into practice

Chapter 1
Interpreting Variables and Expressions

Essential Understanding 1*a*
Expressions are powerful tools for exploring, reasoning about, and representing situations.

Big Idea 2. Variables

Variables are tools for expressing mathematical ideas clearly and concisely. They have many different meanings, depending on context and purpose.

Variables and expressions serve as the bridge by which students cross from arithmetic to algebra (Schoenfeld and Arcavi 1988). Although variables and expressions are pivotal concepts in both middle school and high school mathematics, they are topics that many students find challenging, and many, in fact, do not develop a thorough understanding of these topics. To support students in building a deeper understanding of these crucial concepts, teachers must allow students to work on numerous tasks set in many contexts and engage students in working with and discussing the different meanings of *variable* highlighted by Lloyd, Herbel-Eisenmann, and Star in *Developing Essential Understanding of Expressions, Equations, and Functions for Teaching Mathematics in Grades 6–8* (2011). This chapter supports teachers in initiating this important work by focusing on ways to help students come to a deeper understanding of variables and expressions, and, in particular, the idea that Lloyd, Herbel-Eisenmann, and Star (2011) capture in Essential Understanding 1*a* and Big Idea 2—that expressions and variables are powerful tools.

Working toward Essential Understanding 1*a* and Big Idea 2

Although variables are typically not introduced in a formal way until the middle grades, the school mathematics curriculum begins to acquaint students with the foundational ideas behind the concept of a variable much earlier. For example, a typical problem in the primary grades might be __ + 3 = 5. This same problem might also be found in a first-year algebra course, written as $x + 3 = 5$. In both instances, the equation includes a specific unknown quantity, represented by the blank or the x. That a variable may stand for a specific unknown value is one of the two meanings captured by Essential Understanding 2*b*: "Using variables permits writing expressions whose values are not known or vary under different circumstances" (Lloyd, Herbel-Eisenmann, and Star 2011). The Common Core State Standards for Mathematics (CCSSM) expect students in grade 6 to understand both of these meanings, enabling them to

> Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number, or, depending on the purpose at hand, any number in a specified set. (National Governors Association Center for Best Practices and Council of Chief State School Officers [NGA Center and CCSSO] 2010, 6.EE.6, p. 44)

Reflect 1.1 invites you to consider the effect of different ways of reading an expression with the variable x on your students' understanding of the meaning of x.

Reflect 1.1

Read the equation $x + 3 = 5$ aloud.

What are other ways to read this equation?

What impact might these different ways of reading the equation have on students' understanding of x as an unknown *quantity*?

Consider the following problem—the Pizza and Drink task:

> A slice of pizza costs p dollars and a drink costs d dollars. What does $3p + 2d$ represent?

This task asks students to define the meaning of a particular expression in a given context—purchasing slices of pizza and drinks. As you examine the task, think about how you would respond to the questions in Reflect 1.2.

Reflect 1.2

How do you think middle-grades students might respond to the question in the Pizza and Drink task?

What is the correct response to this question?

What are some incorrect ways in which middle-grades students might interpret the meaning of the coefficients, variables, and terms in the expression?

What strategies might you use to counter these errors?

The Pizza and Drink task was designed to reveal students' abilities to correctly interpret a mathematical expression. The task includes variables used purposefully to reveal whether students can correctly interpret the meaning of the variables as part of the mathematical expression. Although the variables are defined within the task, the task was designed with the anticipation that students' misconceptions related to the meaning of a variable might arise in their written description and, further, that their difficulties in understanding the meaning of the coefficients and the expression as a single quantity might show up as well.

A middle-grades student named Brady provided the response to the Pizza and Drink task shown in figure 1.1. In his thorough, correct response, Brady addressed several important aspects of the expression and the variables and terms in it.

First, Brady correctly interpreted the coefficient 3 as the number of slices of pizza and 2 as the number of drinks purchased. Second, he discussed the variables p and d as representing the cost of one slice of pizza and the cost of one drink, respectively. Further, he was able to determine the multiplicative relationship between the coefficient and the variable. Finally, Brady discussed the expression in its entirety as the total cost of purchasing the drinks and the slices of pizza. Of the 53 responses that we collected from students for this task, only 17 students (32.1%) attended to all of these aspects of the expression.

Interpreting the coefficients proved to be a common stumbling block for these students. Consider, for example, Wendy's response, shown in figure 1.2. Wendy acknowledged the meanings of the two terms in the expression and explained how she would be able to substitute a value for p and a value for d to find a total price. However, the explanation of $3p + 2d$ that Wendy gave was largely procedural—Wendy explained the process of working with the coefficients and terms in the expression but without communicating real understanding of what the coefficients represented.

A slice of pizza costs *p* dollars and a drink costs *d* dollars. What does 3*p* + 2*d* represent?

I think that 3p represent 3x the amount of each pizza and 3 is the number of pizza's there are. 2d represents the number of drinks and "d" represents the price of it. 2 represents the number of drinks. Therefore, this question is asking me or telling me that there are 3 pizzas at the amount and then 2 drink's multiplied by the price of the drinks so when you do this equation, the sum would be the total amount of money you would pay for 3 pizzas and 2 drinks. If each pizza costs \$2.50, and each drink costs a \$1.50, there would be 3 pizzas at \$2.50 so \$2.50 x 3 = \$7.50 for the cost of the pizza. Then there are 2 drinks at a price of 1.50 so \$1.50 x 2 = \$3.00 and the price of the drinks. If this equation had an ~~eq~~ equal sign, the total ~~number~~ price you would pay would be \$10.50.

$$\begin{array}{r} 2.50 \\ +2.50 \\ +2.50 \\ \hline 7.50 \end{array} \qquad \begin{array}{r} 1.50 \\ +1.50 \\ \hline 3.00 \end{array}$$

Fig. 1.1. Brady's correct and complete response to the Pizza and Drink task

A slice of pizza costs *p* dollars and a drink costs *d* dollars.
What does $3p + 2d$ represent?

3p + 2d. This represents the cost of the pizza multiplied by 3 & the cost of the drinks multiplied by 2. Once those were figured out then you would have to add them together. 3p - a number like this would stand for multiplication because any number that is placed by a variable must have to be multiplied by the variable or number. But when you have a variable next to a number, then you can automatically tell that you would have to multiply the two of them together. Since there are 3p & 2D, once you mutiply them together, then you would have to add them together because they have the addition sign in between them.

Fig. 1.2. Wendy's procedural explanation of the meaning of $3p + 2d$, including how to calculate with coefficients when given specific values for *p* and *d*

By contrast, 13 of the 53 students (24.5%) did explicitly discuss the meaning of the coefficients, focusing on multiplication, but their explanations of the meaning of the expression $3p + 2d$ were limited. For example, in the solution shown in figure 1.3, the student, Marco, noted that $3p$ and $2d$ are related to multiplication. However, his explanation was superficial; Marco referred only to the operation indicated by the terms and did not interpret the meaning of the terms or of the expression $3p + 2d$.

> A slice of pizza costs *p* dollars and a drink costs *d* dollars.
> What does 3*p* + 2*d* represent?
>
> 3P, stands for 3 pizzas multiplied by P dollars.
> 2d, represents 2 drinks multiplied by d dollars.

Fig. 1.3. Marco's response to the Pizza and Drink task, including explanations of the coefficients

Another common misconception that emerged in students' responses was the notion that the variables represented pizzas (or slices of pizza) and drinks purchased, even though the task clearly defined the variables p and d as the *cost* of a slice of pizza and the *cost* of a drink, respectively. Figure 1.4 shows work from one student, Kennedy, who had this misconception. Students who made this error interpreted the variables p and d as representing objects—p for a pizza (or slice of pizza) and d for a drink—rather than as quantities. Of the 53 students, 17 (32.1%) interpreted the terms in this manner.

> A slice of pizza costs *p* dollars and a drink costs *d* dollars.
> What does 3*p* + 2*d* represent?

> I made 3 pizza's and 2 drinks. Since the p (pizza) and the d (drinks) I made 3 pizza and 2 drinks. Since the 3p = 3 pizza and then add 2d = 2 drinks.

Fig. 1.4. Kennedy's work on the Pizza and Drink task, mistaking the coefficients for the numbers of pizzas and drinks

As you may have observed in your own teaching, students frequently select the first letter of the word in a problem context as the variable to use in representing the problem situation abstractly and symbolically. Depending on the problem's context, this practice may result in the number of cows being abstracted and represented symbolically as c, often misunderstood as *cows* rather than *number of cows*, or in the number of days being represented as d, often misunderstood as *days* rather than as *number of days*. This practice often tends to objectify the variable in the student's mind rather than to emphasize its representation of a quantity.

To address this common misunderstanding, teachers can encourage students to use symbols that are different from the first letters of the word in the problem statement when they are creating expressions with variables that represent unknown quantities. If students were to use n, for instance, to represent *the number of cows*, they might not be as prone to interpret expressions such as $4n$ as 4 cows. Avoiding writing or saying things such as "c represents cows," when what is really meant is that c represents the *number* of cows, can also be useful. By the same token, in establishing what the variable represents, saying, for example, "k represents the *number* of miles," is valuable. Such minor adjustments can go a long way toward dispelling this misconception.

When students think of variables as objects, they may also develop a limited understanding of multiplicative relationships among coefficients and variables. The term $3p$ indicates 3 times the quantity p. However, if students hear the term discussed only as "$3p$," they may develop a notion that the term indicates three known objects rather than the product of an unknown quantity and the particular value 3. The p in this case is incorrectly interpreted as a particular item, such as a pizza, rather than a quantity, such as the cost of a slice of pizza. This incorrect assumption is connected with similar statements that students hear daily, such as "3 apples," "5 dogs," and so on. In these cases, the apples and dogs are units rather than unknown or varying quantities. Thus, students may tend to consider a particular number of objects rather than a product of a number and an unknown or variable quantity.

This particular misunderstanding of terms also stems from students' early experiences with letters of the alphabet used as symbols in mathematics. Consider students' most common early encounters with letter symbols used as abbreviations, such as m, g, L, and so forth, in elementary school. In such instances, it is quite appropriate for students to see the symbol as an object with the numeral telling how many; they correctly see "4 m," for example, as meaning "4 meters." In this case, "m" is simply an abbreviation for the unit of measure and is not an unknown quantity.

Other problems arise in later grades when students encounter symbols such as π, e, and i, each of which is a specific number represented by a letter. These experiences underscore the need to be aware of these potential conflicts in meaning and the importance of helping students develop a deep understanding of variables in the middle grades.

The previous discussion has focused on common misunderstandings and misconceptions that students have about a variable and its coefficient. Similar misconceptions may arise when students encounter a variable in an expression that includes other operations besides multiplication. For example, in the task shown in figure 1.5, the expression $k - 2$ could be read as "k minus 2," a way of reading the expression that emphasizes the operation of subtracting. Or, alternatively, it could be read as "a number that is 2 less than k," a reading that emphasizes the result (Booth 1988). The expression $k - 2$ represents a quantity that has a specific relationship to k. Reflect 1.3 explores the impact of the two ways of reading the expression on students' approaches to a task involving $k - 2$.

Mark the number line below where you would expect to find $k - 2$. Explain how you decided where to place your mark.

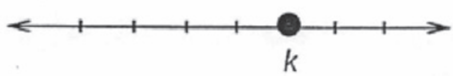


Fig. 1.5. The $k - 2$ task, used to determine students' ability to identify the point on the number line for the given expression

Reflect 1.3

Figure 1.5 presents a task that shows a number line with *k* marked and asks students to mark *k* – 2 on the line. Examine the task, and consider reading "*k* – 2" in two ways: as "*k* minus 2" and as "a number that is 2 less than *k*."

How might each way affect students' interpretations of the expression and their abilities to solve the task?

What do you think might be some solutions that students would offer to the task?

The task in figure 1.5 was selected for several reasons. First, it affords an opportunity to uncover students' difficulties in interpreting the relationship given by the expression $k - 2$. Second, the task presents a variable that can be interpreted in more than one way—a feature of the task that we discuss later. First, however, to focus on the light that the task can shed on students' difficulties in interpreting $k - 2$, consider Joaquin's response, shown in figure 1.6. Joaquin's response is representative of responses from 13 of the 53 students (24.5%) who completed the task. These students indicated that they were either unsure whether the "– 2" referred to subtraction or whether it served as a coefficient for k. This confusion led these students to place the quantity $k - 2$ in various incorrect positions along the number line.

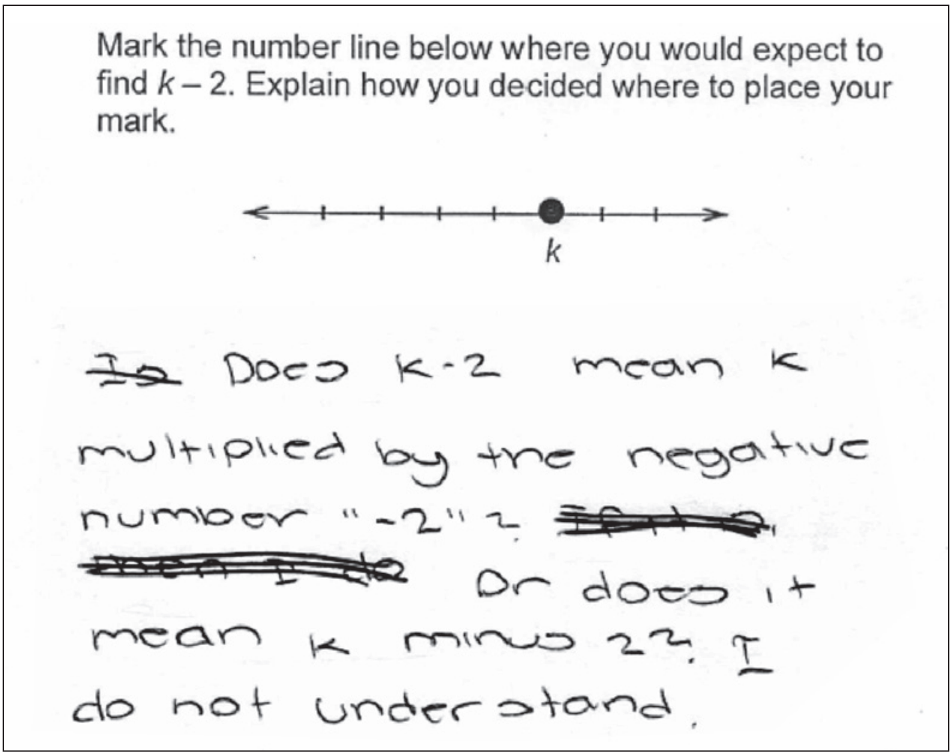


Fig. 1.6. Joaquin's response to the $k - 2$ task, demonstrating uncertainty about how to interpret the expression

As discussed earlier, one way to address this confusion is to attend carefully to how expressions are read and written out in words. CCSSM addresses the interpretation and evaluation of expressions with variables in the following standard for students in grade 6: "Write, read, and evaluate expressions in which letters stand for numbers" (NGA Center and CCSSO 2010, 6.EE.2, p. 43). In the case of $k - 2$, students could interpret the expression as "the difference of k and 2," "k take away 2," "the value that is 2 less than k," and so on. Students' flexibility in negotiating the various meanings depends on how routinely they encounter and use multiple descriptions in discussions and written tasks.

Tasks such as the $k - 2$ task are effective in encouraging students to move from known values to a generalization using variables. These tasks can also help students to develop a better understanding of the operation or operations involved before generating the written expression.

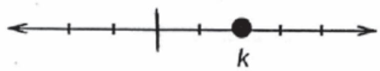
As noted earlier, a second reason for selecting this task is that the variable can be interpreted in more than one way. The variable k might represent a varying quantity, or it might be a specific unknown. This ambiguity provides a special opportunity to examine students' interpretations of the variable and their ability to discuss their thinking. The responses from this group of 53 students suggested that some students had trouble thinking of the variable in terms of either definition. Students commonly assigned the variable a value of 0, as shown in Lucy's and Kevin's responses in figure 1.7, instead of supposing that the value of k was unknown or could vary.

When students are first starting to reason about expressions, they commonly substitute specific values for the variables, and in fact they are often encouraged to do so. In this case, many students assigned k the specific value of 0 either before or after determining where to place $k - 2$. Lucy's response (fig. 1.7a) provides some evidence that Lucy understood the relative value of k and $k - 2$, since she placed a mark two units to the left of k and wrote that "you have to go 2 down the number line." She went on, however, to assign k a value of 0, although why she thought this was necessary to do is unclear. Kevin (fig 1.7b), worked the other way around, first assigning k a value of 0 and then reasoning that he needed to go "2 spots before it."

In responding to errors such as these, you might ask your students how they determined that k was 0 and have them reread the question carefully. Sometimes this is all that is necessary to prompt students to reevaluate their thinking. You might also ask whether -2 and $k - 2$ are equivalent, particularly in instances such as Kevin's solution. This question might help students to think through other quantities that $k - 2$ could represent.

Another strategy that might motivate students to rethink their solutions is to present several different solutions to the class and open a small-group or whole-class discussion regarding the merits of these solutions. This technique would offer students the opportunity to justify and critique their own thinking as well as their classmates' thinking. In so doing, students might be exposed to alternative solutions that counter their own, leading them to correct their own errors.

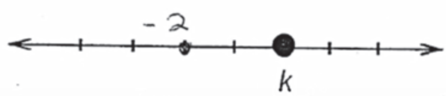
Mark the number line below where you would expect to find $k - 2$. Explain how you decided where to place your mark.



I think that the mark would go where I put it because since it is k-2, you would have to go 2 down the number line. So, I counted 2 down from k and marked where I stopped. Since I put my mark there, k would be equal to 0

(a) Lucy's response

Mark the number line below where you would expect to find $k - 2$. Explain how you decided where to place your mark.



I placed my mark 2 spots before k because if K was a zero than -2 would be 2 spots before it. This method works because if you decide what K stands for you know where to place the point from there. I knew to place -2 2 spots behind K because on a number line negative numbers go behind 0 and positive numbers go after 0. So, since I established that K = 0, -2 would be 2 spaces after K.

(b) Kevin's response

Fig. 1.7. Responses to the *k* – 2 task from two students, (a) Lucy and (b) Kevin, who assigned *k* a value of 0

Strategies such as these can help students develop metacognitive strategies that are crucial to their success in mathematics. Reflect 1.4 encourages you to consider ways in which you could extend the $k - 2$ task to help students think more flexibly about the value of k.

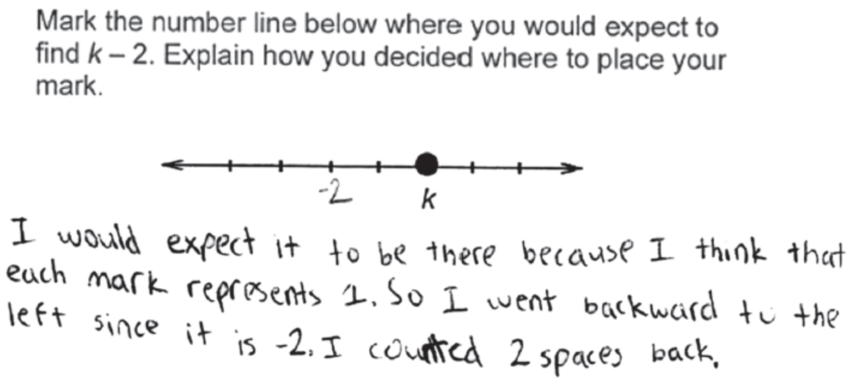
Reflect 1.4

How could you extend the $k - 2$ task to help students in grades 6–8 consider the value of k specifically as a varying quantity?

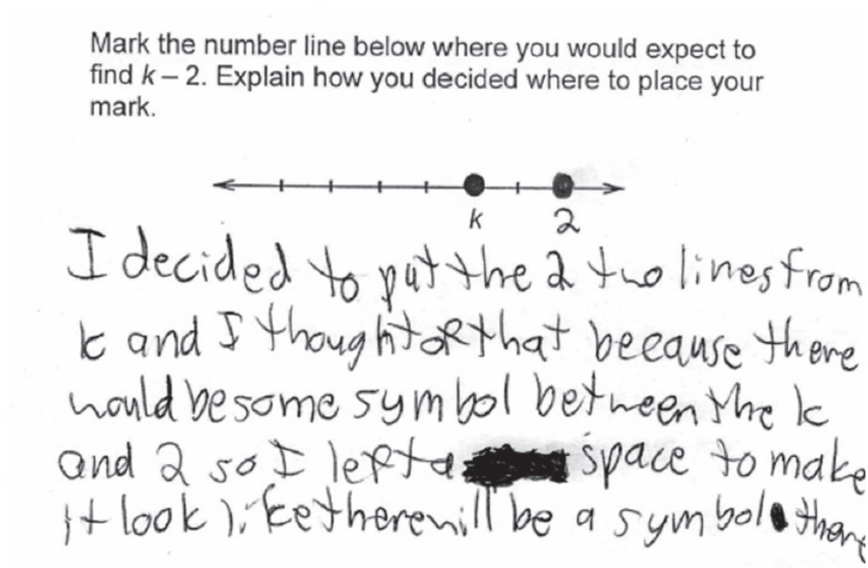
How would you support their development in thinking about generalized quantities?

In this group of 53 students, the $k - 2$ task uncovered a common and potentially detrimental line of reasoning. This reasoning stemmed from considering the expression $k - 2$ as two separate terms rather than as a single quantity. Many students first considered the k and then considered the 2. Their interpretations of the meaning of the 2 varied. In some cases, students interpreted it as the subtraction of 2; in others, as the number –2; and in others, as positive 2—all without ever discussing the single quantity $k - 2$. This type of reasoning tended to focus on two separate, seemingly unrelated quantities, as was evident in the students' omission of a label "$k - 2$" on the number line. Instead, many students counted from the point labeled as k and then simply added a label for –2 or 2 at a second point rather a label for $k - 2$. Figure 1.8 presents two examples of such thinking in the responses of Monica and Daniel.

Interpretations of $k - 2$ like Monica's and Daniel's do not support the growth of students' understanding of an expression as a powerful tool for representing situations, as emphasized in Essential Understanding 1*a*. Such interpretations show a lack of awareness of the quantity that is 2 less than k and instead give evidence of a less sophisticated focus on each term individually. By probing student work further and providing opportunities for students to share ideas, you may be able to counter these kinds of approaches and interpretations. Encouraging students to avoid using "negative two" and "minus two" interchangeably can also be worthwhile because doing so perpetuates students' confusion about whether the minus sign indicates an operation or is part of a coefficient of negative two, as previously discussed.



(a) Monica's response



(b) Daniel's response

Fig. 1.8. Responses from two students, (a) Monica and (b) Daniel, who did not identify the quantity $k - 2$ but instead marked the points –2 and 2, respectively

To investigate students' understanding of this task further, teachers can present other problem situations. For example, students might describe a situation for which the expression $k - 2$ would be an appropriate model. Students' responses to this type of question can reveal how deeply they understand this expression. In the case of $k - 2$, students might say something like, "I had some money and gave away two dollars," or "This was the amount of money I had left after I gave away two dollars." These two responses provide opportunities to discuss interpreting the expression as an operation or as a result. Showing students a variety of responses from their classmates or others to these types of questions can give students additional exposure to the various meanings of expressions and uses of variables.

Focusing on variables and expressions in the transition from arithmetic to algebra presents a wealth of opportunities to engage students in the eight practices and processes emphasized in CCSSM's Standards for Mathematical Practice (NGA Center and CCSSO 2010). The practices of reasoning abstractly and quantitatively, modeling with mathematics, looking for and making use of structure, and looking for and expressing regularity in repeated reasoning (practices 2, 4, 7, and 8, respectively; see fig. 1.9) are especially well aligned with the tasks and ideas presented in this chapter.

Common Core State Standards for Mathematics

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Fig. 1.9. CCSSM Standards for Mathematical Practice (NGA Center and CCSSO 2010, pp. 6–8)

As you design tasks for your students, pay close attention to ways in which these tasks might bring out the important ideas of algebra, such as doing and undoing thinking and generalizing mathematical ideas. Ensuring that the tasks emphasize these practices and ideas will help your students build a strong foundation in early algebra, making the notoriously difficult transition to algebra more natural.

Summarizing Pedagogical Content Knowledge to Support Essential Understanding 1*a* and Big Idea 2

Teaching the mathematical ideas discussed in this chapter requires specialized knowledge related to the four components presented in the Introduction: learners, curriculum, instructional strategies, and assessment. The four sections that follow summarize some key ideas for each of these components in relation to Essential Understanding 1*a* and Big Idea 2. Although we separate the components to highlight their importance, we also recognize that they are connected and support one another.

Knowledge of learners

Understanding variables and expressions is foundational to students' transition from arithmetic to algebra, but students frequently form a number of misconceptions about these concepts. For example, students tend to draw on their prior experiences with letters of the alphabet used in mathematics as abbreviations for units, and often this connection prevents them from reasoning about variables and expressions as quantities. Students may also misinterpret the meaning of coefficients when interpreting and evaluating expressions. Further, when considering expressions, students may see a collection of discrete quantities without making any connection to the single quantity to which the expression refers. Teachers should think about the common misconceptions discussed in this chapter and work to counter them through the selection and enactment of purposefully chosen mathematics tasks.

Knowledge of curriculum

The definition of a variable most commonly found in textbooks is as an unknown quantity. Problems in these textbooks should be supplemented with tasks that also present variables as varying quantities or require students to think of variables in terms of generalized quantities with no specific values attached to them. Furthermore, teachers should encourage students to read and write expressions in ways that promote the notion of expressions and variables as quantities rather than as objects or units.

Knowledge of instructional strategies

The careful use of language is a critical consideration in helping students develop understanding of variables and expressions. Teachers should vary their descriptions of expressions, focusing on both expressions as operations and expressions as quantities. Students should be presented with multiple interpretations of expressions in both written and spoken forms to help them become accustomed to viewing expressions in both ways. Finally, teachers should routinely present students with multiple interpretations of expressions, both correct and incorrect, to foster mathematical discussions, counter misconceptions, promote students' justification of their own reasoning, and encourage students to develop metacognitive strategies for checking their understanding.

Knowledge of assessment

Assessments should be designed to allow teachers to determine students' understanding of expressions and variables as single quantities. Students should be asked not only to determine an expression for a given scenario but also to determine a scenario for a given expression. Designing assessment items to reveal and challenge common misconceptions, such as the notion of a variable as a unit rather than a quantity, can be useful as well as constructive. Teachers might ask students to define an expression such as $3p + 2d$ in the Pizza and Drink task (see fig. 1.1) to gain insight into misconceptions held by students. Varying the wording on assessments is an effective way to promote student flexibility in understanding the various ways to interpret expressions.

Conclusion

To help students develop a deep understanding of variables and expressions, teachers must provide students with opportunities to discuss and define these mathematical tools in the classroom. The language of instruction is particularly important for countering student misconceptions and fostering a nuanced understanding of these powerful tools and important concepts. A deep understanding of these concepts will form the basis for further work with expressions and equations, as discussed in Chapter 2.