Number and Measurement

Introduction

"Number and measurement, which receive substantial attention in kindergarten through grade 8, are foundational for high school mathematics; without reasoning skills in these areas, students will be limited in their reasoning in other areas of mathematics" (NCTM 2009, p. 21).

This chapter presents eight activities that relate to the content area of number and measurement. The activities range in level of difficulty, topic, and context. For example, one activity (Hall 2008) focuses on finding the best value for a box of popcorn and would be appropriate for students early in high school. Another (Herman, Milou, and Schiffman 2004) asks students to consider fractions and decimals in different bases and would be more engaging for students in the later years of high school. The table below presents this chapter's number and measurement activities.

We chose the activities in this chapter because they exemplify activities that illustrate four essential elements of number and measurement and can help develop mathematical habits of mind. *Focus in High School Mathematics* (NCTM 2009) suggests four important elements of reasoning and sense making within number and measurement:

Number and Measurement Activities

Author and title	Mathematical topic(s)	Context(s)	Materials
Albrecht (2001), "The Volume of a Pyramid: Low-Tech and High-Tech Approaches"	Volume, volume of a pyramid	Beginning with the volume of pyramids built with cubes and moving to the volume of pyramids	Cubic blocks; spreadsheet software (high-tech); hollow pyramids (low-tech); hollow prism (low-tech); water, sand, rice, or small pasta (low-tech); student activity sheets
Çağlayan (2006), "Visualizing Summation Formulas"	Summation, perfect squares	Relating the sum of consecutive odd integers to perfect squares	1-inch grid paper, 1-inch tiles, student activity sheets
Hansen and Lewis (2007), "Finding a Parking Spot for the Binomial Theorem"	Binomial theorem, Pascal's triangle	Finding a parking spot at a baseball game	Graphing calculator, student activity sheets
Herman, Milou, and Schiffman (2004), "Unit Fractions and Their 'Basimal' Representations: Exploring Patterns"	Decimal representations, working in bases other than 10	Looking at the period of repeating decimal representations in base 4 and base 10	Calculator, student activity sheets
Hill (2002), "Print-Shop Paper Cutting: Ratios in Algebra"	Ratios, remainders, scale drawings	Working in a print shop to minimize waste	5 × 7 inch paper, 17 × 22 inch paper, rulers, tape, staplers, student activity sheets
Olson (1991), "A Geometric Look at Greatest Common Divisor"	Greatest common divisor	Relating greatest common divisor to a geometric area model	Graph paper, scissors, student activity sheets
Slowbe (2007), "Pi Filling, Archimedes Style"	Area, π	Approximating π with polygons inscribed in a unit circle	Programmable calculator, student activity sheets
Willcutt (1973), "Paths on a Grid"	Pascal's triangle, permutations	Taxicab geometry	Student activity sheets

1. *Reasonableness of answers and measurements.* Judging whether a given answer or measurement has an appropriate order of magnitude and whether it is expressed in appropriate units
2. *Approximations and error.* Realizing that all real-world measurements are approximations and that unsuitably accurate values should not be used for real-world quantities; recognizing the role of error in subsequent computations with measurements
3. *Number systems.* Understanding number-system properties deeply; extending number system properties to algebraic situations
4. *Counting.* Recognizing when enumeration would be a productive approach to solving a problem and then using principles and techniques of counting to find a solution

Many of this chapter's eight articles address more than one key element, and all articles do so through engaging activities.

Hall (2008) addresses reasonableness of answers and measurements in the activity previously mentioned in which students compare the cost value of various brands of popcorn. A component of this key element is for students to judge whether an answer is given with appropriate units—an important aspect of this activity. "Students can improve their problem-solving skills if they understand that finding the unit of measure for the problem's answer is the *first* step" (Hall 2008, p. 609). Through participation in the activity, students take a close look at units related to the amount of popcorn produced per dollar. Wagner (2003) also addresses this element by affording students the opportunity to work with square roots to solve problems about areas and side lengths of squares.

Slowbe (2007) also uses geometric figures to bring the idea of approximation and error in measurement to the forefront in an activity in which students generate the digits of π by using calculations of areas of regular polygons inscribed in a unit circle. "With sufficiently large *n*, we can obtain decimal approximations that become arbitrarily close to the exact value of π" (Slowbe 2007, p. 485).

The key element of number systems is the focus of the activity by Herman, Milou, and Schiffman (2004), in which students conjecture about the relationships between unit fractions and their decimal representations in both base ten and base four. Finally, the counting element appears in the activities by Hansen and Lewis (2007) and by Willcutt (1973). In the activity by Hansen and Lewis, students count the number of paths to a parking space and relate it to Pascal's triangle. In Willcutt's activity, students extend their initial ideas about the number of paths from one point to another on a grid to consider three-dimensional space. This counting activity "presents an excellent model for the need to simplify unwieldy mathematical problems, to look for patterns, and to formulate generalizations based on the results of the data collected for the simple cases" (Willcutt 1973, p. 303).

You can also find the reasoning that *Focus in High School Mathematics* (NCTM 2009) recommends in all the activities in this chapter. The activity by Albrecht (2001) develops the reasoning habit of *analyzing a problem*. Students use blocks and spreadsheets to find patterns to help them write the formula for the volume of a pyramid. Using the spreadsheet was integral for the teacher in this case: "My geometry class had not used spreadsheets before, and the students enjoyed the experience of using the efficiency of technology to compare hundreds—and even thousands—of shapes with ease" (Albrecht 2001, p. 58). In Çağlayan's (2006) activity, students *implement a strategy* to find that "every perfect square is the sum of consecutive odd integers" (p. 70). *Seeking and using connections* appears in Olson's (1991) activity, in which students "examine an arithmetic concept, greatest common divisor, from a geometric standpoint" (p. 202). Finally, Hill (2002) invites students to revisit initial assumptions as part of *reflecting on a solution* in an activity set in a print shop in which they are looking for the most efficient ways to make cuts from a large piece of paper.

REFERENCE

National Council of Teachers of Mathematics (NCTM). *Curriculum and Evaluation Standards for School Mathematics*. Reston, Va.: NCTM, 1989.

———. *Focus in High School Mathematics: Reasoning and Sense Making*. Reston, Va.: NCTM, 2009.

The Volume of a Pyramid:
Low-Tech and High-Tech Approaches

Masha Albrecht

This lesson came about spontaneously during a geometry unit on volume. I had used the lesson shown here in **activity sheet 1,** in which students use cubic blocks to rediscover the formulas for volumes of right prisms, that is, $V = Bh$ and $V = lwh$. This lesson was a simple review for my tenth-grade class, and they completed it easily before the end of the period. With the wooden cubes still on their desks, most of them used the remaining time to build towers and other objects. I noticed that many students piled the cubes into bumpy pyramidal shapes. Because the next day's lesson involved studying the volume of pyramids, I wondered whether these bumpy shapes could be useful for discovering the volume of a real pyramid with smooth sides. Students could compare the volumes of these "pyramids of cubes" with the volumes of corresponding right prisms and perhaps discover the ratio 1/3 to obtain the formula for the volume of a pyramid, $V = (1/3)Bh$. As it turns out, the ratio of 1/3 does not become evident right away. To my students' delight, we found that using a spreadsheet is an excellent way to investigate this problem. My geometry classes had not used spreadsheets before, and the students enjoyed the experience of using the efficiency of technology to compare hundreds—and even thousands—of shapes with ease.

Prerequisites: Students with only very basic mathematical knowledge can benefit from this lesson. Students should have some skill at describing a pattern with an algebraic equation and some familiarity with a spreadsheet. However, I used this lesson with students who had no previous spreadsheet experience.

Grade levels: Although I originally used this lesson with a regular tenth-grade geometry class, the lesson is appropriate for students at different levels and with different abilities. A prealgebra class could do the low-tech part of the lesson, in which students find patterns by using blocks, but they would need help with the formulas for the spreadsheet. Eleventh-grade or twelfth-grade students with more advanced algebra skills could be left on their own to find the spreadsheet formulas and could be given the difficult challenge of finding the closed formula for the volume in the "pyramid of cubes" column on **activity sheet 2.** A calculus class could find the limit of the ratio column as n goes to infinity before they check this limit on the spreadsheet.

Materials: The entire lesson works well in a two-hour block or in two successive fifty-minute lessons, with the low-tech lesson in the first hour and the high-tech spreadsheet lesson in the second. Cubic blocks are needed for the low-tech lesson. Because approximately forty blocks are needed for each group of four students, large classes will need many blocks. If you do not have enough blocks, groups can share. Simple wooden blocks work best; plastic linking cubes do not work as well, because their extruding joints can get in the way when students build the pyramids.

Spreadsheet software is needed for the high-tech lesson. If you are using a separate computer lab, sign out the lab for the second hour of this activity.

For the low-tech extension lesson, the following additional materials are needed: a hollow pyramid and prism with congruent bases and heights, as well as water, sand, rice, or small pasta.

TEACHING SUGGESTIONS

Sheet 1: Using cubic blocks—volume of prisms

This activity sheet is elementary, and more advanced students can skip it. Have students work in groups, with one set of blocks per group. Often one student quickly sees the answers without needing manipulatives, but the other group members are too shy to admit that they need to build the shapes. Require that each group build most of the solids, even if students protest that this activity seems easy.

Sheet 2: Using cubic blocks—volume of pyramids

Students may initially have difficulty understanding what the "pyramids of cubes" look like. Make sure that they build the one with side length 3 correctly. After using the blocks to build a few of the shapes, students recognize the patterns and start filling in the table without using the blocks. Calculating decimal answers for the last column of ratios instead of leaving answers in fraction form helps students look for patterns. Have a whole-class discussion about questions 4, 5, and 6 after students have had a chance to answer these questions in smaller groups, but do not reveal the answers to these questions. Students discover the answers when they continue the table on the spreadsheet.

The last row of the table, where students generalize the results for side length n, is optional. On the spreadsheet, students do not need the difficult closed formula for the second column. They can instead use the recursive formula, which is easier and more intuitive. The solutions include more explanation.

Sheet 3: Using a spreadsheet—volume of pyramids

This activity sheet is designed for students who have some spreadsheet knowledge. Having one pair of students work at each computer is useful if at least one student in each pair knows how to use computers and spreadsheets. For students who have no experience with spreadsheets, you can use this activity sheet as the basis for a whole-class discussion while demonstrating the process on an overhead-projection device. Do not bother photocopying **activity sheet 3** for students who are familiar with spreadsheets. Instead ask them to continue the table from **activity sheet 2,** and give them verbal directions as needed.

SOLUTIONS

Sheet 1, part 1:

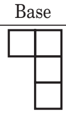
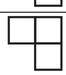
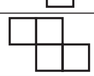
1)

Length	Width	Height	Volume
2 units	2 units	4 units	**16 cubic units**
1 unit	2 units	3 units	**6 cubic units**
2 units	2 units	**2 units**	8 cubic units
0.5 units	2 units	2 units	**2 cubic units**

2) $V = lwh$

Sheet 1, part 2

1)

Base	Area of the base	Height	Volume
	**4 square units**	2 units	**8 cubic units**
	**3 square units**	3 units	**9 cubic units**
	**4 square units**	**3 units**	12 cubic units
	**1 1/2 square units**	4 units	**6 cubic units**

2) $V = Bh$, where B is the area of the base.

Sheet 2:

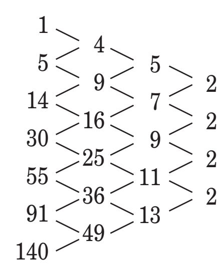
1) 8 cubic units;

2) 5 cubic units

3)

Length of Side	Volume of Cubic Solid	Volume of "Pyramid of Cubes"	Volume of "Pyramid" Divided by Volume of Cubic Solid
1	1	1	1
2	8	5	5/8 = 0.625
3	27	14	14/27 0.518
4	64	30	30/64 0.469
5	125	55	55/125 = 0.440
6	216	91	91/216 0.421
7	343	140	140/343 0.408
8	512	204	204/512 0.398
9	729	285	285/729 0.391
10	1000	385	385/1000 = 0.385
n (if you can)	n^3	Volume of previous n + n^2 or $(1/3)n^3 + (1/2)n^2 + (1/6)n = (n/6) \cdot (n+1)(2n+1)$	$[(n/6)(n+1) \cdot (2n+1)]/n^3$

Some students may be interested in a derivation of the closed formula in the last cell of the "pyramid of cubes" column. One way to derive the formula from the information in the chart is to begin by establishing that the formula is a cubic function. Students who are familiar with the method of finite differences can see that the relationship is a cubic because the differences become constant after three iterations.



When students know that the formula is cubic, they know that they can write it in the form $f(n) = an^3 + bn^2 + cn + d$, where n is the side length. Because the four constants a, b, c, and d are unknown, they can be treated as variables for now. Students can use the first four rows of the data in the table to see that $f(1) = 1$, $f(2) = 5$, $f(3) = 14$, and $f(4) = 30$. They can write the following system of equations:

$$a(1)^3 + b(1)^2 + c(1) + d = 1$$
$$a(2)^3 + b(2)^2 + c(2) + d = 5$$
$$a(3)^3 + b(3)^2 + c(3) + d = 14$$
$$a(4)^3 + b(4)^2 + c(4) + d = 30$$

That system is equivalent to the following system:

$$\begin{aligned} a + b + c + d &= 1 \\ 8a + 4b + 2c + d &= 5 \\ 27a + 9b + 3c + d &= 14 \\ 64a + 16b + 4c + d &= 30 \end{aligned}$$

However students solve this system, they find that $a = 1/3$, $b = 1/2$, $c = 1/6$, and $d = 0$, from which students can obtain the formula shown in the chart. The system is actually not very difficult to solve by hand using linear combinations.

4) Accept any reasonable answer at this point. Such answers might be similar to, "The ratio gets smaller as the shapes get bigger." In fact, the ratio in the last column approaches 1/3, or 0.33333….

5) Again, accept any reasonable answer. The ratio approaches 1/3 because the pyramid of cubes becomes a closer approximation of an actual smooth-sided pyramid. The size of the cubic blocks does not change as the pyramids become larger, so bumps created by the edges of the blocks are less significant as the "pyramid" becomes larger. If students are familiar with the notion of a limit, you can discuss how the limit of these larger and larger shapes is an infinitely large pyramid with completely smooth sides.

6) Although the ratio in the last column keeps getting smaller, it never reaches 0. Let students discuss this result, but do not reveal the answer.

Sheet 3:

1) and 2)

Length of Side	Volume of Cubic Solid	Volume of "Pyramid"	Volume of "Pyramid" Divided by Volume of Cubic Solid
1	1	1	
2	8	5	

3) Although the formulas are displayed here, numbers should show in the cells on the students' spreadsheets.

	A	B	C	D
1	Length of Side	Volume of Cubic Solid	Volume of "Pyramid"	Volume of "Pyramid" Divided by Volume of Cubic Solid
2	1	1	1	=C2/B2
3	2	8	5	

4)

	A	B	C	D
1	Length of Side	Volume of Cubic Solid	Volume of "Pyramid"	Volume of "Pyramid" Divided by Volume of Cubic Solid
2	1	1	1	=C2/B2
3	2	8	5	=C3/B3

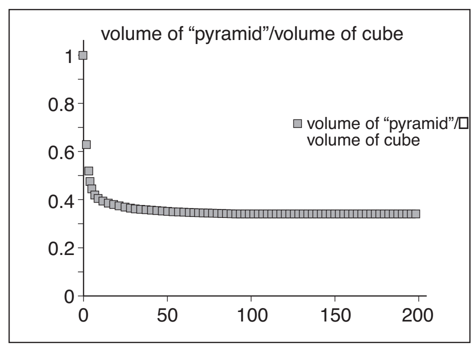
5)

	A	B	C	D
1	Length of Side	Volume of Cubic Solid	Volume of "Pyramid"	Volume of "Pyramid" Divided by Volume of Cubic Solid
2	1	1	1	=C2/B2
3	2	8	5	=C3/B3
4	=A3+1	=A4^3	=C3+A4^2	=C4/B4

6) A few sample rows are shown here.

Length of Side	Volume of Cubic Solid	Volume of "Pyramid"	Volume of "Pyramid" Divided by Volume of Cubic Solid
196	7 529 536	2 529 086	0.335888692
197	7 645 373	2 567 895	0.335875699
198	7 762 392	2 607 099	0.335862837
199	7 880 599	2 646 700	0.335850105
200	8 000 000	2 686 700	0.3358375

7) How students create this graph varies depending on the spreadsheet software and the platform. To select the side-length column and the nonadjacent ratio column, first select one column, then select the other while holding down the control key. Excel users should look for the Chart Wizard icon on the menu bar, click on this icon after selecting the side length and ratio column, and follow the menu choices until the appropriate graph appears.



8) The numbers in the last column get closer and closer to 1/3.

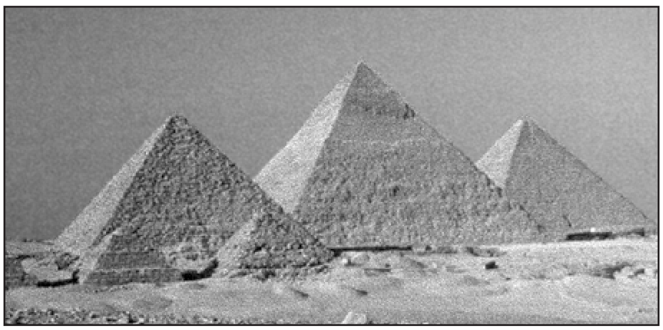
9) No. The ratio will always be higher than 1/3.

10) $V = (1/3)Bh$.

POSSIBLE EXTENSIONS

My students enjoyed moving away from the computers for this low-tech finale. If you have a hollow pyramid-and-prism set that has congruent bases and congruent heights, have students use the pyramid as a measuring device to fill the prism with water, sand, rice, or pasta. They should find that three pyramids of water or sand fill the prism exactly to the brim.

I ended the lesson by giving students a picture of some Egyptian pyramids from a book on architecture. The caption to the picture includes measurements, so students can calculate the volume of one of the actual pyramids.



The pyramid of Cheops, the biggest of the three pyramids at Giza, measures 230.5 meters (756 feet) at its base and is 146 meters high. The slope is 51° 52'. At the center is the pyramid of Chephren. Although it is 215 meters (705 feet) at its base and 143 meters (470 feet) high, it appears higher because of its steeper slope (52° 20'). The pyramid of Mycerinus, in the foreground, is the smallest of the three. It measures 208 meters (354 feet) at its base and 62 meters (203 feet) in height, with a slope of 51°.

REFERENCE

Norwich, John Julius. *World Atlas of Architecture.* New York: Crescent Books, 1984.