4 A Taxonomy of Approaches to Learning Trajectories and Progressions

JOANNE LOBATO
C. DAVID WALTERS
San Diego State University, United States

The current landscape of mathematics and science education looks quite different than it did during the major curriculum reforms of the late 1980s and early 1990s. The reform-oriented curricula generated during that time embodied an exciting vision of classrooms that put student reasoning, problem solving, explaining, and justifying at the center. However, these curricula typically lacked a basis in fine-grained understandings of how student ideas evolved over time for each topic addressed in the particular program. This is not a criticism of these programs but rather an assessment of the state of the field at that time. Since then, research on learning trajectories and progressions has developed to a point where research-informed curriculum development is now possible for more topics and grade levels (Clements, 2007; Duschl, Maeng, & Sezen, 2011). Furthermore, numerous advocates point to the potential for learning trajectories and progressions to help align curriculum, standards, assessment, instructional decision making, and professional development (Confrey, Maloney, & Nguyen, 2014; Daro, Mosher, & Corcoran, 2011; Duncan & Hmelo-Silver, 2009; National Research Council, 2007). Indeed, learning trajectories and progressions have "captured the imaginations and rhetoric of school reformers and education researchers as one possible elixir for getting K–12 education 'on track' " (Shavelson & Karplus, 2012, p. 13).

Although there are historical forerunners (e.g., Gagne's learning hierarchies and the research on levels of sophistication of children's addition and subtraction strategies in the Cognitively Guided Instruction project; Carpenter & Moser, 1984; White, 1974), scholarship explicitly identified as trajectories or progressions research has recently seen a rapid expansion. The timeliness and importance of learning trajectories and progressions research is demonstrated by the publication of several books (Alonzo & Gotwals, 2012; Clements & Sarama, 2009; Maloney, Confrey, & Nguyen, 2014), special journal issues (Clements & Sarama, 2004; Duncan & Hmelo-Silver, 2009), conferences sponsored by the National Science Foundation (Learning Progressions Footprint Conference, 2011; Learning Progressions in Science Conference, 2009), and policy reports (Corcoran, Mosher, Rogat, 2009; Daro et al., 2011; Heritage, 2008; National Research Council, 2007) on the topic. The collection of learning progressions created to organize and elaborate the Common Core State Standards in Mathematics (CCSSM; Common Core Standards Writing Team, 2013b; National Governors Association Center for Best Practices & Council of Chief State School Officers [NGA Center & CCSSO], 2010) has propelled this topic into public awareness by practitioners, state education departments, and curriculum developers (Achieve, 2015; Yettick, 2015).

One focus of this work has been to craft a universal definition of a learning trajectory/progression or to identify unifying features (Confrey, Maloney, & Corley, 2014; Duncan & Hmelo-Silver, 2009; Hess, 2008). For example, the National Research Council (2007) characterized learning progressions as "descriptions of the successively more sophisticated ways of thinking about a topic that can follow one another as children learn about and investigate a topic over a broad span of time" (p. 214). Indeed this seems broad and general enough to capture most approaches in mathematics and science education.

However, the extensive literature review conducted for this chapter revealed a variety of significantly different approaches to learning trajectories and progressions. Although differences in approaches do not seem to have been problematized by the field in general, a few important distinctions have emerged in recent scholarship. For example, Empson (2011) notes a distinction between a "teacher-conjectured possible progression" and a "researcher-documented progression of actual learners" (p. 574). Another group of researchers (Ellis, 2014; Ellis, Weber, & Lockwood, 2014; E. Weber, Walkington, & McGalliard, 2015) distinguish between learning trajectories, which document the emergence of student thinking as it becomes more sophisticated, and learning progressions, which document students' movement through benchmarks that are predetermined as a result of researchers' rational analysis of particular content. Though this distinction is an important one, the association of it with the terms *trajectory* versus *progression* is nonstandard. Learning trajectory is usually the term of choice by mathematics educators and can be traced to a seminal article by Simon (1995) in the *Journal for Research in Mathematics Education*. Learning progression is commonly used by science educators and can be traced to a 2004 special issue of the *Canadian Journal of Science, Mathematics, and Technology Education*. (A notable exception—and one that likely influenced Ellis et al., 2014—is the use of learning progression in the CCSSM work.) Furthermore, the differences in approaches revealed by our literature review do not lie cleanly along the trajectory versus progression distinction but rather cut across disciplinary boundaries of mathematics and science education. Thus, we use trajectory and progression interchangeably in this chapter and the acronym LT/P to refer to learning trajectory/progression.

The phenomenon being captured by LT/Ps is multidimensional; thus, there are many aspects on which they can vary. LT/Ps can have different objects of learning. For example, the elements of an LT/P may be, among many things, cognitive conceptions (e.g., Battista, 2004), forms of discourse (e.g., Jin & Anderson, 2012), observable strategies (e.g., Vermont Mathematics Partnership's Ongoing Assessment Project, 2014b), or textbook tasks (e.g., Wang, Barmby, & Bolden, 2015). LT/Ps can focus on the learning of individuals (e.g., Steffe, 2004), the emergent mathematical practices of a collective classroom (e.g., Cobb, McClain, & Gravemeijer, 2003), or an intertwining of teaching and learning (e.g., Clements & Sarama, 2009). LT/Ps can also be rooted in a variety of theoretical perspectives, such as Piagetian schemes and operations (e.g., Hunt, Westenskow, Silva, & Welch-Ptak, 2016), hierarchic interactionalism (e.g., Clements & Sarama, 2014), or Cobb and Yackel's (1996) emergent perspective (e.g., Stephan & Akyuz, 2012). And, LT/Ps can vary in scale, from addressing a single concept (e.g., partitive reasoning with fractions; Norton & Wilkins, 2010) to spanning multiple topics and grade levels (e.g., Smith, Wiser, Anderson, & Krajcik, 2006).

We argue that these and other differences are important for several reasons. First, a researcher's stance regarding the various dimensions identified above informs the choice of a research method, such as cross-sectional interviews, one-on-one teaching experiments, or the retrospective analysis of classroom data. Second, the audience for the LT/P (e.g., teachers, researchers, or policy makers) will affect how the LT/P is presented (e.g., whether instructional tasks or pedagogical actions are a part of the LT/P or whether it is focused only on the development of an object of learning). Third, the approach taken to LT/P research affects the benefits and trade-offs of that work (see the Taxonomy of Seven Approaches to LT/Ps section below for a detailed discussion of the associated advantages and limitations of each approach). Fourth, clearly articulating one's stance on a variety of dimensions promotes effective communication and helps researchers avoid talking past each other. Finally, being aware of the variety of approaches used in LT/P research across both science and mathematics education may open up new avenues for researchers to consider.

Consequently, the bulk of this chapter is devoted to the presentation of a taxonomy of seven approaches to LT/Ps. We named the approaches as follows: (1) cognitive levels, (2) levels of discourse, (3) schemes and operations, (4) hypothetical learning trajectories, (5) collective mathematical practices, (6) disciplinary logic and curricular coherence, and (7) observable strategies and learning performances. The taxonomy does not represent seven conceptions of the same phenomenon but rather seven different approaches taken by researchers who identify their work as being a learning trajectory or progression. By considering a variety of approaches to LT/Ps, we are not arguing that one is better than another. There are benefits as well as tradeoffs that each approach entails given the purposes for which it was developed. Our point instead is that the historical development of LT/P research has arrived at a stage in which the "it" underlying an LT/P investigation can no longer be assumed to be a generally agreed upon construct. Rather, the stance researchers take across a variety of dimensions should be clarified in reports of research.

Before presenting the taxonomy, we convey the methods used to arrive at the categories in the taxonomy and to locate articles for review. The chapter ends with a section called Crosscutting Issues in which we report on efforts to validate LT/Ps, use them with teachers, and identify challenges facing research in this area. Throughout this chapter, it is important to remember that research on LT/Ps is not the same thing as research on learning. The extensive body of LT/P research that we reviewed does not reflect the variety of theoretical perspectives from which research on knowing and learning is currently being conducted (e.g., see Cobb's 2007 review of learning theories and philosophical foundations). In particular, Vygotskian, activity-theoretic, embodied, and situated perspectives are not well represented in LT/P research. Additionally, there have been recent contributions to ways of conceiving of thinking and learning (e.g., Sfard's, 2008, *commognitive* perspective, as reviewed in Herbel-Eisenmann, Meaney, Bishop, & Heyd-Metzuyanim, 2017, this volume) and to specific learning processes (e.g., Norton & D'Ambrosio's, 2008, development of the *zone of potential construction* and its comparison to the *zone of proximal development*), but this research has not been framed in terms of LT/Ps (and thus is excluded from the review). Finally, we embrace an interdisciplinary approach, drawing upon research in science education as well as in mathematics education, to inform the construction of the taxonomy and to generate key examples in several categories. We believe that reaching beyond our domain yields generative examples that can beneficially influence approaches to LT/Ps in mathematics education.

Literature Review Methods

To identify mathematics education articles from refereed journals that present or validate learning trajectories or that examine related theoretical issues, we conducted a search of the following: *Canadian Journal of Science, Cognition & Instruction, Educational Studies in Mathematics, International Journal of Science and Mathematics Education, Journal for Research in Mathematics Education, The Journal of Mathematical Behavior, Journal of the Learning Sciences, Mathematical Thinking and Learning, Mathematics and Technology Education, Mathematics Education Research Journal, The Mathematics Enthusiast,* and, *ZDM—The International Journal on Mathematics Education.* We searched according to the following key words: learning trajectory, developmental, longitudinal, and progression. Because LT/Ps have not been a focus of attention in recent surveys of research in mathematics education, we did not restrict the dates for inclusion in these searches. As we gathered the articles, we also looked at their references to locate additional pieces, and we conducted a search on Google Scholar. These efforts yielded numerous conference reports, books, book chapters, monographs, policy reports, and dissertations. We constrained our focus to the learning of mathematics K–16, thus excluding studies of teacher learning and noticing, except for reports of the use of LT/Ps with or by teachers, which expanded our search to the *Journal of Mathematics Teacher Education.*

Our goal in identifying relevant science education articles was to inform and extend mathematics education research on LT/Ps. Because the condition of an exhaustive search was lightened, we proceeded in a different manner. Specifically, we started with the science education resources that had been gathered by the organizers of the 2011 Learning Progressions Footprint Conference, which included numerous journal articles and the book emanating from the 2009 Learning Progressions in Science Conference (Alonzo & Gotwals, 2012). Examining the references of these papers led to additional articles and to a search of the *Journal of Research in Science Teaching* and *Science Education* using the same search terms identified above. Note also that four of the journals identified above publish articles from science education as well as mathematics education.

During the initial analytical pass, as we read abstracts and skimmed articles, it became clear that there was much greater variation in the approach to LT/Ps than had previously been reported and that this variation could be productively captured through the creation of a taxonomy. Thus, a structure for the chapter emerged; we would present different approaches to LT/Ps via a taxonomy, using specific articles to elaborate and illustrate each approach, while discussing the other articles in respective sections devoted to the validation of LT/Ps, use with teachers, and critiques.

To create the taxonomy, we initially sorted the articles by field—mathematics versus science—because we thought that learning progressions in science education were fairly uniform and differed significantly from learning trajectories in mathematics. But that impression did not bear up under scrutiny. We soon realized the categories cut across the disciplines. Using open coding from grounded theory (Strauss, 1987), we began with a few similar articles that presented LT/Ps as a set of qualitatively distinct types of cognition that occur within a hierarchy of levels of increasing sophistication. We named this initial category the "cognitive levels" approach. Then we employed the constant comparative method of grounded theory (Glaser & Strauss, 1967), assessing each new article in terms of the initial category, inducing new categories when needed. To achieve a parsimonious taxonomy, we refined categories to be broad enough to accommodate multiple LT/Ps. At the same time, we dimensionalized each category by initially creating a general characterization, a list of features, and examples for each category. Later we returned to the articles to identify the research methods that were used, as well as the purpose for which the LT/P was developed and its affordances and constraints. As a result of this process, we created a taxonomy of seven approaches to LT/Ps, which we present next.

Taxonomy of Seven Approaches to Learning Trajectories and Progressions

Approach 1: Cognitive Levels

Characterization and example. In the *cognitive-levels* approach to LT/Ps, researchers identify qualitatively distinct types of cognition (typically conceptions or ways of reasoning) that occur within a hierarchy of levels of increasing sophistication. The LT/Ps speak to the learning of domain-specific content, such as linear measurement (Barrett, Clements, Klanderman, Pennisi, & Polaki, 2006), angle concepts (Mitchelmore & White, 2000), integers (Bishop, Lamb, Philipp, Whitacre, & Schappelle, 2014), or thermal equilibrium (Clark, 2006). This type of research can include weak hierarchies (e.g., Battista, 2004), in which cognitive milestones are ranked in order of sophistication but class inclusion relationships are not assumed, or strong hierarchies (e.g., van Hiele levels of geometry; Burger & Shaughnessy, 1986), in which a particular level assumes a student has progressed through all previous levels.

As an example of this approach, consider Battista's (2004) LT/P for area and volume measurement. It is an integration of results from a series of empirical studies (Battista, 1999; Battista & Clements, 1996; Battista, Clements, Arnoff, Battista, & Borrow, 1998) and consists of seven levels of sophistication in elementary students' two-dimensional and three-dimensional spatial reasoning (as described in Table 4.1). As an example of Level 1 reasoning, consider a student who is shown the 6×4 rectangle from Figure 4.1a, along with a plastic inch square the same size as one of the squares indicated in the rectangle. The student is asked to predict how many plastic squares it takes to completely cover the rectangle. One student pointed and counted at imagined squares but got lost several times in her counting. She then pointed in a somewhat random path (as shown in Figure 4.1b) to arrive at an (incorrect) answer of 30 squares. The researchers placed this student in Level 1 because she did not appear to be able to locate squares by coordinating row and column dimensions in an array (a lack of a units-locating process) or compose squares to form rows or columns (a lack of an organizing-by-composites process).

TABLE 4.1. Authors' Tabular Representation of Battista's Learning Trajectory

Cognitive level	Description
Level 1	Absence of units-locating and organizing-by-composites processes
Level 2	Beginning use of the units-locating and the organizing-by-composites processes
Level 3	Units-locating process becomes sufficiently coordinated to recognize and eliminate double-counting errors
Level 4	Use of organizing-by-composites process to structure an array with maximal composites, but insufficient coordination for iteration
Level 5	Use of units-locating process sufficient to correctly locate all units, but less-than-maximal composites employed
Level 6	Complete development and coordination of both the units-locating and the organizing-by-composites processes.
Level 7	Students' spatial structuring and enumeration schemes become sufficiently abstract

Note. Adapted from "Applying Cognition-Based Assessment to Elementary School Students' Development of Understanding of Area and Volume Measurement" by M. T. Battista, 2004, *Mathematical Thinking and Learning, 6*(2), pp. 185–204.

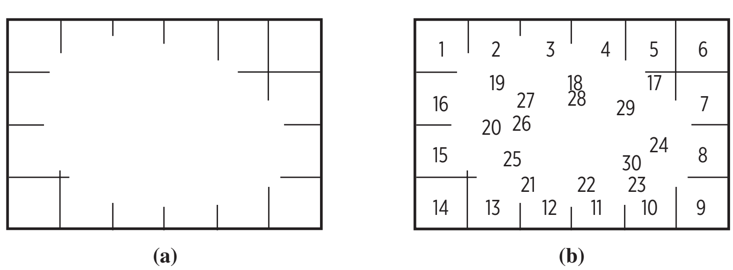

FIGURE 4.1. *Display presented to students* (a) *and one student's subsequent pointing and counting* (b). *From "Applying Cognition-Based Assessment to Elementary School Students' Development of Understanding of Area and Volume Measurement" by M. T. Battista, 2004,* Mathematical Thinking and Learning, 6*(2), p. 193.*

Features. LT/Ps from a cognitive-levels approach typically include a beginning level or lower anchor marked by informal reasoning and an ending level or upper anchor of the conventional content targeted by instruction. Additionally, learners may enter at different levels, jump levels, or fall back to a previous level (Battista, 2011). The landmarks within an LT/P are characterized in a variety of ways by different researchers. One way is in terms of mental actions (e.g., use of the units-locating process in Battista, 2004, or "coordinating iterated-unit items and side lengths" in Barrett et al., 2006, p. 197). A second way is in terms of fine-grained ideas or intuitions, what Minstrell (2001, p. 373) calls "facets of thinking" (e.g., atoms are spherical or an upward sloping graph means the object represented is going uphill; Clark, 2006; Minstrell, 2000). A third approach is to cluster meanings according to types of abstraction (e.g., situated angle concepts, contextual angle concepts, and abstract angle concepts; Mitchelmore & White, 2000). Additionally, some of these LT/Ps include only productive conceptions as milestones (e.g., Mitchelmore & White, 2000), whereas others include misconceptions or partially productive understandings (e.g., Level 1 in Battista's trajectory above or Bishop et al., 2014).

Methods. The primary method used to identify cognitive-levels LT/Ps is cross-sectional interviews over multiple grade levels. For example, Barrett et al. (2006) conducted clinical interviews with 38 students across grades 2–10. Mitchelmore and White (2000) sampled across schools as well as grade levels (grades 2, 4, 6, and 8), conducting interviews with 192 students. Consequently, the levels in these LT/Ps are "compilations of empirical observations of the thinking of many students" (Battista, 2004, p. 187). Though the researchers may posit that instruction plays a key role, in the sense that the levels are not assumed to be achieved solely as a natural result of maturation, instruction is backgrounded in the presentation of the LT/P. This is due, in large part, to the fact that study participants are drawn from cross-sectional studies where they encounter different instructional environments.

Purpose, benefits, and trade-offs. A main purpose of the cognitive-levels LT/Ps is for diagnostic assessment. For example, Battista developed the Cognition-Based Assessment (CBA) program, which produced six volumes for teachers (across a variety of elementary school topics) articulating how to conduct and use formative assessment that is grounded in a learning trajectory. In the book on geometry shapes, Battista (2012) presents the LT/P described previously in this section, along with a set of assessment tasks with typical student responses to help teachers recognize student reasoning associated with each cognitive milestone. Similarly, Minstrell (2001) developed a web-based formative assessment program, called Diagnoser (http://www.diagnoser.com/), which identifies a large set of facets representative of students' ideas across a variety of topics in physics, biology, and chemistry. Diagnoser uses sets of multiple-choice questions in which each alternative response is tied to a particular facet. Students receive feedback tied to the inferred facet as they work. Teachers can access reports of their students' thinking, which in turn can inform the use of facet-driven instructional resources, also available on the project web site.

One trade-off of the cognitive-levels approach to LT/Ps is that the learning mechanisms by which a student uses knowledge at one level to construct understanding at a higher level are backgrounded or underspecified. Researchers sometimes make conjectures regarding such processes; however, the methods used do not tend to offer data to support or reject particular conjectures. A second trade-off concerns the method of developing LT/Ps via cross-sectional interview methods, with most study participants being drawn from traditional classrooms. As a result, less will be known about the progression of understanding that is possible had the students experienced innovative instructional approaches.

Finally, E. Weber and Lockwood (2014) contend that most LT/Ps have focused on students' understanding of specific mathematical content and have not accounted for broader characteristics of that content knowledge (what Harel, 2008, calls students' ways of thinking). For example, in the domain of mathematical functions, specific content understanding includes the idea that the rate of change of the rate of change of a quadratic function is constant. However, a way of thinking may entail different conceptions of covariation, such as focusing on changes in one quantity followed by a change in the other quantity versus simultaneous changes in both quantities. Similarly, LT/Ps have not tended to tackle the development of scientific habits of mind or mathematical practices, such as explaining or conjecturing (Empson, 2011). This is beginning to change with the presentation of LT/Ps on covariation (Ayalon, Watson, & Lerman, 2015; Ellis, Özgür, Kulow, Williams, & Amidon, 2015), generalizing (Blanton, Brizuela, Gardiner, Sawrey, & Newman-Owens, 2015), and defining (Kobiela & Lehrer, 2015). We next turn to a type of LT/P that attempts to identify levels of sophistication in another type of practice, namely ways of communicating.

Approach 2: Levels of Discourse

Characterization and example. Rather than focusing on cognitive landmarks, the *levels-of-discourse* approach describes increasingly sophisticated ways of communicating. Gee defines discourse as "a socially accepted association among ways of using language, of thinking, and of acting that can be used to identify oneself as a member of a socially meaningful group or 'social network' " (1991, p. 3). Gee differentiates between primary discourse, which is acquired within the contexts of home and community, and secondary discourse, which is learned in social institutions such as schools and workplaces. Levels-of-discourse LT/Ps characterize students' narrative accounts and arguments along a continuum of sophistication, for which the upper anchor is representative of practitioners' secondary discourse practices (e.g., scientists or mathematicians), and the lower anchor is representative of informal primary discourse practices. Whereas the LT/Ps outlined in the cognitive-levels approach are fine-grained with many levels to capture subtle differences in individuals' cognitive abilities, the LT/Ps described in this approach tend to be at a macro scale with fewer levels.

Some of the LT/Ps in the cognitive-levels approach characterize the levels in terms of misconceptions or the lack of some understanding. In contrast, the lower levels of levels-of-discourse progressions describe how students' initial narrative accounts serve as a foundation for more sophisticated secondary discourse practices. Additionally, the LT/Ps describe the primary discourse practices that are evident and utilized by students rather than characterizing the students' narrative from a deficit perspective. For instance, Gunckel, Mohan, Covitt, and Anderson (2012) noted that a cognitive-levels trajectory was insufficient for capturing the linkages between young students' and older students' narrative accounts, as well as identifying the connections of both of these to model-based narratives that represent scientific literacy. In general, levels-of-discourse LT/Ps emphasize the quality and tenor of the narrative account of a scientific process (Jin & Anderson, 2012), argument (Berland & McNeill, 2010), or group discussion (Erduran, Simon, & Osborne, 2004). Progress along the levels of discourse can thus be thought of as "movement towards mastering a secondary Discourse" (Gunckel, Mohan, et al., 2012, p. 55).

As an example of an LT/P from this approach, consider Jin and Anderson's (2012) four levels of discourse related to narrative accounts of carbon-transforming processes (see Table 4.2). The trajectory emphasizes how the student communicates about energy rather than the content of the account. An instance of Level 1 discourse can be seen in the following student's description of how a baby uses food for energy:

> Because the food helps make energy for the girl so then she can like learn how to walk and crawl and stuff. And then it will also help the baby so it will be happy, be not mean and stuff. (p. 1164)

Here the student employed a primary discourse to create a force-dynamic narrative account of energy's role in a biological process (to explain what the food allows the baby to do), without appealing to scientific principles, which would have indicated a Level 2 narrative account.

Features. Researchers describe levels of discourse in a variety of ways. As in the example given above, levels may represent increasing use of a secondary discourse as reflected by the narrative accounts that students provide about a scientific concept, such as energy in socio-ecological systems (Jin & Anderson, 2012), the water cycle (Gunckel, Covitt, Salinas, & Anderson, 2012), or biochemical processes (Mohan, Chen, & Anderson, 2009). Other LT/Ps leverage Toulmin's (1958/2003) model of argumentation. For example, Berland and McNeill (2010) describe increasingly complex levels of scientific argumentation across the dimensions of instructional

Table 4.2. Authors' Tabular Representation of Jin and Anderson's LT/P

Discourse level	Description
Level 1: Force-dynamic narrative accounts	Students utilize everyday language from their primary discourse to describe events in terms of actors, enablers, purposes, and results.
Level 2: Force-dynamic narrative accounts with hidden mechanisms	Students' narrative accounts begin to incorporate a sense of physical necessity that includes some key aspects of scientific ideas about energy and represent a slight shift toward a secondary discourse.
Level 3: School science narrative accounts	Such accounts leverage secondary discourse resources, including language about atoms, molecules, forms of energy, and conservation laws. These include many more scientific facts about matter and energy than seen in Level 2.
Level 4: Qualitative model-based narrative accounts	Students successfully use energy as an analytical tool in their narrative accounts and utilize scientific principles, indicating a mastery of a secondary discourse.

Note. Adapted from "A Learning Progression for Energy in Socio-Ecological Systems" by H. Jin and C. W. Anderson, 2012, *Journal of Research in Science Teaching, 49*(9), 1149–1180.

context, argumentative product, and argumentative process. Although Berland and McNeill's LT/P characterizes an individual's level of argumentation, Erduran et al. (2004) leverage Toulmin's scheme to provide an LT/P that characterizes the tenor and sophistication of argumentation among a group of students.

Methods. Research targeting the creation of an LT/P characterizing levels of discourse is typically conducted across multiple grade levels. For example, Jin and Anderson (2012) collected written data from students in fourth grade through high school. Although such research (i.e., written assessments given outside of class and collected from a cross-section of multiple grade levels) is typical for levels-of-discourse progressions, Berland and McNeill (2010) conducted research inside the classroom in an effort to account for how differences in instruction might affect students' arguments. Researchers often set the upper anchor for the trajectory using available standards (e.g., the *Framework for K–12 Science Education,* a standards guide published by the National Research Council, 2012), whereas they capture lower levels of the trajectory through linguistic analysis of students' written and oral accounts about the topic. For example, Gunckel, Covitt, et al. (2012) used written assessments to inform each iteration of their learning progression, and Erduran et al. (2004) analyzed audio recordings of whole class and small group discussions.

Purpose, benefits, and trade-offs. Gee (1991) characterizes literacy as the mastery of a particular discourse that is acquired through increased participation in communities where that discourse is common and valued. Thus, in a broad sense, LT/Ps in the levels-of-discourse approach allow researchers to focus on the degree to which learners are participating in a community of practice (e.g., Erduran et al., 2004; Mohan et al., 2009). By analyzing students' narrative accounts, researchers are better able to understand "the pathways that students take through the learning progression from their primary Discourse to a secondary Discourse of scientific model-based reasoning" (Gunckel, Mohan, et al., 2012, p. 72). In a narrower sense, this research provides a tool by which instruction can be aligned with state standards. For example, based on their analysis of student data and subsequent development of their learning progression, Gunckel, Mohan, et al. argue that current instructional efforts are insufficient for helping students reach the upper anchor.

One trade-off is that framing this approach in terms of primary and secondary discourses may resonate more with science education than mathematics education researchers. Perhaps this is because of the preponderance of vocabulary words (such as energy, force, or matter) that have contrasting meanings in everyday versus formal scientific discourses, because of the centrality of narrative accounts in providing scientific explanations, or because of the emphasis in science education policy documents on helping students participate in scientific discourse (Kuhn, 2010; National Research Council, 2007, 2012).

Two decades ago, Richards (1991) identified four types of mathematics discourse used in different communities: (1) research math, which is the spoken mathematics of mathematicians; (2) inquiry math, which is used by mathematically literate adults; (3) journal math, which is the language of mathematical publications; and (4) school math, which includes the discourse of traditional math classes. Although there has been research identifying changes in the nature of discourse practices in mathematics (e.g., Wood, Williams, & McNeal, 2006), this work has not been characterized in terms of LT/Ps. More common has been the use of Toulmin's scheme to understand students' argumentation (e.g., K. Weber, Maher, Powell, & Lee, 2008); however, when this scheme has been used with LT/Ps, it has been in the methodological service of establishing collective mathematical practices (which we review in Approach 5 below). The closest example of a levels-of-discourse LT/P that we could find in the mathematics education literature is a recent study that presents what Pöhler and Prediger (2015) call a "lexical learning trajectory" (p. 1697). In it, the researchers identify six levels of increasingly sophisticated use of vocabulary related to percent problems, from students' informal language in the everyday register to their extended reading vocabulary in the academic school register.

Approach 3: Schemes and Operations

Characterization and example. Researchers taking a *schemes-and-operations* approach generate a model of students' initial schemes and mental operations (in the sense of Piaget, 1950/2001) and infer the modifications of students' schemes over time (Hackenberg, 2014). In contrast to the identification of levels (cognitive or discursive), researchers seek evidence of both the general and fine-grained learning processes involved as students use prior knowledge as a foundation for constructing or modifying their schemes. This research is conducted from the theoretical perspective of Piagetian scheme theory, in which a scheme is conceived as a researcher's construct used to model students' concepts (meaning the "goal directed regularities in a person's functioning"; Hackenberg, 2010, p. 386). According to von Glasersfeld (1995), a scheme consists of three parts: (1) a set of internal conditions that need to be satisfied for the rest of the scheme to be triggered (meaning that a person recognizes a situation as something he or she has encountered before); (2) ways of operating, mentally or physically; and (3) an anticipation of the outcome of the activity. According to Norton and McCloskey (2008), schemes are activated holistically rather than step-by-step like a strategy. In fact, a single scheme may give rise to several different mathematical strategies.

As an example of an LT/P from this approach, consider Hackenberg and Tillema's (2009) research on the learning of fraction multiplication concepts by two pairs of grade 6 students participating in an 8-month after-school teaching experiment. We present Figure 4.2 below to make several points about the nature of the learning

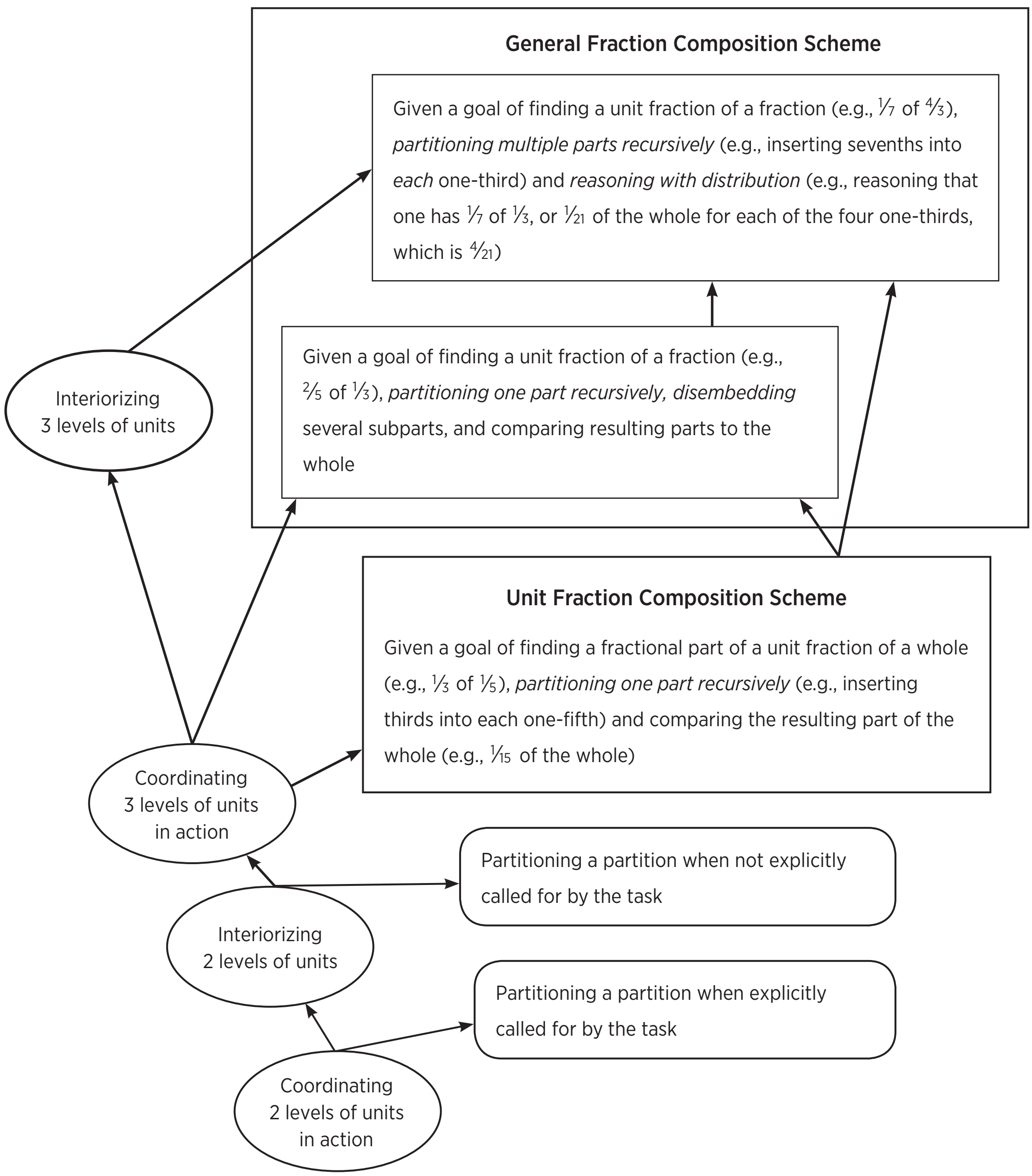

FIGURE 4.2. *Authors' interpretation and illustration of a learning trajectory leading to the general fraction composition scheme, as described by Hackenberg and Tillema. Adapted from "Students' Whole Number Multiplicative Concepts: A Critical Constructive Resource for Fraction Composition Schemes" by A. J. Hackenberg and E. S. Tillema, 2009,* The Journal of Mathematical Behavior, 28*(1), pp. 1–18.*

trajectory. First, the centerpiece of the trajectory is the construction and modification of schemes. For example, one scheme in this trajectory is the *unit fraction composition scheme,* which involves perceiving a situation as calling for taking a unit fractional part of a unit fraction of a whole (e.g., finding $1/5$ of $1/7$ of one whole), separating the whole into seven equal parts, partitioning one of the sevenths into five equal parts, and then comparing the resulting part of a partition to the whole (as $1/35$ of one whole). Second, there is a focus on mental operations, such as *disembedding,* which is the imagistic pulling of a fraction from a whole while maintaining the whole (Olive, 1999). Third, evidence is provided to support an explanation for the students' evolution of schemes. Specifically, the researchers draw upon two resources to identify these learning processes (indicated by the ellipses in Figure 4.2)—students' mental coordinations (of different levels of units) and reflective abstraction (especially the upper level of such, which is called *interiorization,* or taking the result of activity as something to be operated on).

Features. Schemes-and-operations LT/Ps have been identified as the result of research into the learning of a variety of topics in mathematics, including whole number sequences (Steffe, Cobb, & von Glasersfeld, 1988), whole number multiplication (Steffe, 1992), fraction meanings and operations (Hackenberg, 2010; Norton, 2008; Olive, 1999; Steffe & Olive, 2010), algebraic reasoning (Hackenberg, 2013, 2014), trigonometry (Moore, 2013), proportional reasoning (Nabors, 2003), calculus (E. Weber & Thompson, 2014), and combinatorial reasoning (Tillema, 2014).

Because of the shared theoretical perspective of Piagetian constructivism, there is less variation among the trajectories from this approach than the other approaches articulated in this taxonomy. One underlying theoretical assumption is that the way students respond to particular activities and instructional interactions is influenced by their existing conceptual structures. Consequently, researchers working from this approach often create LT/Ps based on *epistemic subjects,* which refers to the common operations across people at the same developmental level (Beth & Piaget, 1966). Thus, this research would not assume a single trajectory for a given domain; rather, LT/Ps would vary across different epistemic subjects (i.e., students starting at different developmental levels).

Methods. The primary method used in the generation of schemes-and-operations LT/Ps is that of the teaching experiment (Steffe & Thompson, 2000), performed typically with a researcher-teacher working in parallel sessions with sets of individuals or pairs of students in a learning or tutoring environment. Although some instructional features, such as the interactions between researcher and students or the particular tasks posed, may be included in the narrative presentation of learners' schemes and operations, the focus is on students' reasoning and on the inferences of cognitive structures that seem to fit with the regularities in the students' behaviors (as exhibited in talk and written work). Due to the fine-grained nature of the analysis, the number of participants is typically small (from 1 to 12) and the time periods long (often weekly sessions for 2–3 years).

Purpose, benefits, and trade-offs. A key aim of this approach is to contribute to basic scientific research on the learning of particular content in mathematics through microanalysis of the evolution of individuals' understanding. Contributions include the elaboration of subtypes of Piaget's general learning process of accommodation (e.g., two of these subtypes are metamorphic accommodation and functional accommodation; Steffe, 1991), content-specific instantiations of elements of Piaget's other learning process of reflective abstraction (e.g., the interiorization of multiplication operations; Hackenberg & Tillema, 2009), and the articulation of important mental operations involved in the learning of particular mathematics topics (e.g., unitizing, iterating, splitting, and partitioning in the learning of fractions; Norton & Wilkins, 2010).

One trade-off is the use of small *n,* which limits researchers' ability to generalize from a study sample to a population. Instead, the main conception of generality present in this research is providing useful constructs to other researchers (Steffe & Thompson, 2000). A second trade-off has been a lack of explicit attention to the teacher's pedagogical actions. This may have been deliberate, so that the emphasis would be on the logic of children's mathematics rather than on the logic of adult mathematics for children (L. P. Steffe, personal communication, October 28, 2002). There have been a few efforts to extend the schemes-and-operations approach by coding the types of instructional moves that seem to support the construction of students' ideas (Barrett & Clements, 2003; Tzur, 1999). However, the presentation of instructional supports is not usually an explicit and prominent component of these LT/Ps, as it is in the next approach.

Approach 4: Hypothetical Learning Trajectory

Characterization and example. Although LT/Ps in the first three approaches focus primarily on learners, this approach includes instructional supports for learning and was originally conceived as part of a model of teachers' decision making (Simon, 1995). Specifically, Simon introduced the term *hypothetical learning trajectory* (HLT) to capture the result of a process in which a teacher posits a conjecture regarding her students' current understanding of a targeted concept (including potential challenges for them) and then develops learning activities that she thinks will support them in constructing more sophisticated ways of reasoning toward a particular learning goal. Simon and Tzur (2004) later highlighted the importance of and principles for selecting tasks that promote students' development of more sophisticated mathematical concepts. Building on this work, Clements and Sarama (2004) define learning trajectories as

> descriptions of children's thinking and learning in a specific mathematical domain and a related, conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking, created with the intent of supporting children's achievement of specific goals in that mathematical domain. (p. 83)

As an example, consider a portion of one trajectory from Clements and Sarama's (2009) book for teachers containing an interrelated web of 10 learning trajectories for pre-K to grade 5 across a variety of domains including number and operations, measurement, and geometry. Each LT/P consists of three components: (1) an overarching learning goal, (2) levels of thinking, and (3) instructional tasks. For example, the LT/P on the composition of number and multidigit addition and subtraction is guided by the following goal from *Curriculum Focal Points* (National Council of Teachers of Mathematics, 2006): "Children develop, discuss, and use efficient, accurate, and generalizable methods to add and subtract multidigit whole numbers . . . [and] understand why the procedures work (on the basis of place value and properties of operations)" (p. 14). Levels of children's reasoning and instructional activities designed to support them in achieving each level of thinking are presented in a two-column format (see Table 4.3 for an example). This approach highlights the context-dependent nature of learning; that is, what children learn is sensitive to the instructional tasks in which they engage (Ellis, 2014). For example, in classrooms dominated by part-whole tasks, children are likely to learn to think of fractions in terms of counting parts, but in classrooms dominated by equal sharing tasks, children are likely to learn to think about fractions in terms of multiplicative relationships between quantities (Empson, 2011).

Features. A central feature of this approach is the ongoing modification of the LT/P. This characteristic arises from the purpose for which Simon (1995) developed the notion of an HLT, namely to offer a conceptualization of teaching as being informed by a constructivist perspective on learning. Because constructivism asserts that knowing is interpretative in nature (von Glasersfeld, 1995), the teacher needs to seek information regarding how students interpret a learning activity. As a result, the teacher revises the activities, goals, and conjectures about students' understandings. However, a critical reader may argue that the LT/Ps provided to teachers (e.g., as shown in Table 4.3) appear to be preplanned roadmaps (Meletiou-Mavrotheris & Paparistodemou, 2015). Yet Clements and Sarama (2004) write that "a priori learning trajectories are always hypothetical. . . . The teacher must construct new models of children's mathematics as they interact with children around the instructional tasks" (p. 85).

Table 4.3. Portion of a Learning Trajectory for Composing Number and Multidigit Addition and Subtraction

Levels of thinking	Instructional tasks
Composer to 10. Knows number combinations to totals of 10. Quickly names parts of any whole or the whole given parts.	*Finger games.* Ask children to show 6 with their fingers. Tell their partner how they did it. Then show 6 in a different way. Now make 6 with the same number on each hand. Repeat with other numbers and other conditions (e.g., "you can't use thumbs").
Composer with 10s and 1s. Understands two-digit numbers as tens and ones.	*Composing 10s and 1s.* Show students connecting cubes—4 tens and 3 ones—for 2 seconds (e.g., hidden under a cloth). Ask how many they saw. Discuss. Show cubes. Repeat with new amounts.
Deriver. Uses flexible strategies and derived combinations, including break-apart-to-make-10. Can simultaneously think of three numbers within a sum and can move part of a number to another, aware of the increase in one and the decrease in another.	*Addition and subtraction.* Present all types of single-digit problems, such as "What is 7 plus 8?" Ask students to describe their thinking. Share different methods. Sample responses: "7 + 7 = 14, so 7 + 8 is 15" or "7 is 2 and 5. Add 2 and 8 to make 10. Then add 5 more to get 15."

Note. Adapted from *Learning and Teaching Early Math: The Learning Trajectories Approach* by D.H. Clements and J. Sarama, 2009, New York, NY: Routledge, pp. 101–104.

This ongoing iterative process of conjecture and revision is reflected in the following terminology: Prior to instruction or analysis a teacher or researcher has a planned or hypothetical learning trajectory, whereas the coproduction of mathematical knowledge during instruction or the results of retrospective analysis by researchers is often known as an actual learning trajectory (Clements & Sarama, 2004; Leikin & Dinur, 2003; Simon, 1995; E. Weber & Lockwood, 2014).

Methods. Although Simon originally offered an HLT as a teacher-conjectured construct, the LT/Ps of Clements and Sarama (2009) and others are a result of research. Multiple phases are required due to the need to integrate developmental progressions with supports for instruction. For example, Meletiou-Mavrotheris and Paparistodemou (2015) detail a two-phase research method. In Phase I, baseline data are collected regarding children's initial concepts in a particular domain via interviews and written assessments. An initial HLT is then constructed, based on the results of Phase 1 and existing research literature, to guide instruction in a researcher-taught teaching experiment. Ongoing and retrospective analysis of the teaching experiment data result in an actual learning trajectory. Clements, Wilson, and Sarama (2004) report a similar method, but with more phases, to develop an LT/P on the composition and decomposition of geometric figures. An initial developmental progression was created by noting regularities in how several case-study students interacted with a software version of pattern blocks called Shapes (Sarama, Clements, & Vukelic, 1996). Instructional tasks were then written that were hypothesized to guide children through each of the levels of the developmental progression. The activities were pilot-tested with groups of increasing size (from individuals to classroom research with eight teachers), with revisions during and after each test. The result was an LT/P, which was then validated in an interview study on a larger scale.

Purpose, benefits, and trade-offs. The trajectories in this approach serve the function of being an important teaching resource. As a result, they have gained considerable attention as a tool for improving mathematics instruction (Daro et al., 2011). Their power comes from the interrelatedness of a developmental progression of children's ways of thinking and a route through a set of instructional tasks. In contrast, many LT/Ps from other approaches focus on only one of these components—a domain-specific developmental progression (e.g., Battista's cognitive milestones shown in Table 4.1) or an instructional progression (e.g., Stephens & Armanto's, 2010, analysis of a Japanese textbook's progression for relational thinking). However, in the hypothetical-learning-trajectory approach, the starting point in teacher planning is the creation of conjectures regarding what students understand initially and what they may be able to learn next. Instructional tasks are selected, not only on the basis of generic task features, such as high cognitive demand or student interest, but also because of an inferred quality of being able to engender the next level of sophistication of student thinking. In turn, hypotheses about student learning are based on the particular tasks employed and how teachers organize students' engagement with these tasks.

One trade-off of this approach, at least as it has currently been enacted, is that explicit conjectures regarding the learning processes that enable students to leverage one understanding to develop more sophisticated understanding are not typically part of the LT/Ps communicated to teachers (Simon et al., 2010). Second, by targeting learning goals specified in standards documents (probably as a means for tapping into goals that teachers already value), the LT/Ps don't speak to what is possible when mathematical goals not included in current standards are embraced, along with associated innovative forms of instruction (Lobato, Hohensee, Rhodehamel, & Diamond, 2012). Finally, although these LT/Ps include instructional tasks, the next approach broadens the instructional supports to include characteristics of the social environment of the classroom, such as social norms and sociomathematical norms.

Approach 5: Collective Mathematical Practices

Characterization and example. Whereas the LT/Ps in the previous four categories capture the evolution of increasing mathematical or scientific sophistication at the individual level, the trajectories in this approach document the progress of a community. This line of research is informed largely by the theoretical approach called the *emergent perspective* (Cobb & Yackel, 1996), in which individual constructs (such as beliefs and conceptions) are coordinated with collective constructs (such as social norms and classroom practices). A learning trajectory from this approach consists of a "sequence of *classroom mathematical practices* together with conjectures about the means of supporting their evolution from prior practices" (Cobb, 1999, p. 9). Specifically, classroom mathematical practices are students' ways of operating, arguing, and using tools that function in the class as if they are taken-as-shared. The phrase *taken-as-shared* is used, rather than *shared*, to emphasize that the claim does not pertain to one individual's understanding but rather to ways of operating that no longer need justification and have become

TABLE 4.4. Three of the Five Classroom Mathematical Practices for Integer Addition and Subtraction Identified by Stephan and Akyuz

Classroom mathematical practice	Normative ways of reasoning
Practice 1: Interpreting net worth as a positive or negative quantity	• Net worth is a combination of a positive and a negative value (when the assets and debts are both nonzero). • When a negative value is greater (in absolute value) than a positive, the combination is negative.
Practice 2: Using zero as a point of reference for calculations	• Referencing zero to determine net worth • Referencing zero to compare two net worths • Referencing zero to add or subtract integers • Cancelling equal positive and negative quantities
Practice 3: Comparing integers using a vertical number line	• Higher (in absolute value) negative numbers are farther away from zero. • Structuring the gap between two integers to find the difference

Note. Adapted from "A Proposed Instructional Theory for Integer Addition and Subtraction" by M. Stephan and D. Akyuz, 2012, *Journal for Research in Mathematics Education*, *43*(4), pp. 428–464.

institutionalized in the microculture of the classroom community (Rasmussen & Stephan, 2008). Thus, normative ways of reasoning are not a feature of an individual but rather are a property of the collective.

By a collective, researchers refer not to a majority of students in the class, but rather to a quality of a group. An everyday life example will illustrate. Consider a married couple in which the wife is energetic and scattered while the husband is methodical and serious. When they interact as a couple with others, they are quite funny, a trait that neither exhibits individually. In a similar way, teachers experience each of their classes as a social entity with qualities that are different from other classes and that transcend the characteristics of the individual students in the class. Cobb and Yackel (1996) maintain that the link between collective mathematics practices and individual psychological conceptions is indirect but reflexive. This means that as ways of operating become accepted into the community, they influence, but do not determine, individual students' conceptions. Conversely, the collective mathematical practices emerge through the sharing and negotiation of individually held ideas (Cobb et al., 2003). In this sense *practice* is conceived as an emergent phenomenon rather than an already-established system into which individuals are enculturated (such as Liberian tailoring or Mayan midwifery; Brown, Collins, & Duguid, 1989; Lave, 1991).

As an example, consider Stephan and Akyuz's (2012) LT/P of five classroom mathematical practices for integer addition and subtraction, presented as the result of a classroom teaching experiment with 20 grade 7 students in a public middle school, where the first author had been a full-time teacher for 3 years. She cotaught the class with another teacher who had taught full-time for 10 years. Table 4.4 shows an excerpt of three of the five practices, each of which is described generally and characterized by a cluster of normative ways of reasoning. For instance, in Practice 3 students compare integers in the form of net worths using a vertical number line (VNL). They were initially given the task of comparing two debts, –$20,000 for Paris and –$22,000 for Nicole (see Figure 4.3), and they were asked who has a greater net worth. This resulted in debate, with one student claiming that Nicole was worth more and placing –$22,000 above –$20,000 on a VNL. This claim was discussed and rejected, ultimately with students coming to the mathematical idea that higher negative numbers (in absolute value) were further away from zero on the VNL. This idea was not challenged in later discussions and, thus, functioned as if taken-as-shared. Being able to represent numbers correctly on the VLN also helped students find the difference between the numbers (in this case a value of $2,000) by exploiting the gap on the number line.

Features. The nature of mathematical practices that researchers have documented has changed over time.

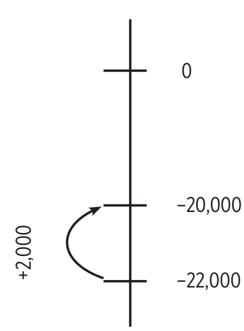

FIGURE 4.3. *Authors' recreation of students' use of the vertical number line as described by Stephan and Akyuz. Adapted from "A Proposed Instructional Theory for Integer Addition and Subtraction" by M. Stephan and D. Akyuz, 2012,* Journal for Research in Mathematics Education, *43(4), pp. 428–464.*

Initially, practices were observable actions or strategies, which avoided the mentalistic language of individual conceptions (e.g., Bowers, Cobb, & McClain, 1999). Other researchers expanded this work by describing what can function as if shared using the more cognitive language of constructed relationships (Roy, 2008), interpretations (Rasmussen, Stephan, & Allen, 2004), meanings (Tobias, 2009), ideas (Stephan & Rasmussen, 2002), and disciplinary practices (such as symbolizing; Rasmussen, Wawro, & Zandieh, 2015). Additionally, whereas earlier work focused on one normative way of acting per practice, Rasmussen and colleagues cluster related normative ways of reasoning as a practice (Rasmussen & Stephan, 2008; Rasmussen et al., 2015).

Research has identified the collective progress of K–8 classrooms engaged in the study of a variety of math topics, including measurement and arithmetic (Gravemeijer, Bowers, & Stephan, 2003), integers (Stephen & Akyuz, 2012), place value (Bowers et al., 1999), and statistics (Cobb et al., 2003). The approach has been extended to undergraduate mathematics classrooms, for a variety of topics in linear algebra and differential equations (Rasmussen et al., 2004; Stephan & Rasmussen, 2002; Wawro, 2011), as well as for topics in courses for prospective teachers (e.g., circles, rational numbers, and whole number concepts; Akyuz, 2014; Roy, 2008; Tobias, 2009). More recently, this approach has expanded to document a progression of collective scientific practices for undergraduate chemistry classrooms (Cole et al., 2012).

Additionally, LT/Ps from this approach uniquely include a description of how instructional supports, such as tasks, contexts, representations, and computer tools, are enacted according to particular classroom norms and discourse practices. Thus, work from this approach also details other collective constructs, including social norms (customary routines of behavior for proper conduct that guide the behavior of group members) and sociomathematical norms (rules guiding the community's behavior that are particular to an environment that supports the learning of mathematics). For example, in their progression of classroom mathematical practices in a grade 8 classroom learning to analyze bivariate data in statistics, Cobb et al. (2003) documented two important sociomathematical norms, namely what counts as a different solution and as an acceptable argument. They wove throughout the presentation of the classroom mathematical practices the ways in which these sociomathematical norms appeared to support the emergence of those practices.

Methods. To identify classroom mathematical practices, Rasmussen and Stephan (2008) present the *documenting collective activity* method (DCA), by extending and systematizing methods proposed by Cobb and colleagues (Cobb & Gravemeijer, 2008; Cobb, Stephan, McClain & Gravemeijer, 2001). The DCA method is a three-phase process in which the unit of analysis in the videorecorded classroom data is students' collective discourse.

In the first phase, researchers generate an argumentation log, recording the structure of each argument made in whole class discussion, using Krummheier's (1995) adaptation of Toulmin's (1958/2003) model of argumentation. Toulmin's model breaks each argument into several components: (*a*) a claim (an assertion or conclusion put forward publicly), (*b*) data (the evidence used to support the claim), (*c*) a warrant (the logic by which the data relates to the claim), and (*d*) backing (statements that further bolster the warrant). In the second phase of the DCA method, the argumentation log is used to establish normative ways of reasoning when either of two criteria are met: (1) when a previously challenged claim no longer requires backings or warrants or (2) if a piece of information changes its function in the Toulmin model (e.g., a statement that was previously a claim later becomes data in the core of another argument) and this shift in function is not challenged. In either case, the reasoning is said to function "as if shared" (Rasmussen & Stephan, 2008, p. 196). A third criterion was added later, namely when an idea is used repeatedly as data or warrant across multiple arguments (Cole et al., 2012). Finally, in the third phase, the researchers cluster mathematically related normative ways of reasoning into a sequence of classroom mathematical practices.

Purpose, benefits, and trade-offs. A major benefit of this approach to LT/Ps is that it embraces the way that most teachers experience their classrooms—as collectives. Teachers know that the character of one class can differ from another and that it is possible to characterize the mathematical development of each class. Cobb and Yackel (1996) trace their own history of moving from teaching experiments with individuals to working in classrooms. Initially, they tried to account for the conceptual reorganizations of individuals as they interacted with the teacher and their peers. Soon the researchers realized that they were missing regularities in communal behavior, both in terms of the obligations and roles of students and teachers and in terms of shifts in reasoning over time that initially required explanation from students but eventually were established as practices that no longer needed justification.

They also found that individualistic psychological accounts of learning were inadequate for the development of instructional theory and design. Specifically, accounts of students' mathematical development as it occurs in the social context of the classroom informs ongoing instructional development efforts; in turn, one way to assess the viability of an instructional sequence is to document both the classroom mathematical practices and individuals' ways of participating in and contributing to them (Cobb, 2003). Much of the research on LT/Ps in this approach has been conducted within the Dutch instructional theory of *realistic mathematics education* (RME). RME is a domain-specific developmental approach to instructional design that specifies several heuristics by which students move from informal representations and models for solving problems in *realistic* settings (meaning realizable or imaginable by the learner, rather than real world or authentic) to more formal conventional and abstract strategies (van den Heuvel-Panhuizen, 2003). According to Cobb and Gravemeijer (2008), an instructional theory in which activities and resources are justified explicitly in terms of principles and learning trajectories enables other researchers to customize and adapt a particular instructional sequence to the setting in which they are working; such adaptations can, in turn, inform the instructional theory, "thereby making the production of design-based knowledge a cumulative activity" (p. 77).

One trade-off is that the method used to establish classroom mathematical practices requires an inquiry-oriented classroom with a substantial amount of public discussion in which the social norms of explaining and justifying one's thinking and agreeing with or debating others' ideas are present. Without these norms the actual learning trajectory would simply consist of the teacher's a priori sequence of activities for the class. Furthermore, the criterion for the establishment of a mathematical practice as the absence of challenging talk is reasonable only under the assumption that the teacher and the other students are continuing to press other students to react to claims that are made in the public space. In more traditional classrooms the absence of voiced disagreement may be attributable to many other factors including the violation of social norms or lack of engagement (Hall, 2001).

A second trade-off is that this line of research is still in its infancy in terms of understanding the complex relationship between individual interpretations and normative ways of reasoning (see Tabach, Hershkowitz, Rasmussen, & Dreyfus, 2014, as an example of such work). A diversity of student ideas is acknowledged through use of the metaphor that students participate differentially in classroom mathematical practices (Cobb, 2003). However, it is unclear how far this metaphor goes when that practice is emergent. Can individual interpretations be so different from normative ways of reasoning that there are actually two or more practices in the classroom at the same time? This could, in turn, push on the notion of there being a single trajectory of mathematical classroom practices. Finally, in the few studies in which individual ways of reasoning are coordinated with collective practices (e.g., Bowers et al., 1999), the analyses conducted from the psychological perspective tends not to specify cognitive learning mechanisms but rather to either establish the appropriation of normative ways of reasoning or to document changes in individual students' reasoning as a result of instruction (Cobb et al., 2001).

Approach 6: Disciplinary Logic and Curricular Coherence

Characterization and example. LT/Ps from the *disciplinary-logic-and-curricular-coherence* approach are generated by reflecting upon experts' knowledge of the domain, synthesizing research from studies of student knowledge and learning, drawing upon scholarly writing on the nature and structure of disciplinary knowledge, and unpacking the constructs identified as targets of learning in standards documents. That is, these LT/Ps are typically *informed by research* versus being the *product of research* (as is the case in each of the previously described approaches). LT/Ps of this type provide a macro view of how student thinking and proficiency in a particular domain may develop over several years. Thus, a much longer time frame is used in the disciplinary logic and curricular coherence approach than in other approaches, with the exception of some of the LT/Ps from the hypothetical learning trajectory approach (e.g., Clements & Sarama, 2009).

The most prominent examples in this approach are the 14 LT/Ps produced by the Common Core Standards Writing Team (2013b). CCSSM, accepted by 43 states, form the backbone of these LT/Ps. Research indicates that CCSSM represents an increase in cognitive demand, focus, and coherence in comparison to the state standards in place prior to CCSSM (Porter, McMaken, Hwang, & Yang, 2011; Schmidt & Houang, 2012). Furthermore, CCSSM significantly expands the process standards of the *Principles and Standards of School Mathematics* (National Council of Teachers of Mathematics, 2000) into a set of eight mathematical practices (e.g., reasoning quantitatively, constructing viable arguments, and expressing regularity through repeated reasoning).

As an example, consider the LT/P for fractions in grades 3–5 (Common Core Standards Writing Team, 2013a). It is presented in a two-column format, in which the left column contains a narrative describing the progression of ideas and skills regarding fractions that students are expected to engage in across the three grade levels. In the right column are the corresponding content standards from CCSSM, along with illustrative diagrams. The narrative emphasizes the logical structure of mathematics and important disciplinary connections. For example, in explaining the meaning of the standard, "Understand a fraction a/b as the quantity formed by a parts of size $1/b$" (NGA Center & CCSSO, 2010, 3.NF.a.1), the writers elaborate a connection between whole numbers and fractions. Specifically, in the same sense that 1 is the basic building block of the whole numbers, unit fractions are the basic building blocks of fractions. For example, the whole number 3 can be conceived as 3 iterates of a unit interval of 1, and similarly the fraction $3/5$ can be formed by iterating the unit fraction $1/5$ three times, so that $3/5$ is thought of as 3 one-fifths.

The LT/P also elaborates how ideas build from one year to the next. For example, part of understanding fraction equivalence in grade 3 involves being able to express 1 as different fractions (e.g., as $2/2 = 3/3 = 4/4$). In grade 4, students use that understanding to convert an improper fraction to a mixed number by decomposing the fraction into a sum of a whole number and a proper fraction. For example, knowing that $1 = 3/3$, a student can see that $5/3 = 3/3 + 2/3 = 1 + 2/3 = 1\,2/3$. Finally, elements from the CCSSM practice standards are interwoven with the content standards and illustrated with fraction content in this LT/P. For example, specifying the referent (or whole) for a fraction involves the mathematical practice of attending to precision. In Figure 4.4 below, if the left square is the whole, then the grey shaded area represents $3/2$ of that whole. If the whole is the entire outer rectangle, then the shaded area is $3/4$ of the whole.

Features. In addition to the Common Core Learning Progressions, there are several other LT/Ps that were created by drawing upon existing research, disciplinary structure, and standards. These include progressions on biodiversity (Songer, Kelcey, & Gotwals, 2009), atomic structures/electrical forces (Stevens, Delgado, & Krajcik, 2009), matter (Smith et al., 2006), and genetics (Duncan, Rogat, & Yarden, 2009). They all describe learning levels over a large time scale, from 3 years for the biodiversity progression (grades 4–6) to 9 years for the progression on matter (grades K–8). The science LT/Ps draw upon the following standards documents: *National Science Education Standards* (National Research Council, 1996), *Benchmarks for Science Literacy* (American Association for the Advancement of Science [AAAS], 1993), and *Atlas for Science Literacy* (AAAS & National Science Teachers Association, 2001). Most make explicit the intertwining of content and inquiry-oriented practices (e.g., Songer et al., 2009), and all include learning performances and assessment tasks aligned with the progressions (e.g., Smith et al., 2006). The fact that the biodiversity and atomic structures/electrical forces LT/Ps were validated empirically and then revised could productively inform the Common Core Math Progressions reviewed above. Similarly in mathematics, Bernbaum Wilmot, Schoenfeld, Wilson, Champney, and Zahner (2011) created an LT/P for function representations in grades 6–12, based on a literature review by a panel of experts, followed by a large-scale validation (details provided in the Validation of LT/Ps section).

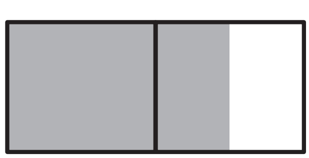

FIGURE 4.4. *The shaded portion is $3/2$ of the area of the left square but $3/4$ of the area of the entire rectangle.*

Purpose, benefits, and trade-offs. One of the chief purposes of this type of LT/P is to inform curricular organization and textbook content so that it is coherent and cohesive (Wiser, Smith, & Doubler, 2012). Furthermore, these progressions can help teachers understand how what they teach can lay the foundation for mathematical ideas to be developed in later grades—an important element of mathematical knowledge for teaching that Ball, Thames, and Phelps (2008) refer to as "horizon knowledge" (p. 403). An added benefit is that the levels identified in the LT/Ps can serve as reference points for assessments, thus helping ensure the alignment of curriculum materials, instruction, and assessment (Corcoran et al., 2009). Finally, most of the trajectories in this approach were written to be accessible to a wide audience, including teachers, administrators, teacher educators, test developers, curriculum designers, parents, and policy makers (Daro et al., 2011).

One trade-off is that LT/Ps from this approach can lack consistency due to the process of picking and choosing ideas from a variety of research that was conducted with different goals and from different theoretical perspectives. For example, a key idea developed in the Common Core fractions learning progression involves the elaboration of the previously mentioned standard, "Understand a fraction a/b as the quantity formed by a parts of size $1/b$." This statement appears to identify a meaning from the psychological perspective of a child, rather than being a statement of adult mathematics for children; as such, it represents a laudable advance over many state standards in effect prior to CCSSM (Norton & Boyce, 2013).

It is consistent with what many researchers call a partitive fraction scheme (Nabors, 2003; Norton & McCloskey, 2008). However, the Common Core learning progression did not continue to draw upon similar research to articulate how students move to more sophisticated understandings, such as a reversible fraction scheme (e.g., being able to partition a nonunit fraction a/b into a parts of $1/b$), an iterative fraction scheme (e.g., being able to produce a whole from any fraction), or the construction of fraction equivalence (Hackenberg; 2013; Steffe, 2004; Steffe & Olive, 2010). Instead, fraction equivalence in the Common Core learning progression is developed by appealing to the disciplinary logic of multiplying a fraction by a fraction equivalent of 1 (e.g., $a/b \times n/n = a{\times}n/b{\times}n$), which is illustrated in an area model with the claim that students will "see" that when the whole is partitioned into n times as many pieces, there are n times as many smaller unit fraction pieces. The claim of transparent "seeing" is inconsistent with research demonstrating the need for students to construct three levels of units to understand fraction equivalence (e.g., Norton, 2008).

A second trade-off is that trajectories aligned with standards are dependent upon the nature and quality of those standards. For example, the previously identified grade 3 standard for fractions fits firmly in the definition of a mathematical concept as the meanings, interpretations, images, ideas, connections, ways of comprehending situations, and explanations regarding why particular procedures work, which can be leveraged productively in students' mathematical development (Lobato, 2014). However, using this definition of concept to examine the algebra and functions strands in CCSSM (for grades 6–8 plus high school), Lobato found that only 17.5% of the associated content standards articulated particular mathematical concepts, while an additional 13.5% had a "conceptual feel" (often using verbs such as "explain" or "interpret" but lacking specificity regarding the particular meanings or connections desired), leaving nearly 70% of the algebra standards focused on skills. Although the development of important skills is essential, the underspecification of productive meanings and their connection to procedures as generalizations of reasoning can have a deleterious effect on the formulation of LT/Ps, which in turn, shape instruction, curriculum development, and assessment.

Approach 7: Observable Strategies and Learning Performances

Characterization and example. We conclude the presentation of the taxonomy with a shorter section devoted to an approach that is less distinct from the others but nonetheless represents an important consideration for future writers of LT/Ps. In this approach, each level in the LT/P is described in terms of observable behaviors, strategies, or other learning performances.

As an integrated math/science example, consider work by Lehrer and Schauble (2012a, 2012b) to identify a complex K–grade 6 progression on modeling. It consists of three interlocking strands: *change* (in organisms, populations, and systems), *variation* (which includes directed and random processes that create distributions), and *ecology* (a system of relationships governing the relative abundance and distribution of organisms). Each strand consists of benchmarks defined in terms of learning performances and illustrated with student examples. Part of the change strand is shown in Table 4.5. Each learning performance is described in terms of an observable behavior using action verbs rather than the language of mental conceptions.

Features. This type of LT/P typically identifies proficiency levels in terms of strategies or other observable behaviors. For example, the Vermont Mathematics Partnership Ongoing Assessment Project (VMP OGAP) produced LT/Ps for (*a*) multiplicative reasoning, (*b*) fractions, and (*c*) proportionality. Each presents three to four levels of student strategies, illustrated with examples from student work that will resonate with teachers (VMP OGAP, 2013, 2014a, 2014b). In other words, the levels of proficiency may be seen to interact with other variables, such as task complexity. For instance, in a pre-K–6 LT/P on equipartitioning (a foundational construct of rational number), Confrey and colleagues present a two-dimensional matrix comprised of 16 ordered proficiency levels along the vertical axis and several task parameters along the horizontal axis (Confrey & Maloney, 2010; Confrey, Maloney, & Corley, 2014). In a similar fashion, Sherin and Fuson (2005) offer an LT/P of children's strategies for single-digit multiplication, for which they assert that "strategy use by individuals, in a particular circumstance, will be very sensitive to the number-specific resources available . . . and will vary across cultural and instructional contexts" (p. 348). Thus, Sherin and Fuson conceive of growth in strategy development being driven primarily by the learning of number-specific computational resources rather than changes in general cognitive capabilities related to number.

Trajectories in this approach differ in terms of assumptions made regarding the relationship between cognition and strategies or other learning performances. For example, Steinthorsdottir and Sriraman (2009), in a study with Icelandic students using curriculum informed by a proportional reasoning trajectory with four levels

TABLE 4.5. Authors' Recreation of an Excerpt From the Change Strand of the LT/P on Modeling by Lehrer and Schauble

	Level 6 of 8: Rate Describe change as rate or changing rate	
	Learning performances	**Examples**
6A	Coordinate time elapsed with counts or measures of change but without expressing the relationship as a rate.	"My plant grew 3 mm between days 5 and 7, and then it grew 7 mm between days 8 and 11."
6B	Determine the rate of change by dividing the difference between two measurements of one attribute by the difference in time.	"My plant grew 12 mm in 3 days, so it grew 4 mm per day."
6C	Interpret graph/table of rate of change.	Student reads graph as showing that her plant grew 6 mm per week during the first week but 9 mm per week during the second. Student concludes that rates of growth differ at different points in the plant's life cycle.
6D	Compare rates of change across more than one organism and justify reasoning.	Student appeals to graph to claim that one plant grew faster than another "overall," but goes on to explain that there were periods during growth when the first plant was growing faster.
6E	Coordinate rate description with a qualitative inscription.	Coordinate rate graph with pressed plant display.

Note. Adapted from "Supporting Inquiry About the Foundations of Evolutionary Thinking in the Elementary Grades" by R. Lehrer and L. Schauble, 2012, in S. M. Carver and J. Shrager (Eds.), *The Journey From Child to Scientist: Integrating Cognitive Development and the Education Sciences,* pp. 171–206, Washington, DC: American Psychological Association.

of strategies (developed by Carpenter et al., 1999), write that "there is no consensus on whether the framework of Carpenter et al. is simply a classification schema for students' solution strategies or whether the framework presents a developmental trajectory" (pp. 7–8). The researchers go on to assert that they interpret it as the latter, viewing levels of conceptual development as being expressed in the strategies that students employ to solve problems. Other researchers foreground proficiencies but background cognitive conceptions that cut across different performance levels (e.g., Confrey, Maloney, Nguyen, & Rupp, 2014). Finally, because this type of trajectory may be either a product of research (e.g., Lehrer & Schauble, 2012b) or informed by research (in the sense of LT/Ps from Approach 6), there is no prototypical method that we can report.

Purpose, benefits, and trade-offs. A primary benefit of describing the elements of an LT/P in terms of observable strategies and behaviors is the increased power to communicate with teachers, making these types of trajectories particularly well suited as tools for teacher professional development. As Empson (2011) explains, "We know that teachers can learn to differentiate students' strategies and use what they learn about students' thinking to successfully guide instruction" (p. 586). Indeed, in an early forerunner to research on learning trajectories, Carpenter and Moser (1984) created a framework of growth in children's strategies for solving single-digit addition and subtraction problems, which later formed the core of the very successful cognitively guided instruction (CGI) professional development program (see Sowder, 2007, for a review of the extensive CGI literature). There was another visible, well-funded project at the same time and working in the same domain—the Interdisciplinary Research on Number Project, which produced schemes-and-operations trajectories (Steffe et al., 1988). History suggests greater saliency using strategies rather than schemes as the unit for communication with teachers regarding children's thinking. Indeed, when a group of secondary teachers, working with student data from a researcher, set out to create an LT/P on exponential functions, they focused on ordering student strategies for solving exponential tasks (Brendefur, Bunning, & Secada, 2014).

A second benefit of describing levels in an LT/P in terms of learning performances is increased precision and explicitness over the more ambiguous language of cognitive conceptions (a sentiment expressed by a number of researchers attending the Learning Progressions Footprint Conference sponsored by the National Science Foundation). As a result, LT/Ps in this approach are well suited for providing diagnostic or formative assessment information to teachers (Petit, 2011). By encapsulating knowledge states in learning performances, they also lend themselves to the design of measures used to validate learning trajectories (Confrey, Maloney, Nguyen, & Rupp, 2014).

A trade-off is that downplaying the language of understanding can leave conceptions underspecified and can overlook how a single conception can give rise to multiple strategies (Ellis, 2014). Also, a correct performance can be produced by a nondesirable conception. For example, Level 6B in the previously cited LT/P by Leherer and Schauble (2012b) states, "Determine the rate of change by dividing the difference between two measurements of one attribute by the difference in time" (see Table 4.5). Thus, the judgment of whether or not a student has conceived of a rate of change is based on the performance of dividing two measurements, such as 12 mm ÷ 3 days (to arrive at 4 mm per day). However, students can perform the arithmetic operation of division without mentally conceiving of a multiplicative relationship between the two quantities—a necessity for forming a ratio or rate (Lobato, 2008; Lobato & Ellis, 2010).

Crosscutting Issues

There are several themes in the literature that cut across the different approaches to LT/Ps presented in the taxonomy. This section reviews the literature on how researchers have validated LT/Ps and how they have studied the use of LT/Ps by and with teachers. We conclude with a discussion of general critiques of LT/Ps and possible alternatives.

Validation of LT/Ps

The most common approach to validating an LT/P involves the use of item response theory (IRT) based on Rasch models (Kennedy & Wilson, 2007; M. Wilson, 2009; M. Wilson & Carstensen, 2007). A benefit of Rasch analysis is its affordance of providing an estimate of learners' proficiency and item difficulty using the same scale (for a popular book on this topic, see Bond & Fox, 2015). Studies have validated LT/Ps on equipartitioning (Confrey & Maloney, 2010; Confrey, Maloney, Nguyen, & Rupp, 2014), functions (Bernbaum Wilmot et al., 2011), force and motion (Fulmer, 2015), biodiversity (Gotwals & Songer, 2013), integration across three facets of energy (Lee & Liu, 2009), and carbon cycling (reported in Corcoran et al., 2009).

There are roughly four components of the validation process using Rasch models. First, there is a *construct map*, which M. Wilson (2009) defines as "a well thought out and researched ordering of qualitatively different levels of performance focusing on one characteristic" (p. 718). This may correspond to the LT/P, or in a particularly complex domain, each level of the trajectory may represent a coordination of multiple construct maps. Thus, the types of LT/Ps being validated using Rasch models typically have been generated either through a synthesis of existing literature and domain expertise or by cross-sectional studies via written assessments or interviews (i.e., they often come from the following approaches in the taxonomy: cognitive levels, levels of discourse, disciplinary logic and curricular coherence, and observable strategies/learning performances).

Second, assessment measures are developed and must be related to the levels in the LT/P and to any associated instruction. To this end, it is helpful if each level is translated into an observable performance (in the spirit of the observable strategies/learning performances approach). Qualitative methods, such as the use and analysis of clinical or think-aloud interviews, can inform the development of items for assessments designed to measure students' levels in the trajectory (Confrey, Maloney, Nguyen, & Rupp, 2014). Additionally, an outcome space is created, which is "the set of categorical outcomes into which student performances are categorized for all the items associated with a particular progress variable" (M. Wilson, 2009, p. 721). This can include scoring guides, complete with rubrics, exemplars, and the identification of each item with a level in the LT/P. Third, data are collected, often on a large scale. For example, Confrey, Maloney, Nguyen, and Rupp field-tested their assessment items with 4,800 students in K–grade 8 in North Carolina. Bernbaum Wilmot et al. (2011) administered their assessment with 2,356 students in 125 classrooms (grades 6–12). Triangulation may occur with selected students also participating in interviews to explain their thinking on the paper-and-pencil assessment.

Finally, the data are analyzed using Rasch modeling and Wright Maps, which plot students' proficiency levels on the LT/P against item difficulty on the assessment. As an example, consider the Wright Map for the functions LT/P from Bernbaum Wilmot et al. (2011), shown in Figure 4.5. The left-most column shows the six levels of the LT/P identified in order (e.g., with Extended Abstract as the most sophisticated level). The right-hand side of the map shows 8 of the 12 items on the assessment. The Xs represent the proficiency of 688 students as distributed across the sample. If the estimated order of the levels of difficulty correspond to the theoretical expectations embedded in the progression, then "the theoretical framework as a developmental learning progression holds true" (Bernbaum Wilmot et al., 2011, p. 277). Although the six levels in the LT/P were not consistent across all assessment items, the authors concluded that there was sufficient evidence to validate the progression, and they used the item information formatively to revise and improve the assessment. Thus, there is usually an iterative process of validation and trajectory refinement, with each informing the other.

College Readiness Assessment—Wright Map			Distribution of item difficulty (*n* = 8)							
Level on the Connections Construct Map	**Estimate**	**Distribution of student proficiency (*n* = 688)**	**HEXAGON PATTERN**	**EQUIVALENT FUNCTIONS**	**EDUCATIONAL ACCOMPLISH-MENTS**	**GENDER GAP**	**COST OF POSTAGE**	**STAIRCASE TOOTHPICK**	**24 HOUR CRUDE OIL**	**100 YEARS CRUDE OIL**
Extended Abstract (EA) Relational (R)							R			
	4								EA	
										EA
	3					MS				
								MS		
					EA					
								US		
	2	X							R	R
		XXX		R	R					
Multistructural (MS) Unistructural (US)		XXXXXX	EA						MS	MS
		XXXXXXXXX	R		MS	US				
	1	XXXXXXXXXXXXXXXXXX					MS			
		XXXXXXXXXXXXXX	MS							
		–XXXXXXXXXXXXXXXXXXX	US	MS		PS	US	PS		US
	0	XXXXXXXXXXXXXXXXX			US					
		XXXXXXXXXXXXX					PS	PA		
		XXXXXXXXX		US					US	
Pre-structural (PS) Pre-algebraic (PA)		XXXXX		PA PS	PS		PA		PS	
	–1	XX	PS							
		XXX	PA						PA	
		X								
	–2	X								PS PA
					PA	PA				
EACH X REPRESENTS 4 STUDENTS; EACH ROW REPRESENTS .255 LOGITS										

FIGURE 4.5. *Wright Map for an LT/P on mathematical functions. From "Validating a Learning Progression in Mathematical Functions for College Readiness" by D. Bernbaum Wilmot, A. Schoenfeld, M. Wilson, D. Champney, and W. Zahner, 2011,* Mathematical Thinking and Learning, 13*(4), p. 276.*

The power of this type of construct-validation method is the empirical confirmation of the ordering of the progression, the generation of high-quality assessments, and the development of a rigorous process of reliably scoring students' work on formative assessments (Bernbaum Wilmot et al., 2011). This type of validation requires an impressive level of interdisciplinary collaboration and a scale of field trials that is difficult to achieve (Confrey, Maloney, Nguyen, & Rupp, 2014).

One trade-off is that the Rash model assumes a unidimensional variable in the learning progression (Lee & Liu, 2009; see the Critiques and Alternatives Section below for an articulation of concerns regarding this requirement). Furthermore, the model assumes that students at a particular level consistently respond to all assessment tasks at that level—an assumption that is questioned on both theoretical and empirical grounds (Battista, 2011; Steedle & Shavelson, 2009). Finally, Stacey and Steinle (2006), in a careful analysis of the mismatch between Rasch models and their diagnostic test of decimal understanding, argue that developing items in a "hierarchy of difficulty to represent a single underlying ability dimension" (p. 86) fails to capture crucial aspects of the nature of student reasoning. We turn next to research involving the use of LT/Ps with teachers to further explore, among several issues, the types of materials and information that would inform teachers' understanding of their students' progress.

Research on Use of LT/Ps With Teachers

Despite the potential worth of LT/Ps as instructional tools, how teachers interpret, use, and create LT/Ps has only recently become an object of research attention. One cluster of studies investigated the effect of exposing both practicing and prospective elementary teachers to the equipartitioning learning trajectory (reviewed above in the observable strategies/learning performances approach). Specifically, P. H. Wilson, Mojica, and Confrey (2013) examined how professional development with 33 K–2 teachers and 56 prospective elementary teachers influenced their ability to make sense of children's thinking during clinical interviews. Although less than one half of the practicing teachers and one third of the prospective teachers were able to make reasonable inferences regarding the children's thinking before gaining an understanding of the equipartitioning learning trajectory, nearly all were able to do so afterwards. Additionally, they were able to use children's actions and words as evidence of particular types of thinking (versus evaluating whether or not a response was correct). P. H. Wilson (2009) then followed 10 of the practicing teachers into their classrooms and found evidence that the LT/P helped teachers select instructional activities, identify what students needed to learn next, and interact with students in class discussions. The research team also investigated the claim that LT/Ps can deepen teachers' content knowledge. In a yearlong professional development experience with 24 additional K–5 teachers, the researchers found that the teachers did participate in mathematics content discussions; however, the discussions did not occur without the use of learning activities at higher levels of the LT/P and the teachers' participation was mediated by their own mathematical knowledge for teaching (P. H. Wilson, Sztajn, Edgington, & Confrey, 2014).

Other research has investigated teachers' interpretations of LT/Ps embedded in curricular resources. Specifically, Land and Drake (2014) followed three expert teachers (of grades 1, 2, and 4) who were using a reform-oriented curriculum, *Investigations in Number, Data, and Space* (TERC, 2008). The *Investigations* units incorporate several mathematical strands that build upon each other. An LT/P for each strand is presented in the teachers' materials, where evolving student ideas and strategies across grades levels are articulated and illustrated with student responses to key instructional activities. Given that this curriculum was informed by a synthesis of existing research literature, the LT/P fits within the disciplinary logic and curricular coherence approach. The three teachers also had participated for several years in CGI professional development, where they had exposure to strategy-oriented trajectories (like those from the observable strategies/learning performances approach). The researchers found that the teachers conceived of four types of progressions: (1) *math subtopics* (e.g., students represent easy fractions, followed by finding fractions of a group, and then later are able to add and compare fractions); (2) *instructional activities* (e.g., a sequence of increasingly more complex math tasks); (3) *number choices* (e.g., teachers would re-pose tasks, using increasingly more difficult number combinations); and (4) *student strategies* (teachers' descriptions of student solutions from least to most sophisticated). Evidence showed that each of the teachers made use of all four progression types and that the progressions were embedded within each other.

Finally, a few studies have investigated the nature of LT/Ps generated by teachers. Amador and Lamberg (2013) conducted case studies with four grade 4 teachers, which involved an extensive series of interviews plus regular classroom observations. In contrast to the hypothetical learning trajectories of Simon (1995), the three veteran teachers formulated *testing trajectories*. The teachers' knowledge of the content of three high-stakes tests, along with their beliefs about test preparation, drove their mathematical goals for students and dictated their task selection and instructional decisions. In contrast, the novice teacher selected problem-solving tasks (versus questions posed in test formats), gathered information about students' understanding, and modified her planned instruction accordingly. Although she was aware of the testing pressure, she dealt with it by focusing on the student understanding related to tested content. Suh and Seshaiyer (2015) found lesson study to be a more hospitable environment for teachers to create a hypothetical learning trajectory. Specifically, a vertical team of six teachers (in grades 3, 6, and 8) met regularly to plan and teach a research lesson on linear patterns and representations at each grade level. The lessons were rooted in the teachers' modifications of an initial problem involving linear inequalities for different grade levels. Working together, the teachers set grade-appropriate learning goals and made conjectures regarding student ideas, strategies, and conceptual challenges. The vertical teaming provided opportunities for teachers to think beyond their grade levels, which was crucial for creating a progression of student ideas. The lesson-study environment supported an emphasis on student thinking as the foundation for pedagogical responses and for subsequent modifications to the learning trajectory.

Critiques and Alternatives

Researchers have made a number of critical observations regarding LT/Ps, despite their potential. First, issues of equity, diversity, race, language, and cultural heterogeneity are not being addressed satisfactorily in current LT/P research (Anderson et al., 2012). These concerns include how research is framed theoretically, who participates in the research, the types of tasks that are employed, and the way the work is translated for policy makers and practitioners. For example, much of the work on LT/Ps uses existing standards to set upper anchors. This can result in insufficient variation and a lack of innovation in instruction and curriculum. Alternatively, research by Bang and colleagues (Bang et al., 2014; Bang & Medin, 2010; Bang, Medin, Washinawatok, & Chapman, 2010; Bang, Warren, Rosebery, & Medin, 2012), set in the context of creating science education experiences for children from indigenous communities, provides a model for how community members can be enlisted to set learning goals, how the range of student experiences that are viewed as relevant to the classroom can be expanded, and how the epistemological perspectives of the community regarding key scientific constructs can be engaged.

Second, researchers have raised concerns about the assumptions regarding knowledge growth present in many LT/Ps (Lesh & Yoon, 2004; Salinas, 2009). Sikorski and Hammer (2010) assert that conceiving of student learning as proceeding as a sequence of conceptual attainments that get ever closer to canonical scientific understanding is at odds with the manner in which progress is characterized in the discipline. For example, the formation of wrong ideas in science has often been generative for later progress. On the occasions when student misconceptions are included in LT/Ps, they are often included only in lower levels and without an explanation regarding how the partially correct or nonnormative idea could serve as a resource for later development. One alternative is a two-dimensional image of an LT/P as a web or network of schemes that combines vertical with horizontal learning (Steffe, 2014). Although vertical learning, as the conceptual reorganizations that constitute a new scheme, is consistent with the metaphor of a path or progression, horizontal learning involves using a scheme in a novel way to establish a new scheme that is at the same level as the current scheme (Steffe, 2004).

Third, levels in an LT/P are supposed to represent periods of consistency in students' knowledge; yet several studies have found that students can appear to be on two levels simultaneously (Alonzo & Steedle, 2008; Steedle & Shavelson, 2009). Furthermore, levels-based LT/Ps appear to be at odds with current conceptions of student knowledge as being context-sensitive and embedded in social and material environments (Shavelson & Karplus, 2012; Sikorski & Hammer, 2010). An alternative to a step-like structure of an LT/P is that of a landscape, which evokes a space of numerous connected elements and allows for clustering concepts that adhere to a particular big idea, as well as highlighting connections among ideas (Salinas, 2009).

Finally, questions have been raised regarding the unidimensionality of most LT/Ps, namely that modeling learners' progress along a single dimension is a potentially misleading simplification (Corcoran et al., 2009). According to Lesh and Yoon (2004), research has demonstrated that "mathematical constructs (and conceptual systems) develop along a variety of dimensions" (p. 207). As an alternative to unidimensional LT/Ps, Thompson, Carlson, Byerley, and Hatfield (2014) invoke the metaphor of a *cloud* to make the point that many complex ideas in mathematics (such as proportional reasoning) involve parallel developments in multiple interacting domains (e.g., multiplication and division, measurement, fractions, scaling, ratio, and covariation). Thinking about the formation and evolution of clouds of interrelated ideas has implications for instruction and assessment. Specifically, each conceptual component of a cloud is a context for exhibiting ways of thinking that are related to other components. Instead of using Rasch or IRT models to assess students' progress along a single dimension, methods need to be developed to "profile the state of a student's cloud" (Thompson et al., 2014, p. 15).

Conclusion

The construction of a taxonomy developed from a review of the literature on LT/Ps, along with a discussion of cross-cutting issues and challenges, paves the way for a more comprehensive approach to research in this area. By drawing attention to the advantages and trade-offs of each approach, we hope to advance a conversation that leads to the next generation of approaches to LT/P research. By highlighting significant differences in current approaches to LT/P research, we hope to help researchers consider new ways to conceptualize and present their work. And, by encouraging creators of LT/Ps to be explicit about their stance on each of multiple dimensions underlying their work, we hope to facilitate communication across different lines of research.

References

Achieve. (2015). *The role of learning progressions in competency-based pathways.* Washington, DC: Author. Retrieved from http://www.achieve.org/files/Achieve-LearningProgressionsinCBP.pdf

Akyuz, D. (2014). Mathematical practices in a technological setting: A design research experiment for teaching circle properties. *International Journal of Science and Mathematics Education, 14*(3), 549–573. doi:10.1007/s10763-014-9588-z

Alonzo, A. C., & Gotwals, A. W. (Eds.). (2012). *Learning progressions in science: Current challenges and future directions.* Rotterdam, The Netherlands: Sense.

Alonzo, A., & Steedle, J. T. (2008). Developing and assessing a force and motion learning progression. *Science Education, 93*(3), 389–421. doi:10.1002/sce.20303

Amador, J., & Lamberg, T. (2013). Learning trajectories, lesson planning, affordances, and constraints in the design and enactment of mathematics teaching. *Mathematical Thinking and Learning, 15*(2), 146–170. doi:10.1080/10986065.2013.770719

American Association for the Advancement of Science. (1993). *Benchmarks for science literacy: Project 2061.* New York, NY: Oxford University Press.

American Association for the Advancement of Science & National Science Teachers Association. (2001). *Atlas of science literacy: Project 2061.* Washington, DC: Author.

Anderson, C. W., Cobb, P., Calabrese Barton, A., Confrey, J., Penuel, W. R., & Schauble, L. (2012). Learning Progressions Footprint Conference Final Report. Washington, DC. Retrieved from http://create4stem.msu.edu/sites/default/files/story/files/NSF%20LP%20Conf%20rpt%203-5-12%20-1.pdf

Ayalon, M., Watson, A., & Lerman, S. (2015). Progression towards functions: Students' performance on three tasks about variables from grades 7 to 12. *International Journal of Science and Mathematics Education,* 1–21. doi:10.1007/s10763-014-9611-4

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389–407. doi:10.1177/0022487108324554

Bang, M., Curley, L., Kessel, A., Marin, A., Suzukovich, E. S., & Strack, G. (2014). Muskrat theories, tobacco in the streets, and living Chicago as Indigenous land. *Environmental Education Research, 20*(1), 37–55. doi:10.1080/13504622.2013.865113

Bang, M., & Medin, D. (2010). Cultural processes in science education: Supporting the navigation of multiple epistemologies. *Science Education, 94*(6), 1008–1026. doi:10.1002/sce.20392

Bang, M., Medin, D., Washinawatok, K., & Chapman, S. (2010). Innovations in culturally based science education through partnerships and community. In M. S. Khine & I. M. Saleh (Eds.), *New science of learning* (pp. 569–592). New York, NY: Springer.

Bang, M., Warren, B., Rosebery, A. S., & Medin, D. (2012). Desettling expectations in science education. *Human Development, 55*(5–6), 302–318. doi:10.1159/000345322

Barrett, J. E., & Clements, D. H. (2003). Quantifying path length: Fourth-grade children's developing abstractions for linear measurement. *Cognition and Instruction, 21*(4), 475–520. doi:10.1207/s1532690xci2104_4

Barrett, J. E., Clements, D. H., Klanderman, D., Pennisi, S. J., & Polaki, M. V. (2006). Students' coordination of geometric reasoning and measuring strategies on a fixed perimeter task: Developing mathematical understanding of linear measurement. *Journal for Research in Mathematics Education, 37*(3), 187–221. doi:10.2307/30035058

Battista, M. T. (1999). Fifth graders' enumeration of cubes in 3D arrays: Conceptual progress in an inquiry-based classroom. *Journal for Research in Mathematics Education, 30*(4), 417–448. doi:10.2307/749708

Battista, M. T. (2004). Applying cognition-based assessment to elementary school students' development of understanding of area and volume measurement. *Mathematical Thinking and Learning, 6*(2), 185–204. doi:10.1207/s15327833mt10602_6

Battista, M. T. (2011). Conceptualizations and issues related to learning progressions, learning trajectories, and levels of sophistication. *The Mathematics Enthusiast, 8*(3), 507–570.

Battista, M. T. (2012). *Cognition-based assessment and teaching of geometric shapes: Building on students' reasoning.* Portsmouth, NH: Heinemann.

Battista, M. T., & Clements, D. H. (1996). Students' understanding of three-dimensional rectangular arrays of cubes. *Journal for Research in Mathematics Education, 27*(3), 258–292. doi:10.2307/749365

Battista, M. T., Clements, D. H., Arnoff, J., Battista, K., & Borrow, C. V. A. (1998). Students' spatial structuring of 2D arrays of squares. *Journal for Research in Mathematics Education, 29*(5), 503–532. doi:10.2307/749731

Berland, L. K., & McNeill, K. L. (2010). A learning progression for scientific argumentation: Understanding student work and designing supportive instructional contexts. *Science Education, 94*(5), 765–793. doi:10.1002/sce.20402

Bernbaum Wilmot, D., Schoenfeld, A., Wilson, M., Champney, D., & Zahner, W. (2011). Validating a learning progression in mathematical functions for college readiness. *Mathematical Thinking and Learning, 13*(4), 259–291. doi:10.1080/10986065.2011.608344

Beth, E. W., & Piaget, J. (1966). General conclusions. In E. W. Beth & J. Piaget (Eds.), *Mathematical epistemology and psychology* (pp. 305–312). Dordrecht, The Netherlands: D. Reidel.

Bishop, J. P., Lamb, L. L., Philipp, R. A., Whitacre, I., & Schappelle, B. P. (2014). Using order to reason about negative numbers: The case of Violet. *Educational Studies in Mathematics, 86*(1), 39–59. doi:10.1007/s10649-013-9519-x

Blanton, M., Brizuela, B. M., Gardiner, A. M., Sawrey, K., & Newman-Owens, A. (2015). A learning trajectory in 6-year-olds' thinking about generalizing functional relationships. *Journal for Research in Mathematics Education, 46*(5), 511–558. doi:10.5951/jresematheduc.46.5.0511

Bond, T., & Fox, C. M. (2015). *Applying the Rasch model: Fundamental measurement in the human sciences* (3rd ed.). New York, NY: Routledge.

Bowers, J., Cobb, P., & McClain, K. (1999). The evolution of mathematical practices: A case study. *Cognition and Instruction, 17*(1), 25–66. doi:10.1207/s1532690xci1701_2

Brendefur, J. L., Bunning, K., & Secada, W. G. (2014). Using solution strategies to examine and promote high-school students' understanding of exponential functions: One teacher's attempt. *International Journal for Mathematics Teaching and Learning.* Retrieved from http://www.cimt.plymouth.ac.uk/journal/brendefur.pdf

Brown, J. S., Collins, A., & Duguid, P. (1989). Situated cognition and the culture of learning. *Educational Researcher, 18*(1), 32–42. doi:10.3102/0013189X018001032

Burger, W. F., & Shaughnessy, J. M. (1986). Characterizing the van Hiele levels of development in geometry. *Journal for Research in Mathematics Education, 17*(1), 31–48. doi:10.2307/749317

Carpenter, T. P., Gomez, C., Rousseau, C., Steinthorsdottir, O. B., Valentine, C., Wagner, L., & Wiles, P. (1999, April). *An analysis of student construction of ratio and proportion understanding.* Paper presented at the annual meeting of the American Educational Research Association, Montreal, Canada.

Carpenter, T. P., & Moser, J. M. (1984). The acquisition of addition and subtraction concepts in grades one through three. *Journal for Research in Mathematics Education, 15*(3), 179–202. doi:10.2307/748348
Clark, D. B. (2006). Longitudinal conceptual change in students' understanding of thermal equilibrium: An examination of the process of conceptual restructuring. *Cognition and Instruction, 24*(4), 467–563. doi:10.1207/s1532690xci2404_3
Clements, D. H. (2007). Curriculum research: Toward a framework for "research-based curricula." *Journal for Research in Mathematics Education, 38*(1), 35–70. doi:10.2307/30034927
Clements, D. H., & Sarama, J. (2004). Learning trajectories in mathematics education. *Mathematical Thinking and Learning, 6*(2), 81–89. doi:10.1207/s15327833mt10602_1
Clements, D. H., & Sarama, J. (2009). *Learning and teaching early math: The learning trajectories approach.* New York, NY: Routledge.
Clements, D. H., & Sarama, J. (2014). Learning trajectories: Foundations for effective, research-based education. In A. P. Maloney, J. Confrey, & K. H. Nguyen (Eds.), *Learning over time: Learning trajectories in mathematics education* (pp. 1–30). Charlotte, NC: Information Age.
Clements, D. H., Wilson, D. C., & Sarama, J. (2004). Young children's composition of geometric figures: A learning trajectory. *Mathematical Thinking and Learning, 6*(2), 163–184. doi:10.1207/s15327833mt10602_5
Cobb, P. (1999). Individual and collective mathematical development: The case of statistical data analysis. *Mathematical Thinking and Learning, 1*(1), 5–43. doi:10.1207/s15327833mt10101_1
Cobb, P. (2003). Investigating students' reasoning about linear measurement as a paradigm case of design research. In M. Stephan, J. Bowers, P. Cobb, & K. Gravemeijer (Eds.), Supporting students' development of measuring conceptions: Analyzing students' learning in social context (pp. 1–16). *Journal for Research in Mathematics Education* monograph series (Vol. 12). Reston, VA: National Council of Teachers of Mathematics.
Cobb, P. (2007). Putting philosophy to work. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 3–38). Charlotte, NC: Information Age; Reston, VA: National Council of Teachers of Mathematics.
Cobb, P., & Gravemeijer, K. (2008). Experimenting to support and understand learning processes. In A. E. Kelly, R. A. Lesh, & J. Y. Baek (Eds.), *Handbook of design research methods in education: Innovations in science, technology, engineering, and mathematics learning and teaching* (pp. 68–95). Mahwah, NJ: Erlbaum.
Cobb, P., McClain, K., & Gravemeijer, K. (2003). Learning about statistical covariation. *Cognition and Instruction, 21*(1), 1–78. doi:10.1207/S1532690XCI2101_1
Cobb, P., Stephan, M., McClain, K., & Gravemeijer, K. (2001). Participating in classroom mathematical practices. *Journal of the Learning Sciences, 10*(1 & 2), 113–163. doi:10.1007/978-90-481-9729-3_9
Cobb, P. & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist, 31*(3/4), 175–190. doi:10.1080/00461520.1996.9653265
Cole, R., Becker, N., Towns, M., Sweeney, G., Wawro, M., & Rasmussen, C. (2012). Adapting a methodology from mathematics education research to chemistry education research: Documenting collective activity. *International Journal of Science and Mathematics Education, 10*(1), 193–211. doi:10.1007/s10763-011-9284-1
Common Core Standards Writing Team. (2013a). *Grades 3–5, number and operations, fractions: A progression of the Common Core State Standards in Mathematics* (*draft*). Tucson, AZ: Institute for Mathematics and Education, University of Arizona. Retrieved from http://commoncoretools.me/wp-content/uploads/2011/08/ccss_progression_nf_35_2013_09_19.pdf
Common Core Standards Writing Team. (2013b). *Preface for the draft progressions* (*draft*). Tucson, AZ: Institute for Mathematics and Education, University of Arizona. Retrieved from http://commoncoretools.me/wp-content/uploads/2013/07/ccss_progression_frontmatter_2013_07_30.pdf
Confrey, J., & Maloney, A. (2010). The construction, refinement, and early validation of the equipartitioning learning trajectory. In K. Gomez, L. Lyons, & J. Radinsky (Eds.), *Proceedings of the 9th International Conference of the Learning Sciences* (Vol. 1, pp. 968–975). Chicago, IL: International Society of the Learning Sciences.
Confrey, J., Maloney, A. P., & Corley, A. K. (2014). Learning trajectories: A framework for connecting standards with curriculum. *ZDM—The International Journal on Mathematics Education, 46*(5), 719–733. doi:10.1007/s11858-014-0598-7
Confrey, J., Maloney, A. P., & Nguyen, K. H. (2014). Learning trajectories in mathematics. In A. P. Maloney, J. Confrey, & K. H. Nguyen (Eds.), *Learning over time: Learning trajectories in mathematics education* (pp. xi–xxi). Charlotte, NC: Information Age.
Confrey, J., Maloney, A. P., Nguyen, K. H, & Rupp, A. A. (2014). Equipartitioning, a foundation for rational number reasoning: Elucidation of a learning trajectory. In A. P. Maloney, J. Confrey, & K. H. Nguyen (Eds.), *Learning over time: Learning trajectories in mathematics education* (pp. 61–96). Charlotte, NC: Information Age.
Corcoran, T., Mosher, F., & Rogat, A. D. (2009). *Learning progressions in science: An evidence-based approach to reform* (Research Report No. RR-63). Consortium for Policy Research in Education. Retrieved from http://www.cpre.org/images/stories/cpre_pdfs/lp_science_rr63.pdf
Daro, P., Mosher, F., & Corcoran, T. (2011). *Learning trajectories in mathematics: A foundation for standards, curriculum, assessment, and instruction* (Research Report No. RR-68). Consortium for Policy Research in Education. Retrieved from http://www.cpre.org/images/stories/cpre_pdfs/learning%20trajectories%20in%20math_ccii%20report.pdf
Duncan, R. G., & Hmelo-Silver, C. E. (2009). Learning progressions: Aligning curriculum, instruction, and assessment. *Journal of Research in Science Teaching, 46*(6), 606–609. doi:10.1002/tea.20316
Duncan, R. G., Rogat, A. D., & Yarden, A. (2009). A learning progression for deepening students' understandings of modern genetics across the 5th–10th grades. *Journal of Research in Science Teaching, 46*(6), 655–674. doi:10.1002/tea.20312

Duschl, R., Maeng, S., & Sezen, A. (2011). Learning progressions and teaching sequences: A review and analysis. *Studies in Science Education, 47*(2), 123–182. doi:10.1080/03057267.2011.604476

Ellis, A. B., (2014). What if we built learning trajectories for epistemic students? In L. Hatfield, K. Moore, & L. Steffe (Eds.), *Epistemic algebraic students: Emerging models of students' algebraic knowing,* WISDOMe Monographs (Vol. 4, pp. 199–207). Laramie, WY: University of Wyoming.

Ellis, A. B., Özgür, Z., Kulow, T., Williams, C. C., & Amidon, J. (2015). Quantifying exponential growth: Three conceptual shifts in coordinating multiplicative and additive growth. *The Journal of Mathematical Behavior, 39,* 135–155. doi:10.1016/j.jmathb.2015.06.004

Ellis, A. B., Weber, E., & Lockwood, E. (2014). The case for learning trajectories research. In S. Oesterle, P. Liljedahl, C. Nicol, & D. Allan (Eds.), *Proceedings of the Joint Meeting of PME 38 and PME-NA 36* (Vol. 3, pp. 1–8). Vancouver, Canada: PME.

Empson, S. (2011). On the idea of learning trajectories: Promises and pitfalls. *The Mathematics Enthusiast, 8*(3), 571–596.

Erduran, S., Simon, S., & Osborne, J. (2004). TAPping into argumentation: Developments in the application of Toulmin's argument pattern for studying science discourse. *Science Education, 88*(6), 915–933. doi:10.1002/sce.20012

Fulmer, G. W. (2015). Validating proposed learning progressions on force and motion using the force concept inventory: Findings from Singapore secondary schools. *International Journal of Science and Mathematics Education, 13*(6), 1235–1254. doi:10.1007/s10763-014-9553-x

Gee, J. (1991). What is literacy? In C. Mitchell & K. Weiler (Eds.), *Rewriting literacy: Culture and the discourse of the other* (pp. 3–11). Westport, CT: Bergin & Gavin.

Glaser, B. G., & Strauss, A. L. (1967). *The discovery of grounded theory: Strategies for qualitative research.* Hawthorne, NY: Aldine.

Gotwals, A. W., & Songer, N. B. (2013). Validity evidence for learning progression-based assessment items that fuse core disciplinary ideas and science practices. *Journal of Research in Science Teaching, 50*(5), 597–626. doi:10.1002/tea.21083

Gravemeijer, K., Bowers, J., & Stephan, M. (2003). A hypothetical learning trajectory on measurement and flexible arithmetic. In M. Stephan, J. Bowers, P. Cobb, & K. Gravemeijer (Eds.), Supporting students' development of measurement conceptions: Analyzing students' learning in social context (pp. 51–66). *Journal for Research in Mathematics Education* monograph series (Vol. 12). Reston, VA: National Council of Teachers of Mathematics. doi:10.2307/30037721

Gunckel, K. L., Covitt, B. A., Salinas, I., & Anderson, C. W. (2012). A learning progression for water in socio-ecological systems. *Journal of Research in Science Teaching, 49*(7), 843–868. doi:10.1002/tea.21024

Gunckel, K. L., Mohan, L., Covitt, B. A., & Anderson, C. W. (2012). Addressing challenges in developing learning progressions for environmental science literacy. In A. C. Alonzo & A. W. Gotwals (Eds.), *Learning progressions in science: Current challenges and future directions* (pp. 39–75). Rotterdam, The Netherlands: Sense.

Hackenberg, A. J. (2010). Students' reasoning with reversible multiplicative relationships. *Cognition and Instruction, 28*(4), 383–432. doi:10.1080/07370008.2010.511565

Hackenberg, A. J. (2013). The fractional knowledge and algebraic reasoning of students with the first multiplicative concept. *The Journal of Mathematical Behavior, 32*(3), 538–563. doi:10.1016/j.jmathb.2013.06.007

Hackenberg, A. J. (2014). Musings on three epistemic algebraic students. In K. C. Moore, L. P. Steffe, & L. L. Hatfield (Eds.), *Epistemic algebra students: Emerging models of students' algebraic knowing,* WISDOMe Monographs (Vol. 4, pp. 81–124). Laramie, WY: University of Wyoming.

Hackenberg, A. J., & Tillema, E. S. (2009). Students' whole number multiplicative concepts: A critical constructive resource for fraction composition schemes. *The Journal of Mathematical Behavior, 28*(1), 1–18. doi:10.1016/j.jmathb.2009.04.004

Hall, R. (2001). Schedules of practical work for the analysis of case studies of learning and development. *The Journal of the Learning Sciences, 10*(1–2), 203–222. doi:10.1207/S15327809JLS10-1-2_8

Harel, G. (2008). DNR perspective on mathematics curriculum and instruction. *ZDM—The International Journal on Mathematics Education, 40,* 487–500. doi:10.1007/s11858-008-0146-4

Herbel-Eisenmann, B., Meaney, T., Bishop, J., & Heyd-Metzuyanim, E. (2017). Highlighting heritages and building tasks: A critical analysis of mathematics classroom discourse literature. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 722–765). Reston, VA: National Council of Teachers of Mathematics.

Heritage, M. (2008). *Learning progressions: Supporting instruction and formative assessment.* Washington, DC: The Council of Chief State School Officers.

Hess, K. (2008). *Developing and using learning progressions as a schema for measuring progress.* Retrieved from http://www.nciea.org/publications/CCSS02_KH08.pdf

Hunt, J. H., Westenskow, A., Silva, J., & Welch-Ptak, J. (2016). Levels of participatory conception of fractional quantity along a purposefully sequenced series of equal sharing tasks: Stu's trajectory. *The Journal of Mathematical Behavior, 41,* 45–67. doi:10.1016/j.jmathb.2015.11.004

Jin, H., & Anderson, C. W. (2012). A learning progression for energy in socio-ecological systems. *Journal of Research in Science Teaching, 49*(9), 1149–1180. doi:10.1002/tea.21051

Kennedy, C. A., & Wilson, M. (2007). *Using progress variables to interpret student achievement and progress* (BEAR Report Series, 2006–12–01). University of California, Berkeley. Retrieved from https://bearcenter.berkeley.edu/sites/default/files/Kennedy_Wilson2007.pdf

Kobiela, M., & Lehrer, R. (2015). The codevelopment of mathematical concepts and the practice of defining. *Journal for Research in Mathematics Education, 46*(4), 423–454. doi:10.5951/jresematheduc.46.4.0423

Krummheuer, G. (1995). The ethnography of argumentation. In P. Cobb and H. Bauersfeld (Eds.), *The emergence of mathematical meaning: Interaction in classroom cultures* (pp. 229–270). Hillsdale, NJ: Erlbaum.

Kuhn, D. (2010). Teaching and learning science as argument. *Science Education, 94*(5), 810–824. doi:10.1002/sce.20395

Land, T. J., & Drake, C. (2014). Enhancing and enacting curricular progressions in elementary mathematics. *Mathematical Thinking and Learning, 16*(2), 109–134. doi:10.1080/10986065.2014.889502

Lave, J. (1991). Situating learning in communities of practice. In L. B. Resnick, J. M. Levine, & S. D. Teasley (Eds.), *Perspectives on socially shared cognition* (pp. 63–82). Washington, DC: American Psychological Association. doi:10.1037/10096-003

Lee, H. S., & Liu, O. L. (2009). Assessing learning progression of energy concepts across middle school grades: The knowledge integration perspective. *Science Education, 94*(4), 665–688. doi:10.1002/sce.20382

Lehrer, R., & Schauble, L. (2012a). Seeding evolutionary thinking by engaging children in modeling its foundations. *Science Education, 96*(4), 701–724. doi:10.1002/sce.20475

Lehrer, R., & Schauble, L. (2012b). Supporting inquiry about the foundations of evolutionary thinking in the elementary grades. In S. M. Carver & J. Shrager (Eds.), *The journey from child to scientist: Integrating cognitive development and the education sciences* (pp. 171–206). Washington, DC: American Psychological Association.

Leikin, R., & Dinur, S. (2003). Patterns of flexibility: Teachers' behavior in mathematical discussion. In *Electronic Proceedings of the Third Conference of the European Society for Research in Mathematics Education.* Retrieved from http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.582.9254&rep=rep1&type=pdf

Lesh, R., & Yoon, C. (2004). Evolving communities of mind in which development involves several interacting and simultaneously developing strands. *Mathematical Thinking and Learning, 6*(2), 205–226. doi:10.1207/s15327833mt10602_7

Lobato, J. (2008). When students don't apply the knowledge you think they have, rethink your assumptions about transfer. In M. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics* (pp. 289–304). Washington, DC: Mathematical Association of America.

Lobato, J., (2014). Why do we need a set of conceptual learning goals in algebra when we are drowning in standards? In K. C. Moore, L. P. Steffe, & L. L. Hatfield (Eds.), *Epistemic algebra students: Emerging models of students' algebraic knowing,* WISDOMe Monographs (Vol. 4, pp. 25–47). Laramie, WY: University of Wyoming.

Lobato, J., & Ellis, A. B. (2010). Developing essential understanding of ratios, proportions, and proportional reasoning for teaching mathematics in grades 6–8. In R. M. Zbiek (Series Ed.) and R. I. Charles (Volume Ed.), *Essential understandings series.* Reston, VA: National Council of Teachers of Mathematics.

Lobato, J., Hohensee, C., Rhodehamel, B., & Diamond, J. (2012). Using student reasoning to inform the development of conceptual learning goals: The case of quadratic functions. *Mathematical Thinking and Learning, 14*(2), 85–119. doi:10.1080/10986065.2012.656362

Maloney, A. P., Confrey, J., & Nguyen, K. (Eds.). (2014). *Learning over time: Learning trajectories in mathematics education.* Charlotte, NC: Information Age.

Meletiou-Mavrotheris, M., & Paparistodemou, E. (2015). Developing students' reasoning about samples and sampling in the context of informal inferences. *Educational Studies in Mathematics, 88*(3), 385–404. doi:10.1007/s10649-014-9551-5

Minstrell, J. (2000). Student thinking and related assessment: Creating a facet-based learning environment. In N. S. Raju, J. W. Pellegrino, M. W. Bertenthal, K. J. Mitchell, & L. R. Jones (Eds.), *Grading the nation's report card: Research from the evaluation of NAEP* (pp. 44–73). Washington, DC: National Academy Press.

Minstrell, J. (2001). Facets of students' thinking: Designing to cross the gap from research to standards-based practice. In K. Crowley, C. Schunn, & T. Okada (Eds.), *Designing for science: Implications from everyday, classroom, and professional settings* (pp. 415–443). Mahwah, NJ: Erlbaum.

Mitchelmore, M. C., & White, P. (2000). Development of angle concepts by progressive abstraction and generalisation. *Educational Studies in Mathematics, 41*(3), 209–238. doi:10.1023/A:1003927811079

Mohan, L., Chen, J., & Anderson, C. W. (2009). Developing a multi-year learning progression for carbon cycling in socio-ecological systems. *Journal of Research in Science Teaching, 46*(6), 675–698. doi:10.1002/tea.20314

Moore, K. C. (2013). Making sense by measuring arcs: A teaching experiment in angle measure. *Educational Studies in Mathematics, 83*(2), 225–245. doi:10.1007/s10649-012-9450-6

Nabors, W. K. (2003). From fractions to proportional reasoning: A cognitive schemes of operation approach. *The Journal of Mathematical Behavior, 22*(2), 133–179. doi:10.1016/S0732-3123(03)00018-X

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics. (2006). *Curriculum focal points for prekindergarten through grade 8 mathematics: A quest for coherence.* Reston, VA: Author.

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common core state standards for mathematics.* Washington, DC: Author. Retrieved from http://www.corestandards.org

National Research Council. (1996). *National science education standards.* Washington, DC: The National Academy Press.

National Research Council. (2007). *Taking science to school: Learning and teaching science in grades K–8.* Committee on Science Learning, Kindergarten Through Eighth Grade. (R. A. Duschl, H. A. Schweingruber, & A. W. Shouse, Eds.). Washington DC: The National Academies Press.

National Research Council (2012). *A framework for K–12 science education: Practices, crosscutting concepts, and core ideas.* Committee on a Conceptual Framework for New K–12 Science Education Standards, Board on Science Education, Division of Behavioral and Social Sciences and Education. Washington, DC: The National Academies Press.

Norton, A. (2008). Josh's operational conjectures: Abductions of a splitting operation and the construction of new fractional schemes. *Journal for Research in Mathematics Education, 39*(4), 401–430.

Norton, A., & Boyce, S. (2013). A cognitive core for common state standards. *The Journal of Mathematical Behavior, 32*(2), 266–279. doi:10.1016/j.jmathb.2013.01.001

Norton, A., & D'Ambrosio, B. (2008). ZPC and ZPD: Zones of teaching and learning. *Journal for Research in Mathematics Education, 39*(3), 220–246. doi:10.2307/30034969

Norton, A., & McCloskey, A. V. (2008). Modeling students' mathematics using Steffe's fraction schemes. *Teaching Children Mathematics, 15*(1), 48–54.

Norton, A., & Wilkins, J. L. M. (2010). Students' partitive reasoning. *The Journal of Mathematical Behavior, 29*(4), 181–194. doi:10.1016/j.jmathb.2010.10.001

Olive, J. (1999). From fractions to rational numbers of arithmetic: A reorganization hypothesis. *Mathematical Thinking and Learning, 1*(4), 279–314. doi:10.1207/s15327833mt10104_2

Petit, M. (2011). Learning trajectories and adaptive instruction meet the realities of practice. In P. Daro, F. A. Mosher, & T. Corcoran (Eds.), *Learning trajectories in mathematics: A foundation for standards, curriculum, assessment, and instruction* (Research Report No. RR-68; pp. 35–40). Consortium for Policy Research in Education. Retrieved from http://www.cpre.org/images/stories/cpre_pdfs/learning%20trajectories%20in%20math_ccii%20report.pdf.

Piaget, J. (2001). *The psychology of intelligence* (M. Piercy & D. E. Berlyne, Trans.). London, United Kingdom: Routledge. (Original work published 1950)

Pöhler, B., & Prediger, S. (2015). Intertwining lexical and conceptual learning trajectories: A design research study on dual macro-scaffolding towards percentages. *Eurasia Journal of Mathematics, Science & Technology Education, 11*(6), 1697–1722.

Porter, A., McMaken, J., Hwang, J., & Yang, R. (2011). Common core standards: The new U.S. intended curriculum. *Educational Researcher, 40*(3), 103–116. doi:10.3102/0013189X11405038

Rasmussen, C., & Stephan, M. (2008). A methodology for documenting collective activity. In A. E. Kelly, R. A. Lesh, & J. Y. Baek (Eds.), *Handbook of design research methods in education: Innovations in science, technology, engineering, and mathematics learning and teaching* (pp. 195–215). Mahwah, NJ: Erlbaum.

Rasmussen, C., Stephan, M., & Allen, K. (2004). Classroom mathematical practices and gesturing. *The Journal of Mathematical Behavior, 23*(3), 301–323. doi:10.1016/j.jmathb.2004.06.003

Rasmussen, C., Wawro, M., & Zandieh, M., (2015). Examining individual and collective level mathematical progress. *Educational Studies in Mathematics, 88*(2), 259–281. doi:10.1007/s10649-014-9583-x

Richards, J. (1991). Mathematical discussions. In E. von Glasersfeld (Ed.), *Radical constructivism in mathematics education* (pp. 13–51). Dordrecht, The Netherlands: Kluwer.

Roy, G. J. (2008). *Prospective teachers' development of whole number concepts and operations during a classroom teaching experiment* (Doctoral dissertation). Retrieved from http://etd.fcla.edu/CF/CFE0002398/Roy_George_J_200812_PhD.pdf

Salinas, I. (2009, June). *Learning progressions in science education: Two approaches for development.* Paper presented at the Learning Progressions in Science (LeaPS) Conference, Iowa City, IA. Retrieved from http://education.msu.edu/projects/leaps/proceedings/Salinas.pdf

Sarama, J., Clements, D. H., & Vukelic, E. B. (1996). The role of a computer manipulative in fostering specific psychological/mathematical processes. In E. Jakubowski (Ed.), *Proceedings of the 18th annual meeting of the North America Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 567–572). Columbus, OH: ERIC Publications. Retrieved from http://files.eric.ed.gov/fulltext/ED400178.pdf

Schmidt, W. H., & Houang, R. T. (2012). Curricular coherence and the Common Core State Standards for Mathematics. *Educational Researcher, 41*(8), 294–308. doi:10.3102/0013189X12464517

Sfard, A. (2008). *Thinking as communication.* Cambridge, United Kingdom: Cambridge University Press.

Shavelson, R. J., & Karplus, A. (2012). Reflections on learning progressions. In A. C. Alonzo & A. W. Gotwals (Eds.), *Learning progressions in science: Current challenges and future directions* (pp. 13–26). Rotterdam, The Netherlands: Sense.

Sherin, B., & Fuson, K. (2005). Multiplication strategies and the appropriation of computational resources. *Journal for Research in Mathematics Education, 36*(4), 347–395. doi:10.2307/30035044

Sikorski, T.-R., & Hammer, D. (2010). A critique of how learning progressions research conceptualizes sophistication and progress. In *Proceedings of the 9th International Conference of the Learning Sciences* (Vol. 1, pp. 1032–1039). Chicago, IL: International Society of the Learning Sciences.

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*(2), 114–145. doi:10.2307/749205

Simon, M. A., Saldanha, L., McClintock, E., Akar, G. K., Watanabe, T., & Zembat, I. O. (2010). A developing approach to studying students' learning through their mathematical activity. *Cognition and Instruction, 28*(1), 70–112. doi:10.1080/07370000903430566

Simon, M. A., & Tzur, R. (2004). Explicating the role of mathematical tasks in conceptual learning: An elaboration of the hypothetical learning trajectory. *Mathematical Thinking and Learning, 6*(2), 91–104. doi:10.1207/s15327833mt10602_2

Smith, C. L., Wiser, M., Anderson, C. W., & Krajcik, J. (2006). Implications of research on children's learning for standards and assessment: A proposed learning progression for matter and the atomic-molecular theory. *Measurement: Interdisciplinary Research & Perspective, 4*(1–2), 1–98.

Songer, N. B., Kelcey, B., & Gotwals, A. W. (2009). How and when does complex reasoning occur? Empirically driven development of a learning progression focused on complex reasoning about biodiversity. *Journal of Research in Science Teaching, 46*(6), 610–631. doi:10.1002/tea.20313

Sowder, J. T. (2007). The mathematical education and development of teachers. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 157–222). Charlotte, NC: Information Age; Reston, VA: National Council of Teachers of Mathematics.

Stacey, K., & Steinle, V. (2006). A case of the inapplicability of the Rasch model: Mapping conceptual learning. *Mathematics Education Research Journal, 18*(2), 77–92. doi:10.1007/BF03217437

Steedle, J. T., & Shavelson, R. J. (2009). Supporting valid interpretations of learning progression level diagnoses. *Journal of Research in Science Teaching, 46*(6), 699–715. doi:10.1002/tea.20308

Steffe, L. P. (1991). The constructivist teaching experiment: Illustrations and implications. In E. von Glasersfeld (Ed.), *Radical constructivism in mathematics education* (pp. 177–194). Dordrecht, The Netherlands: Kluwer.

Steffe, L. P. (1992). Schemes of action and operation involving composite units. *Learning and Individual Differences, 4*(3), 259–309. doi:10.1016/1041-6080(92)90005-Y

Steffe, L. P. (2004). On the construction of learning trajectories of children: The case of commensurate fractions. *Mathematical Thinking and Learning, 6*(2), 129–162. doi:10.1207/s15327833mt10602_4

Steffe, L. P. (2014). Perspectives on epistemic algebraic students. In K. C. Moore, L. P. Steffe, & L. L. Hatfield (Eds.), *Epistemic algebra students: Emerging models of students' algebraic knowing,* WISDOMe Monographs (Vol. 4, pp. 317–323). Laramie, WY: University of Wyoming.

Steffe, L. P., Cobb, P., & von Glasersfeld, E. (1988). *Construction of arithmetical meanings and strategies.* New York, NY: Springer-Verlag.

Steffe, L. P., & Olive, J. (2010). *Children's fractional knowledge.* New York, NY: Springer-Verlag.

Steffe, L., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education,* (pp. 267–306). Mahwah, NJ: Erlbaum.

Steinthorsdottir, O. B., & Sriraman, B. (2009). Icelandic 5th-grade girls' developmental trajectories in proportional reasoning. *Mathematics Education Research Journal, 21*(1), 6–30. doi:10.1007/BF03217536

Stephan, M., & Akyuz, D. (2012). A proposed instructional theory for integer addition and subtraction. *Journal for Research in Mathematics Education, 43*(4), 428–464. doi:10.5951/jresematheduc.43.4.0428

Stephan, M., & Rasmussen, C. (2002). Classroom mathematical practices in differential equations. *The Journal of Mathematical Behavior, 21*(4), 459–490. doi:10.1016/S0732-3123(02)00145-1

Stephens, M., & Armanto, D. (2010). How to build powerful learning trajectories for relational thinking in the primary school years. In L. Sparrow, B. Kissane, & C. Hurst (Eds.), *Shaping the future of mathematics education: Proceedings of the 33rd annual conference of the Mathematics Education Research Group of Australasia.* (pp. 523–530). Fremantle, Australia: MERGA. Retrieved from http://files.eric.ed.gov/fulltext/ED520968.pdf

Stevens, S. Y., Delgado, C., & Krajcik, J. S. (2009). Developing a hypothetical multi-dimensional learning progression for the nature of matter. *Journal of Research in Science Teaching, 47*(6), 687–715.

Strauss, A. L. (1987). *Qualitative analysis for social scientists.* New York, NY: Cambridge University Press.

Suh, J., & Seshaiyer, P. (2015). Examining teachers' understanding of the mathematical learning progression through vertical articulation during lesson study. *Journal of Mathematics Teacher Education, 18*(3), 207–229. doi:10.1007/s10857-014-9282-7

Tabach, M., Hershkowitz, R., Rasmussen, C., & Dreyfus, T. (2014). Knowledge shifts and knowledge agents in the classroom. *The Journal of Mathematical Behavior, 33,* 192–208. doi:10.1016/j.jmathb.2013.12.001

TERC (2008). *Investigations in number, data, and space.* Upper Saddle River, NJ: Pearson/Scott Foresman.

Thompson, P. W, Carlson, M. P., Byerley, C., & Hatfield, N. (2014). Schemes for thinking with magnitudes: A hypothesis about foundational reasoning abilities in algebra. In K. C. Moore, L. P. Steffe, & L. L. Hatfield (Eds.), *Epistemic algebra students: Emerging models of students' algebraic knowing,* WISDOMe Monographs (Vol. 4, pp. 1–24). Laramie, WY: University of Wyoming.

Tillema, E. S. (2014). Students' coordination of lower and higher dimensional units in the context of constructing and evaluating sums of consecutive whole numbers. *The Journal of Mathematical Behavior, 36,* 51–72. doi:10.1016/j.jmathb.2014.07.005

Tobias, J. M. (2009). *Preservice elementary teachers' development of rational number understanding through the social perspective and the relationship among social and individual environments* (Doctoral dissertation). University of Central Florida, Orlando. Retrieved from http://etd.fcla.edu/CF/CFE0002737/Tobias_Jennifer_M_200908_PhD.pdf

Toulmin, S. E. (2003). *The uses of argument.* Cambridge, United Kingdom: Cambridge University Press. (Original work published in 1958)

Tzur, R. (1999). An integrated study of children's construction of improper fractions and the teacher's role in promoting that learning. *Journal for Research in Mathematics Education, 30*(4), 390–416. doi:10.2307/749707

van den Heuvel-Panhuizen, M. (2003). The didactical use of models in realistic mathematics education: An example from a longitudinal trajectory on percentage. *Educational Studies in Mathematics, 54*(1), 9–35. doi:10.1023/B:EDUC.0000005212.03219.dc

Vermont Mathematics Partnership's Ongoing Assessment Project. (2013). *OGAP proportional reasoning framework.* Montpelier, VT: Author. Retrieved from http://margepetit.com/wp-content/uploads/2015/04/OGAPProportionalFramework10.2013.pdf

Vermont Mathematics Partnership's Ongoing Assessment Project. (2014a). *OGAP fraction framework.* Montpelier, VT: Author. Retrieved from http://documents.routledge-interactive.s3.amazonaws.com/9781138816442/OGAP_Fraction_Framework.pdf

Vermont Mathematics Partnership's Ongoing Assessment Project. (2014b). *OGAP multiplicative framework.* Montpelier, VT: Author. Retrieved from http://margepetit.com/wp-content/uploads/2015/04/OGAP-Multiplicative-Framework-6.2014.pdf

von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning* (Studies in Mathematics Education Series, Vol. 6). London, United Kingdom: Falmer.

Wang, Y., Barmby, P., & Bolden, D. (2015). Understanding linear function: A comparison of selected textbooks from England and Shanghai. *International Journal of Science and Mathematics Education,* 1–23. doi:10.1007/s10763-015-9674-x

Wawro, M. J. (2011). Individual and collective analyses of the genesis of student reasoning regarding the invertible matrix theorem in linear algebra (Doctoral dissertation). Retrieved from https://escholarship.org/uc/item/30g0b1g1

Weber, E., & Lockwood, E. (2014). The duality between ways of thinking and ways of understanding: Implications for learning trajectories in mathematics education. *The Journal of Mathematical Behavior, 35,* 44–57. doi:10.1016/j.jmathb.2014.05.002

Weber, E., & Thompson, P. W. (2014). Students' images of two-variable functions and their graphs. *Educational Studies in Mathematics, 87,* 67–85. doi:10.1007/s10649-014-9548-0

Weber, E., Walkington, C., & McGalliard, W. (2015), Expanding notions of "learning trajectories" in mathematics education. *Mathematical Thinking and Learning, 17*(4), 253–272. doi:10.1080/10986065.2015.1083836

Weber, K., Maher, C., Powell, A., & Lee, H. S. (2008). Learning opportunities from group discussions: Warrants become the objects of debate. *Educational Studies in Mathematics, 68*(3), 247–261. doi:10.1007/s10649-008-9114-8

White, R. T. (1974). The validation of a learning hierarchy. *American Educational Research Journal, 11*(2), 121–136. doi:10.2307/1161783

Wilson, M. (2009). Measuring progressions: Assessment structures underlying a learning progression. *Journal of Research in Science Teaching, 46*(6), 716–730. doi:10.1002/tea.20318

Wilson, M., & Carstensen, C. (2007). Assessment to improve learning in mathematics: The BEAR assessment system. In A. Schoenfeld (Ed.), *Assessing mathematical proficiency* (pp. 311–332). London, United Kingdom: Cambridge University Press.

Wilson, P. H. (2009). Teachers' uses of a learning trajectory for equipartitioning (Doctoral dissertation). Retrieved from http://repository.lib.ncsu.edu/ir/handle/1840.16/2994

Wilson, P. H., Mojica, G. F., & Confrey, J. (2013). Learning trajectories in teacher education: Supporting teachers' understandings of students' mathematical thinking. *The Journal of Mathematical Behavior, 32*(2), 103–121. doi:10.1016/j.jmathb.2012.12.003

Wilson, P. H., Sztajn, P., Edgington, C., & Confrey, J. (2014). Teachers' use of their mathematical knowledge for teaching in learning a mathematics learning trajectory. *Journal of Mathematics Teacher Education, 17*(2), 149–175. doi:10.1007/s10857-013-9256-1

Wiser, M., Smith, C. L., & Doubler, S. (2012). Learning progressions as tools for curriculum development. In A. C. Alonzo & A. W. Gotwals (Eds.), *Learning progressions in science: Current challenges and future directions* (pp. 359–403). Rotterdam, The Netherlands: Sense.

Wood, T., Williams, G., & McNeal, B. (2006). Children's mathematical thinking in different classroom cultures. *Journal for Research in Mathematics Education, 37*(3), 222–255. doi:10.2307/30035059

Yettick, H. (2015). Learning progressions: Maps to personalized teaching. *Education Week, 35*(12), S18–S19. Retrieved from http://www.edweek.org/ew/articles/2015/11/11/learning-progressions-maps-to-personalized-teaching.html