Preface

Focus in High School Mathematics: Reasoning and Sense Making (NCTM 2009) captures the direction for high school mathematics for students in the twenty-first century:

> Reasoning and sense making should occur in every mathematics classroom every day. In such an environment, teachers and students ask and answer such questions as "What's going on here?" and "Why do you think that?" Addressing reasoning and sense making does not need to be an extra burden for teachers struggling with students who are having a difficult time just learning the procedures. On the contrary, the structure that reasoning brings forms a vital support for understanding and continued learning. Currently, many students have difficulty because they find mathematics meaningless…. With purposeful attention and planning, teachers can hold all students in every high school mathematics classroom accountable for personally engaging in reasoning and sense making, and thus lead students to experience reasoning for themselves rather than merely observe it. (NCTM 2009, pp. 5–6)

This new publication urges a refocusing of the high school mathematics curriculum on reasoning and sense making, building on the guidelines for teaching and learning mathematics advocated by NCTM in *Principles and Standards for School Mathematics* (NCTM 2000). *Focus in High School Mathematics: Reasoning and Sense Making* makes the case that reasoning and sense making must reside at the core of all mathematics learning and instruction, at all grades. Moving forward from *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics* (NCTM 2006), *Focus in High School Mathematics: Reasoning and Sense Making* also addresses the need for the continuation of a coherent and well-articulated mathematics curriculum at the high school level.

The underlying principles of *Focus in High School Mathematics: Reasoning and Sense Making* are "reasoning habits" that should develop across the curriculum, along with "key elements" organized around five content strands. The book provides a group of examples that illustrate how these principles might play out in the classroom. Historically, NCTM has provided supplementary materials to accompany major publications that present official positions of the Council (e.g., the Teaching with Curriculum Focal Points series for *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics,* the Navigations Series for *Principles and Standards for School Mathematics*, the Addenda Series for *Curriculum and Evaluation Standards for School Mathematics* [NCTM 1989]). In keeping with this tradition, a series of supplementary books, Focus in High School Mathematics, provides additional guidance for ensuring that reasoning and sense making are part of the mathematics experiences of all high school students every day.

This series is intended for secondary mathematics teachers, curriculum specialists, mathematics supervisors, district administrators, and mathematics teacher educators. *Focus in High School Mathematics: Reasoning and Sense Making* underscores the critical role of the Process Standards outlined in *Principles and Standards* and provides a foundation for achieving the principal goals for the mathematical experiences of all secondary school students. Each volume in the Focus in High School Mathematics series presents detailed examples of worthwhile mathematical tasks, along with follow-up discussion. The examples and discussions are intended to help classroom teachers understand what it means to promote sense making and to find ways to increase it in their classrooms. The material could also be used as classroom cases in professional development. In addition, supervisors, curriculum specialists, and administrators might use the examples and discussions to catalyze conversations about shifts in the high school mathematics curriculum to bring them into alignment with the goals of *Focus in High School Mathematics: Reasoning and Sense Making*.

Although the books in the series focus on a particular content strand from *Principles and Standards* (e.g., geometry and measurement, algebra, statistics and probability), they are not intended to outline a curriculum for a particular content area. In fact, many of the examples in the books point to potential connections across content areas.

The authors of the present volume, *Focus in High School Mathematics: Reasoning and Sense Making in Algebra*, would like to acknowledge the following individuals for their critical review of drafts and technical help on the project: Anna Baccaglini-Frank, Bowen Kerins, Charlene Newton, and Kevin Waterman. Valuable advice also came from Dick Askey, Gail Burrill, John Carter, Roger Howe, Bill McCallum, and Glenn Stevens. Susan Forster provided useful assistance as a liaison from the writing group for *Focus in High School Mathematics: Reasoning and Sense Making.* In addition, the authors are grateful to all the teachers who offered opinions and reactions to the manuscript during the 2009 PROMYS for Teachers seminar session at Education Development Center, Newton, Massachusetts. The collective feedback from all of these individuals has made the final document stronger.

General Introduction to the Focus in High School Mathematics Series

Focus in High School Mathematics: Reasoning and Sense Making addresses the need for reasoning to play a larger role in high school mathematics:

> A focus on reasoning and sense making, when developed in the context of strong content, will ensure that students can accurately carry out mathematical procedures, understand why those procedures work, and know how they might be used and their results interpreted.… Such a focus on reasoning and sense making will produce citizens who make informed and reasoned decisions, including quantitatively sophisticated choices about their personal finances, about which public policies deserve their support, and about which insurance or health plans to select. It will also produce workers who can satisfy the increased mathematical needs in professional areas ranging from health care to small business to digital technology. (NCTM 2009, p. 3)

Focus in High School Mathematics: Reasoning and Sense Making provides an outline for how reasoning and sense making might play out in core topic areas of the high school curriculum: numbers and measurement, algebra, geometry, and statistics and probability. The topics and examples contained in this publication and the supporting volumes do not represent an exhaustive list of topics that should be covered in any particular course or curriculum. The examples are meant to illustrate reasoning habits that all students at a variety of grade levels should know by the time they complete high school. As such, they provide multiple entry points for the students and, where appropriate, emphasize connections between several areas of mathematics. The discussions point to key teaching strategies that foster the development of reasoning and sense making. The strategies should be viewed as general and not tied to the particular context or task.

Most teachers and teacher educators would probably nod in agreement that reasoning and sense making are important to consider in the mathematical experiences of their students. However, the purpose of *Focus in High School Mathematics: Reasoning and Sense Making* and the Focus in High School Mathematics series is to highlight these as major goals of the study of secondary mathematics. Although reasoning and sense making may have been a part of secondary mathematics teaching and learning in the past, they are certainly worthy of being discussed in greater depth, and becoming a primary focus of our secondary mathematics teaching, in classrooms today. Therefore, with this shift in emphasis, it is important for NCTM to provide thoughtful examples of worthwhile tasks that can be pursued at a number of levels.

The Role of Teaching

Often, high school mathematics teaching in the United States and Canada has been characterized by two main classroom activities; teachers share information, such as definitions of new terms and procedures for solving mathematics problems, and then students practice and perhaps discuss results of those procedures. Although these activities are important, such practices can lead to learning that is devoid of reasoning and sense making. By contrast, NCTM strongly supports a view of mathematics teaching and learning that focuses on reasoning, as described in *Mathematics Teaching Today* (NCTM 2007): "Teachers … must shift their perspectives about teaching from that of a process of delivering information to that of a process of facilitating students' sense making about mathematics" (p. 5).

A shift of perspective to one that views reasoning and sense making as primary goals for students' learning of mathematics will lead to a shift in choices made by the classroom teacher. For example, the teacher will choose tasks that allow students to see the need for sense making and

provide opportunities for them to demonstrate their reasoning processes. Such tasks should also help students build on their informal knowledge of mathematics and see the logical connections with other areas of mathematics that they have learned. This shift may require changes in the structure of the classroom setting so that students are challenged and encouraged to explore mathematical situations both collaboratively and independently. Students should be expected to make conjectures and develop arguments to support them, connecting earlier knowledge with newly acquired knowledge.

As students are investigating and shaping ideas, they should have opportunities to interact directly and openly with one another and with the teacher. More details about the teacher's and students' roles in the classroom can be found in chapter 1, "Standards for Teaching and Learning," of *Mathematics Teaching Today*, which includes Standards describing characteristics of *worthwhile mathematical tasks* (Standard 3), components of a productive classroom *learning environment* (Standard 4), and suggestions for orchestrating mathematical *discourse* (Standard 5). The Focus in High School Mathematics series provides tasks, examples, and classroom vignettes that illustrate how a teacher might choose tasks and orchestrate classroom discourse to capitalize on student reasoning and promote sense making.

The Role of Technology

Technology is integrated into the examples in these books in a strategic manner to enrich opportunities for students' reasoning and sense making. The power of recent technological tools (e.g., computer algebra systems, dynamic geometry software, and dynamic data representation tools) to enhance reasoning and sense making in mathematics is so great that it would be remiss to omit them from these volumes.

Increasingly, technology is an integral part of society and the research that is conducted in the majority of mathematics-related fields. We support the philosophy of *Focus in High School Mathematics: Reasoning and Sense Making* that "students can be challenged to take responsibility for deciding which tool might be useful in a given situation when they are allowed to choose from a menu of mathematical tools that includes technology. Students who have regular opportunities to discuss and reflect on how a technological tool is used effectively will be less apt to use technology as a crutch" (p. 14). The Focus in High School Mathematics series provides examples that show students using technology to reduce computational overhead, but the books also illustrate the use of technology in experimenting with mathematical objects and modeling mathematical structures.

The Format of the Focus in High School Mathematics Series

Focus in High School Mathematics: Reasoning and Sense Making underscores the need to refocus the high school mathematics curriculum on reasoning and sense making. Companion books provide further insights into how these ways of thinking might develop in three major areas of content in high school mathematics:

- *Focus in High School Mathematics: Reasoning and Sense Making in Algebra*
- *Focus in High School Mathematics: Reasoning and Sense Making in Geometry*
- *Focus in High School Mathematics: Reasoning and Sense Making in Statistics and Probability*

The strand on reasoning and sense making with numbers and measurement discussed in *Focus in High School Mathematics: Reasoning and Sense Making* receives primary attention in *Focus in High School Mathematics: Reasoning and Sense Making in Geometry,* but aspects of this strand are also addressed in the other two content books.

Reasoning Habits

To detail what mathematical reasoning and sense making should look like across the high school curriculum, *Focus in High School Mathematics: Reasoning and Sense Making* provides a list of "reasoning habits." The intent is not to present a new list of topics to be added to the high school curriculum: "Approaching the list as a new set of topics to be taught in an already crowded curriculum is not likely to have the desired effect. Instead, attention to reasoning habits needs to be integrated within the curriculum to ensure that students both understand and can use what they are taught" (p. 9). The reasoning habits are described and illustrated in the examples throughout the companion books in the Focus in High School Mathematics series.

Key Elements

Focus in High School Mathematics: Reasoning and Sense Making identifies "key elements" for each of the strands. These key elements are intended to provide "a lens through which to view the potential of high school programs for promoting mathematical reasoning and sense making" (p. 18).

Content Expectations

As *Focus in High School Mathematics: Reasoning and Sense Making* suggests, readers wishing for more detailed content recommendations should refer to chapter 7, "Standards for Grades 9–12," of *Principles and Standards for School Mathematics* (NCTM 2000). However, for the readers' convenience, each companion volume shows the grades 9–12 expectations of the relevant Standard (Algebra, Geometry, or Data Analysis and Probability) in the appendix, along with the grades 9–12 expectations for the Number and Operations and the Measurement Standards, which are addressed by all three volumes.

Introduction to *Focus in High School Mathematics: Reasoning and Sense Making in Algebra*

Algebra and algebraic reasoning remain at the core of the high school mathematics curriculum. Algebra's dominance in the school curriculum is related to the importance of algebra in more advanced areas of mathematics, the usefulness of algebraic reasoning in all walks of life, and the role of algebra as a tool for the mathematical modeling required in many technological and scientific fields. Algebra's vital role in the school curriculum is reflected in many recent policy documents (e.g., NCTM 2000; Mathematical Association of America 2007; National Mathematics Advisory Panel 2008), as well as in the often-heated debate over whether algebra should be a required course for all eighth-grade students. The executive order that established the National Mathematics Advisory Panel places algebra at the top of the list of topics for the panel to consider, identifying as item (*a*) "the critical skills and skill progression for students to acquire competence in algebra and readiness for higher levels of mathematics" (National Mathematics Advisory Panel 2008, p. 7). The Common Core State Standards for K–12 Mathematics, developed under the leadership of the National Governors Association Center for Best Practices and the Council of Chief State School Officers, provide a strong foundation for algebra, with a focus in early grades on number and operations and in middle grades on ratio and proportional reasoning, supporting high school algebra standards that include reasoning and sense making in addition to procedural fluency. *Principles and Standards for School Mathematics*, published by the National Council of Teachers of Mathematics in 2000, and the recent report *Algebra: Gateway to a Technological Future,* released by the Mathematical Association of America in 2007, outline key standards for algebra and the connections between algebra and other areas of mathematics.

Although formal algebra coursework begins in grade 9 for most students, many recent reports (NCTM 2000; National Mathematics Advisory Panel 2008; Greenes 2008) stress the importance of laying the foundation for algebra and algebraic thinking as early as kindergarten. The building blocks for algebra include fluency with whole numbers and fractions, experience in analyzing properties of patterns and shapes and using proportional reasoning, and several of the algebraic habits that we describe in this book. Many of the early experiences involve the use of manipulatives and concrete models. It is then the job of the teachers of beginning algebra to "proceed carefully, ensuring that their students have some images that will give meaning and coherence while weaning them away from concrete models and examples so that they can exploit the flexibility and breadth of vision that formal algebraic procedures bring" (Barbeau and Brown 1997). This book provides examples of how teachers can build on students' concrete experiences and help them to make connections between these experiences and the more formal aspects of algebra, which are the focus of higher-level mathematics.

Organizing Frameworks for School Algebra

Usiskin (1988, pp. 11–16) describes several conceptions of algebra that he says have influenced the development of the school algebra curriculum and give meaning to purposes for teaching school algebra:

> **Conception 1:** Algebra as generalized arithmetic—variables are pattern generalizers, and key instructions to students are *translate* and *generalize*.
>
> **Conception 2:** Algebra as a study of procedures for solving certain kinds of problems—variables are unknowns or constants, and key instructions are *simplify* and *solve*.

Conception 3: Algebra as the study of relationships among quantities—variables are arguments or parameters, and key instructions are contained in questions such as "What happens to the value of $1/x$ as x gets larger and larger?"

Conception 4: Algebra as the study of structures—variables are marks on paper, and key instructions are contained in commands such as "Factor the polynomial $3x^2 + 4ax - 132a^2$."

Other mathematicians and mathematics educators have outlined similar conceptions of school algebra (Bass 1998; Mathematical Sciences Research Institute 2008; Dossey 1998; Fey and Good 1985; Thompson 2008). In particular, Dossey (1998) describes the different conceptions or models for school as structural, focusing on functions and relations, building models, and linguistic (p.18). He argues that aspects of each conception are necessary in practice and that none of the approaches can survive on its own: "What we really have to do is to think of how to merge them to support a coherent program of the study of algebra with four main goals" (p. 19). The Standard for algebra outlined in *Principles and Standards* supports this view. The elements of the Algebra Standard that cut across pre-K–grade 12 echo these conceptions:

> Instructional programs from prekindergarten through grade 12 should enable all students to—
>
> • Understand patterns, relations, and functions
>
> • Represent and analyze mathematical situations and structures using algebraic symbols
>
> • Use mathematical models to represent and understand quantitative relationships
>
> • Analyze change in various contexts. (NCTM 2000, p. 37).

We believe that an emphasis on reasoning and sense making is essential to students' success in algebra, no matter what conception or approach instruction takes. As *Navigating through Reasoning and Proof in Grades 9–12*, a volume in the recent Navigations Series, states, "instruction that emphasizes reasoning can transform algebra from a procedural and formula-based study, focused on the ability to memorize, apply, and combine processes, to a study that calls for creative and original thinking" (Burke et al. 2008, p. 18). Algebra continues to be an important tool—one that is required in many disciplines and careers—and an emphasis on reasoning and sense making will help students appreciate its value and understand why it continues to be an integral part of the high school curriculum.

Key Elements of Algebraic Reasoning

Focus in High School Mathematics: Reasoning and Sense Making (NCTM 2009) discusses the key elements for reasoning and sense making with algebra in two parts: reasoning and sense making with algebraic symbols, and reasoning and sense making with functions. Key elements of reasoning and sense making with algebraic symbols include:

- *Meaningful use of symbols.* Choosing variables and constructing expressions and equations in context; interpreting the form of expressions and equations; manipulating expressions so that interesting interpretations can be made.
- *Mindful manipulation.* Connecting manipulation with the laws of arithmetic; anticipating the results of manipulations; choosing procedures purposefully in context; picturing calculations mentally.
- *Reasoned solving*. Seeing solution steps as logical deductions about equality; interpreting solutions in context.

- *Connecting algebra with geometry.* Representing geometric situations algebraically and algebraic situations geometrically; using connections in solving problems.
- *Linking expressions and functions.* Using multiple algebraic representations to understand functions; working with function notation. (NCTM 2009, p. 31)

Key elements of reasoning and sense making with functions include the following:

- *Using multiple representations of functions.* Representing functions in various ways, including tabular, graphic, symbolic (explicit and recursive), visual, and verbal; making decisions about which representations are most helpful in problem-solving circumstances; and moving flexibly among those representations.
- *Modeling by using families of functions.* Working to develop a reasonable mathematical model for a particular contextual situation by applying knowledge of the characteristic behaviors of different families of functions.
- *Analyzing the effects of parameters.* Using a general representation of a function in a given family (e.g., the vertex form of a quadratic, $f(x) = a(x - h)^2 + k$), to analyze the effects of varying coefficients or other parameters; converting between different forms of functions (e.g., the standard form of a quadratic and its factored form) according to the requirements of the problem-solving situation (e.g., finding the vertex of a quadratic or finding its zeros). (NCTM 2009, p. 41)

Each chapter in this book deals with one or more of these key elements of algebraic reasoning in greater detail. In addition to the key elements, certain habits of mind are particularly evident in algebraic reasoning.

Specific Habits of Mind in Algebraic Reasoning

In the examples developed in this book, we have noted several algebraic sub-themes within the reasoning habits presented in *Focus in High School Mathematics: Reasoning and Sense Making.* Table 0.1 shows the habits of mind that we believe are of particular importance in algebraic reasoning.

Table 0.1
Habits of Mind in Reasoning and Sense Making in Algebra

Analyzing a problem, *for example,*

– *defining relevant variables and conditions* carefully, including units if appropriate;
– *seeking patterns and relationships;*
– *looking for hidden structure* (for example, finding equivalent forms of expressions that reveal different aspects of a problem).

Implementing a strategy, *for example,*

– *making purposeful use of procedures;*
– *monitoring progress toward a solution,* including reviewing a chosen strategy and other possible strategies generated by oneself or others.

(Continued on the next page)

Table 0.1
Habits of Mind in Reasoning and Sense Making in Algebra—Continued

Reflecting on a solution to a problem, *for example,*

– *interpreting a solution* and how it answers the problem, including making decisions under uncertain conditions;

– *considering the reasonableness of a solution*, including whether any numbers are reported to an unreasonable degree of accuracy;

– *generalizing a solution* to a broader class of problems and looking for connections to other problems.

Overview of This Book

Each of the chapters in this book provides examples of how reasoning and sense making might play out in the high school algebra curriculum. Chapter 1 looks at the ways in which algebra and geometry "talk to each other" about area by exploring three specific contexts: finding area formulas for polygons, maximizing area and connecting it to the arithmetic-geometric mean inequality, and Heron's formula for the area of triangles. Chapter 2 explores the habit of seeking and expressing regularity in repeated calculations in three contexts: word problems, fitting lines to data, and monthly loan payments. Chapter 3 presents ideas about how students might make sense of the algebra of formal expressions in the high school curriculum by exploring the process of factoring and completing the square, combinatorial phenomena, and patterns in the factors of a sequence of polynomials. The epilogue surveys the journey that the readers have made, inviting them to reflect on the examples and classroom vignettes contained in the book and to consider how to make reasoning and sense making integral parts of the high school curriculum and classroom practice.

The examples in each chapter illustrate a range of content and grade levels across the 9–12 spectrum and reflect essential concepts in algebra that the authors believe students should have experience with before graduating from high school. The concepts that are the focus of the examples could arise in algebra courses or other courses, such as geometry or probability and statistics.

Again, readers should note that this book is not an algebra curriculum but instead contains key algebraic ideas that illustrate reasoning and sense making that should be part of any mathematics curriculum in grades 9–12. The examples do not in any way represent an exhaustive list of topics. Readers can refer to *Principles and Standards for School Mathematics* (NCTM 2000) and the references at the end of the book for more detail on recommended goals for school algebra and other related resources.

Chapter 1

Algebra and Geometry

Students can develop ideas about algebra and geometry together throughout the high school curriculum, enriching their study of each. Geometric interpretations of algebraic identities can help them give meaning to and make sense of algebraic symbols and calculations. Conversely, casting geometric phenomena in algebraic terms can give them a way to reason about the geometry, leading to interesting and nontrivial geometric conjectures and their proofs.

Several topics in the high school curriculum invite students to make connections very naturally between algebra and geometry—analytic geometry, vector methods, geometric probability, and conic sections are examples. This chapter looks at the ways in which algebra and geometry "talk to each other" about *area* in the following specific contexts:

1. Finding area formulas for polygons
2. Maximizing area and connecting this work to the arithmetic-geometric mean inequality
3. Working with Heron's formula for the area of a triangle

Activities in these contexts can highlight many of the reasoning habits and key elements of algebra outlined in *Focus in High School Mathematics: Reasoning and Sense Making* (NCTM 2009).

Area Formulas

Most students come to high school knowing that the formula for the area of a rectangle is "area equals base times height." Many teachers revisit the formula briefly to make sure that students understand why it makes sense. One way to do this is by generic example. Consider a rectangle that is 3 units by 5 units and can be partitioned into 15 squares, each with area 1 square unit (see fig. 1.1).

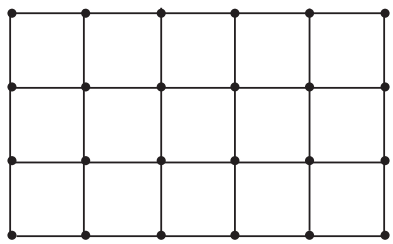

Fig. 1.1. A rectangle that is 3 units by 5 units, partitioned into unit squares