why the reasoning they used to conclude that the shapes in problem 6 have the same perimeter does not work for problem 8 (next page), in which the perimeter of B is greater than that of A. In this case, the sum of the vertical segments on the right side of B is greater than 7, because some segment lengths are "repeated."

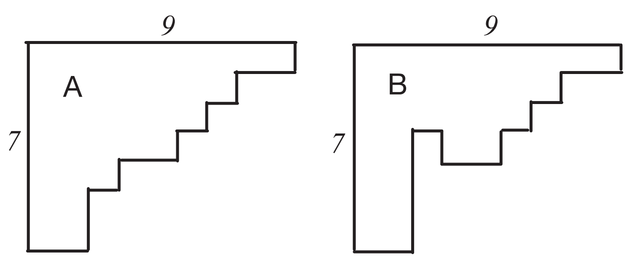
In the shapes below, angles that look like right angles are right angles. In each problem, which shape has the greater perimeter, or are the perimeters equal?



Problem 6.



Problem 7.



Problem 8.

Summary of Student Sense Making and Reasoning About Missing-Side Perimeter Problems

Having students work on and discuss the initial problem set (see fig. 1.2) of missing-side perimeter problems provides students with a great opportunity to think about and discuss mathematical reasoning and sense making. Students in grades 6–8 use a wide variety of reasoning on these problems: invalid and valid, intuitive and formal, unsophisticated and sophisticated. As students work on these problems and participate in the resulting class discussions, there will be numerous opportunities for students to test, evaluate, and improve their mathematical reasoning and sense making. These problems can also encourage and support students in their transition to more formal and rigorous reasoning (algebra, logical deductions, etc.), a transition that is difficult for most students but critical for their success in high school mathematics and beyond. A key element in supporting the development of students' reasoning about these problems is having them reason about perimeters in somewhat more abstract settings (the original problems in fig. 1.2) while providing them with concrete means to test the validity of their reasoning (e.g., with straws or graph paper).

Standards for Mathematical Practice and Process Standards in Sense-Making Episodes

To relate our discussion of students' reasoning and sense making about the concept of length to the CCSSM Standards for Mathematical Practice and NCTM's Process Standards, it is useful to point out how the episodes on the perimeter problems are related to these practices and processes.

In these perimeter problems, students used a number of mathematical practices. They made their own personal sense of the ideas and persevered in solving the problems even though they were nonroutine (SMP 1a, b, i). Most of the students reasoned quantitatively as they made sense of measurement quantities and their interrelationships, and created mental representations of the problem situations; others reasoned abstractly as they used properties of shapes and algebraic representations (SMP 2a, b, d). The students constructed arguments and justifications that were viable to them, with students at the higher levels of reasoning constructing mathematically valid although not completely formal arguments (SMP 3). Students who used properties of rectangles instead of estimation to infer the lengths of shape B in problem 2 strived for precision and made use of geometric structure in their reasoning (SMP 6, SMP 7). Students also used mathematical processes described in the *Principles and Standards for School Mathematics* Process Standards. They solved problems using a variety of strategies and seemed to construct new ideas as they solved the problems (PS 1a, b). Meanwhile teachers helped students check their reasoning so that they might improve it (PS 2). Students communicated their reasoning clearly (PS 3). They interconnected concepts and reasoning related to perimeter, properties of rectangles, length measurement, and, for some, algebra (PS 4). Finally, most of the students used the shape diagrams, annotating and drawing on them, to support their quantitative reasoning (PS 5).

Concluding Remarks on Mathematical Reasoning and Sense Making

To use mathematics to make sense of the world, students must first make sense of mathematics. To make sense of mathematics, students must transition from the initial intuitive, informal reasoning they develop while interacting with the world to precise reasoning that uses formal mathematical concepts, procedures, and symbols. The key to helping students make this transition is providing appropriate instructional tasks that precisely target those concepts and ways of reasoning that students are currently ready to acquire. And the key to providing this support is understanding research-based descriptions of the development of students' increasingly more sophisticated conceptualizations and reasoning about particular mathematical concepts. Understanding the mathematical thinking of students is critical for selecting and creating instructional tasks, asking appropriate questions of students, guiding classroom discussions, adapting instruction to students' needs, understanding students' reasoning, assessing students' learning progress, and diagnosing and remediating students' learning difficulties. This chapter, as well as all the other chapters in this book, uses research on student learning to help teachers monitor, understand, and guide the development of students' reasoning and sense making about core ideas in elementary school mathematics.

Endnote

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