Chapter

1

Ratios, Proportions, and Proportional Reasoning:

The Big Idea and Essential Understandings

A TYPICAL instructional unit or chapter on ratio and proportion shows students different ways to write ratios and then introduces a proportion as two equivalent ratios. Next, students usually encounter the cross-multiplication algorithm as a technique for solving a proportion. Does this customary development of ratio and proportion promote a deep understanding of these ideas? Consider an interview with a student named Bonita to think about what it means to reason proportionally.

Bonita was given a problem about a leaky faucet through which 6 ounces of water dripped in 8 minutes. She needed to figure out how much water dripped in 4 minutes. Bonita set up a proportion and used cross multiplication, as shown in figure 1.1, to arrive at a correct response of 3 ounces. Reflect 1.1 invites you to think about Bonita's work on the problem.

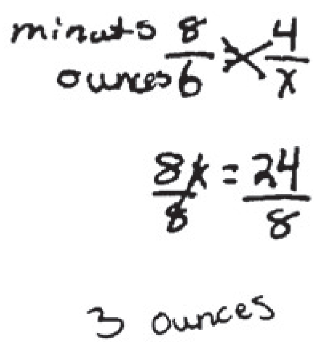

Fig. 1.1. Bonita's work on a proportion problem

Reflect 1.1

Do you think Bonita's work in figure 1.1 shows that she was reasoning proportionally? If so, why do you think so? If not, what do you think she may not have understood?

Bonita's work offers much to like. It is well organized. Bonita labeled the quantities of time and water in her proportion and correctly carried out the cross-multiplication procedure. However, Bonita's responses to three additional tasks suggest that she might not have understood important ideas related to proportional reasoning.

A second task called on Bonita to find the number of ounces that would drip through the same faucet in 40 minutes. To determine whether or not Bonita was procedurally bound to the cross-multiplication method, the interviewer asked her to solve the problem mentally or to use paper and pencil but without applying the algorithm. Bonita was at a loss. She said she couldn't do the problem in her head, and she was unable to do it on paper either. Even after the interviewer changed the specified time from 40 minutes to 16 minutes, Bonita was apparently unable to perform the simple act of doubling mentally or was unaware that doubling would be a reasonable approach.

A third task asked Bonita to solve a problem not posed in the typical form of three numbers given and one missing:

> Crystal placed a bucket under a faucet and collected 6 ounces of water in 20 minutes. Joanne placed a bucket under a second faucet and collected 3 ounces of water in 10 minutes. Were the faucets dripping equally fast or was one dripping faster than the other?

From what you have read so far about Bonita's reasoning, would you expect Bonita to come up with a way to solve this problem? Reflect 1.2 asks you to speculate about Bonita's thinking.

Reflect 1.2

How do you think Bonita approached the third task set for her by the interviewer? Do you think she was able to reason about it proportionally? Why or why not?

Bonita presented two solutions. First, she said that Crystal's faucet was dripping more slowly than Joanne's because "it took its time." This response suggests that Bonita compared only the amounts of time. Because 20 minutes is greater than 10 minutes, Bonita reasoned that the faucet taking more time was dripping more slowly than the faucet taking less time. Then Bonita changed her mind and said that Crystal's faucet was dripping faster because both amounts—time and water—for Crystal's faucet were greater than the corresponding amounts for Joanne's faucet (i.e., $20 > 10$, and $6 > 3$).

Bonita's response indicates that she did not form a ratio between the amount of water and the amount of time. In her first solution, she considered only one quantity—elapsed time. In her second attempt, she applied whole-number reasoning to two disconnected pairs of numbers. Bonita's response illustrates the difficulty that many middle school students experience in conceiving that something may remain the same while the values of the two quantities change.

The fourth and final task presented Bonita with the data shown in figure 1.2. She was told that another girl, Cassandra, had collected the data to see how fast her bathtub faucet was leaking. Cassandra had put a large container under the faucet in the morning and then had checked periodically throughout the day to see how much water was in the container. The interviewer constructed the table with uneven time intervals to approximate actual data collection but provided numbers that readily permitted mental calculations.

Time	Amount of Water
7:00 a.m.	2 ounces
8:15 a.m.	12 ounces
9:45 a.m.	24 ounces
2:30 p.m.	62 ounces
5:15 p.m.	84 ounces
6:00 p.m.	90 ounces
9:30 p.m.	118 ounces

Fig. 1.2. Data collected from a dripping bathtub faucet

To help Bonita comprehend the situation before encountering any difficult questions, the interviewer asked her how much water dripped between 7 a.m. and 8:15 a.m. Although this question required only simple subtraction ($12 - 2 = 10$), Bonita inappropriately set up a proportion and attempted to solve for x, as shown in figure 1.3. This work strongly suggests that Bonita did not understand when it is appropriate to compare numbers by forming a ratio. In sum, although Bonita could correctly execute the proportion algorithm on the first task, her work on the next three tasks demonstrates her poor conceptual understanding.

$$\frac{7.00}{8.15} = \frac{2}{x}$$

$$\frac{7.x}{7} = \frac{16.3}{7}$$

Fig. 1.3. Bonita's work on the bathtub task

If Bonita had understood the ideas behind her work, then she should have been able to reason about the faucet that drips 6 ounces of water in 8 minutes by using at least one of the two following methods.

Proportional Reasoning Method 1

Bonita might have used the method described below to reason about the faucet that dripped 6 ounces in 8 minutes:

- Form a ratio by joining 6 ounces and 8 minutes into a single unit: *6 ounces in 8 minutes.*
- Iterate (repeat) this unit by reasoning that if the faucet drips another 6 ounces in 8 minutes, it does not speed up or slow down since the amounts of time and water are identical. Thus, a faucet that drips 12 ounces in 16 minutes drips at the same rate as one that drips 6 ounces in 8 minutes.
- Similarly, partition, or split, the "6 ounces in 8 minutes" unit in half. A faucet that drips 3 ounces in 4 minutes drips at the same rate as one that drips 6 ounces in 8 minutes.
- Make more challenging partitions. To determine the amount of water that drips in 1 minute, split the unit into eighths by finding $^1/_8$ of 6 ounces, which is $^6/_8$, or $^3/_4$, ounce, and by finding $^1/_8$ of 8 minutes, which is 1 minute. Thus, a faucet that drips $^3/_4$ ounce in 1 minute drips at the same rate as one that drips 6 ounces in 8 minutes.
- Combine the actions of iterating and partitioning. For example, quadruple the "6 ounces in 8 minutes" unit to obtain 24 ounces in 32 minutes. Also partition the "6 ounces in 8 minutes" unit into thirds by finding $^1/_3$ of 6 ounces, which is 2 ounces, and by finding $^1/_3$ of 8 minutes, which is $^8/_3$, or $2\,^2/_3$ minutes. Combine these results to obtain 26 ounces in $34\,^2/_3$ minutes, which is $4\,^1/_3$ times the "6 ounces in 8 minutes" unit.

In this manner, construct a large collection of ratios, all of which represent the same dripping rate: 6 ounces in 8 minutes, 12 ounces in 16 minutes, 3 ounces in 4 minutes, $\frac{3}{4}$ ounce in 1 minute, 26 ounces in $34\frac{2}{3}$ minutes, and so on.

Proportional Reasoning Method 2

Alternatively, Bonita might have reasoned about the faucet dripping 6 ounces of water in 8 minutes by using the following method:

- Compare the two numerical values 6 and 8 (from 6 ounces in 8 minutes) by finding how many times greater 8 is than 6. Eight is $1\frac{1}{3}$ times greater than 6.
- To determine the amount of time that it takes for any amount of water to drip, multiply the value of the water amount by $1\frac{1}{3}$. For example, for 3 ounces of water, it will take $3 \times 1\frac{1}{3}$, or 4, minutes. For 12 ounces, it will take $12 \times 1\frac{1}{3}$, or 16, minutes.
- Construct a collection of ratios by maintaining the factor of $1\frac{1}{3}$. That is, the water amount is always $1\frac{1}{3}$ times greater than the time amount.
- Also compare the values 6 and 8 by finding what fraction 6 is of 8. Six is $\frac{6}{8}$, or $\frac{3}{4}$, of 8.
- To determine the amount of water that drips for any amount of time, multiply the time amount by $\frac{3}{4}$. For example, in 16 minutes, $16 \times \frac{3}{4}$, or 12, ounces, of water will drip. In 4 minutes, $4 \times \frac{3}{4}$, or 3, ounces of water will drip.

One Big Idea and Multiple Essential Understandings

The two methods that Bonita could have used to reason proportionally about the faucet dripping 6 ounces in 8 minutes suggest the following big idea related to ratios, proportions, and proportional reasoning: When two quantities are related proportionally, the ratio of one quantity to the other is invariant as the numerical values of both quantities change by the same factor.

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In the situation of the dripping faucet, the water and time values change; yet, infinitely many water and time pairs represent the same dripping rate (e.g., 6 ounces in 8 minutes, 9 ounces in 12 minutes, 3 ounces in 4 minutes, $\frac{3}{4}$ ounce in 1 minute). Any pair in the collection of water and time pairs can be obtained by iterating and/or partitioning any other pair. For example, 9 ounces in 12 minutes is $1\frac{1}{2}$ groups of 6 ounces in 8 minutes and is equal to 3 groups of 3 ounces in 4 minutes. Furthermore, the ratio of time to water in each pair is constant: the number of minutes is $1\frac{1}{3}$ times the number of ounces. The ratio of water to time is also constant: the number of ounces is $3/4$ the number of minutes.

Although the big idea of proportionality may at first seem straightforward, developing an understanding of it is a complex process for students. It involves grasping many essential understandings:

Essential Understanding 1. Reasoning with ratios involves attending to and coordinating two quantities.

Essential Understanding 2. A ratio is a multiplicative comparison of two quantities, or it is a joining of two quantities in a composed unit.

Essential Understanding 3. Forming a ratio as a measure of a real-world attribute involves isolating that attribute from other attributes and understanding the effect of changing each quantity on the attribute of interest.

Essential Understanding 4. A number of mathematical connections link ratios and fractions:

- Ratios are often expressed in fraction notation, although ratios and fractions do not have identical meaning.
- Ratios are often used to make "part-part" comparisons, but fractions are not.
- Ratios and fractions can be thought of as overlapping sets.
- Ratios can often be meaningfully reinterpreted as fractions.

Essential Understanding 5. Ratios can be meaningfully reinterpreted as quotients.

Essential Understanding 6. A proportion is a relationship of equality between two ratios. In a proportion, the ratio of two quantities remains constant as the corresponding values of the quantities change.

Essential Understanding 7. Proportional reasoning is complex and involves understanding that—

- Equivalent ratios can be created by iterating and/or partitioning a composed unit;
- If one quantity in a ratio is multiplied or divided by a particular factor, then the other quantity must be multiplied or divided by the same factor to maintain the proportional relationship; and
- The two types of ratios—composed units and multiplicative comparisons—are related.

Essential Understanding 8. A rate is a set of infinitely many equivalent ratios.

Essential Understanding 9. Several ways of reasoning, all grounded in sense making, can be generalized into algorithms for solving proportion problems.

Essential Understanding 10. Superficial cues present in the context of a problem do not provide sufficient evidence of proportional relationships between quantities.

The purpose of this chapter is to elaborate and develop these essential understandings, which were implicit in the discussion of Bonita's work. The discussion moves from ratios to proportions (pairs of equivalent ratios) and finally to proportional reasoning (which involves generating an entire set of equivalent ratios). The chart in figure 1.4 illustrates the flow of ideas. Notice that each essential understanding provides a response to a different question. However, the chart is not meant to show the order in which all students develop these ideas.

The initial cluster of essential understandings deals with ratios, because ratios are a building block for the formation of proportions and proportional reasoning. The first essential understanding addresses how ratio reasoning differs from non-ratio reasoning.

Essential Understanding	Question	Topic
1	How does ratio reasoning differ from other types of reasoning?	Ratios
2	What is a ratio?	
3	What is a ratio as a measure of an attribute in a real-world situation?	
4	How are ratios related to fractions?	
5	How are ratios related to division?	
6	What is a proportion?	Proportions
7	What are the key aspects of proportional reasoning?	Proportional Reasoning
8	What is a rate and how is it related to proportional reasoning?	
9	What is the relationship between the cross-multiplication algorithm and proportional reasoning?	
10	When is it appropriate to reason proportionally?	

Fig. 1.4. Organization of the essential understandings developed in chapter 1

Essential Understanding 1

Reasoning with ratios involves attending to and coordinating two quantities.

Attending to two quantities is an aspect of reasoning with ratios that mathematically knowledgeable adults understand so implicitly that they often do not recognize its importance until they become aware of its absence in the reasoning of children. Before children are able to reason with ratios, they typically reason with a single quantity. This type of reasoning is called *univariate reasoning*. Harel and colleagues (1994) offer an example of this reasoning. Sixth-grade students were shown a picture of a carton of orange juice and were told that the juice was made from orange concentrate and water. Next to the carton in the picture were two glasses—a large glass and a small glass—both filled with orange juice from the carton. The sixth graders were asked if they thought that the orange juice from the two glasses would taste equally orangey, or if they thought that the juice in one glass would taste more orangey than the juice in the other.

The results are fascinating. Half the class responded incorrectly that the juice from the two glasses would not be equally orangey. About half of these students said that the juice in the large glass would taste more orangey, and about half chose the small glass as likely to taste more orangey. Their explanations suggest that they either focused on one quantity—the water or the orange concentrate—or attended to both quantities but did not coordinate them. For example, one student explained that the juice in the large glass would taste more orangey "because the glass is bigger, so it would hold more orange" (p. 333). Other students explained that the juice in the small glass would taste more orangey because a smaller volume would allow less water to get in, which would leave more room for the orange concentrate.

The importance of coordinating two quantities becomes clear in the following example, which shows the intellectual achievement that such coordination can represent for children. In a study by Lobato and Thanheiser (2002), students in a class viewed a computer screen with SimCalc Mathworlds software showing two characters—a clown and a frog—capable of being set to walk at constant speeds. The clown was set to walk 10 centimeters in 4 seconds. The children were asked to enter distance and time values for the frog so that it would walk at the same speed as the clown (see fig. 1.5). The simulation software would then show the two journeys simultaneously, thus providing feedback that students could use to determine whether the values that they entered were correct. This activity presented a challenge for the students. Many used a guess-and-check strategy; for example, one student tried 15 centimeters and 8 seconds and then kept adjusting the time until he arrived at 15 centimeters in 6 seconds. Other students used numeric patterns—for example, doubling the 10 and the 4 to obtain 20 centimeters in 8 seconds.

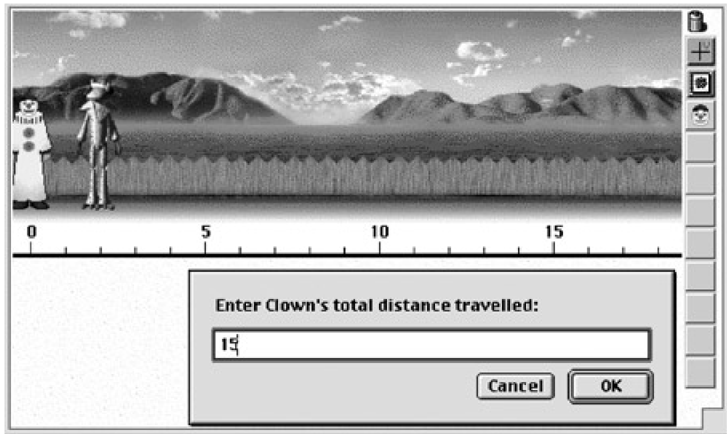

Fig. 1.5. A screen from Roschelle and Kaput's (1996) SimCalc Mathworlds

When the teacher asked the students to explain why walking 20 centimeters in 8 seconds is the same speed as walking 10 centimeters in 4 seconds, one student, Terry, created a drawing that suggests that he had not formed a ratio. Figure 1.6 shows a re-creation of his diagram. He drew lines to represent the distances walked by the two characters without attempting to show that the frog's distance was double the clown's distance. He then relied on calculations, stating, "If you want frog's distance to be 20, then you have to multiply 10 by 2 to get 20. Since you multiplied 10 by 2, you also need to multiply 4 by 2 to get 8." Terry did not explain *why* the time and distance had to be doubled or how multiplying by two could be represented in his drawing.

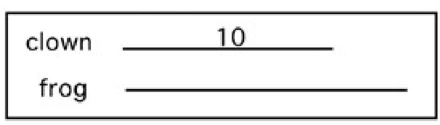

Fig. 1.6. A re-creation of Terry's diagram

Jim, the next student to go to the board, offered a limited explanation that was nearly identical to Terry's. The discussion appeared to stall, when suddenly another student—Brad—had a new idea that he seemed eager to share. Brad explained that doubling works as follows:

> Because the clown is walking the same distance; it's just that he's walking the distance twice... he's walking it once, going li, li, li, li, li, li, [Brad made a "li" sound, evidently to represent time, while his hand retraced the 10 cm line that Terry had drawn], all the way to here [Brad made a vertical hash mark at 10 cm]. Four seconds. Okay. He's going to walk it again. Another four seconds, li, li, li, li, li, li, li, li. Another ten centimeters in four seconds. He's done. (Lobato and Thanheiser 2002, p. 173)

Brad's explanation involved three elements lacking in both Terry's and Jim's work. First, Brad appeared to coordinate time and distance by using sound to represent time while using a hand gesture to represent distance. Second, Brad seemed to coordinate distance and time by forming a "10 centimeters in 4 seconds chunk," which he could repeat. In contrast, Terry seemed to pick one quantity—namely, 20 centimeters—and then produced the other related quantity of 8 seconds. Finally, Brad's image accounted for the frog after the initial 10 centimeters in 4 seconds by noting that the frog walks another 10 centimeters in 4 seconds. By repeating the action of walking 10 centimeters in 4 seconds, the frog will not go faster or slower but will walk at the same speed in both journeys, as well as in the combined journey. In contrast, Terry's explanation did not account for how far the frog walked and in what time after the clown had stopped.

As necessary as it is for students to coordinate two quantities in their reasoning, doing so is not sufficient for understanding ratios. For example, it is possible for students to coordinate two quantities by engaging in a form of reasoning that is different from ratio reasoning—namely, *additive reasoning*. Consider the following situation:

> Jonathan has walked 5 feet in 4 seconds. How long should Rafael take to walk 15 feet if he walks at the same speed as Jonathan?

A seventh grader, Miriam, responded that Rafael should take 14 seconds. She reasoned that 15 feet is 10 more than 5 feet, so you should add 10 seconds to 4 seconds to get 14 seconds. Miriam accounted for both time and distance, but her reasoning was additive because it focused on questions related to "how much more" or "how much less" one quantity is than another. Miriam's work raises the question of what it means to form a ratio.