Chapter 2

Connections: Looking Back and Ahead in Learning

As the NCTM (2000) Content Standards for geometry indicate, geometry not only is an area of mathematical study in its own right but also is connected—through the use of imagery and diagrams as well as specific ideas and results—to ideas and problems across mathematics. Big Idea 2, which captures the importance of perceiving and working with invariance across variation, is the primary insight that we draw on in chapter 2, where we look at the development and connections of geometric thinking both *horizontally*, across other areas of mathematics taught in the high school years, and *vertically*, in the years leading up to and beyond grades 9–12.

Big Idea 2

Geometry is about working with variance and invariance, despite appearing to be about theorems.

Big Idea 4, which recognizes written proof as the culmination of the process of arguing and explaining, extends across content strands and grades in important ways that are treated in separate volumes in the Essential Understanding Series. For details on conjecturing, generalizing, and reasoning across mathematical contexts and grade levels, see *Developing Essential Understanding of Proof and Proving for Teaching Mathematics in Grades 9–12* (Ellis, Bieda, and Knuth, forthcoming) and *Developing Essential Understanding of Mathematical Reasoning for Teaching Mathematics in Prekindergarten–Grade 8* (Lannin, Ellis, and Elliott 2011).

Big Idea 4

A written proof is the endpoint of the process of proving.

Thinking through invariance can involve reframing questions and ways of looking at situations to highlight what is changing and what is staying the same despite other things changing. Although algebraic work employs the notion of variable (often designated by x) to allow working on all instances of something at the same time, geometry has no corresponding symbol. Even though geometers work on whole classes of figures at the same time (e.g., all right triangles, all rectangles with a given perimeter, all angles subtended by the diameter of a circle) and rarely, if ever, work only on a particular figure or configuration, the static geometric diagram is not a counterpart to the algebraic x.

In algebra, variables such as x take on many meanings—for details, see *Developing Essential Understanding of Equations, Expressions, and Functions for Teaching Mathematics in Grades 6–8* (Lloyd, Herbel-Eisenmann, and Star 2011).

DGEs make apparent the specificity and particularity of a single instance of a configuration by allowing its deformation into another like one at the touch of a mouse. Dragging emphasizes the continuity of figures within families subject to the same constraints, such as that *this* point must always lie on *that* line, and the center of *this* circle must always fall on *that* perimeter.

Looking Horizontally at High School Mathematics

In keeping with our approach in chapter 1, we have opted to indicate connections that extend across high school mathematics by means of two case studies: the case of coordinate geometry and the case of trigonometry.

Coordinate geometry

Measurement as attention to attributes of objects is discussed in *Developing Essential Understanding of Number and Numeration for Teaching Mathematics in Prekindergarten–Grade 2* (Dougherty et al. 2010).

We have chosen to include coordinate geometry as a horizontal *extension* in high school geometry, even though it is often taken as an integral part of the geometry strand (as in the NCTM Standards [2000]). We have done so because coordinate geometry provides, in essence, an algebraic way of working with geometric shapes. In this sense, coordinate geometry is not dissimilar to measurement, which assigns numbers based on units to attributes of a geometric object and involves formulas for relating these numbers to, say, an object's area or perimeter. Coordinate geometry marks every point in the plane by an (x, y) coordinate pair and, through this assignment, enables us to work numerically and algebraically with segments (e.g., using the distance formula) and angles (e.g., using the slopes of lines to determine whether they form a right angle).

A major use of the coordinate system in the high school geometry curriculum is in the representation of transformations. For example, it turns the geometric fact that a point and its image, when reflected across the y-axis, are equidistant to the line of reflection into the algebraic fact that the coordinates of the reflected image of a point (x, y) is $(-x, y)$. For special cases, the switch provides nice results, though for others (e.g., rotation around a point that is not the origin), the relationship between the coordinates of a point and its image is far less informative. The use of coordinate geometry to work with transformations can provide a good background for working with matrices in the context of linear transformations, since matrices offer yet another way of representing certain transformations of the plane—a topic that we discuss later in this chapter.

The historical motivation for the coordinate system was to channel the computational power of algebra into geometric problems. A geometric idea could be transformed into algebraic terms that would be easier to work with, and then the result could be brought back to bear on the geometric configuration. Consider the set of questions that we posed in chapter 1 about the intersection of a line with a circle. It seemed visually apparent that this could happen only twice, once, or not at all. But how might we prove this? One way would be to transfer these objects (the line and the circle) into the algebraic frame of systems of linear and quadratic equations in x and y whose solution sets lie in the coordinate plane. To find the common solutions, we can set two expressions for y equal to each other and solve for x. Given that we always obtain a quadratic equation (if we believe the algebra), we can show that there are, indeed, no real number solutions, one, or two. Taking this computation back to the geometric situation, we can now say something about the line and the circle in general. It would be a loss to treat coordinate geometry as a way of moving *away from* geometry, when it can be so useful in working *with* geometry as well.

Trigonometry

One setting in which triangle similarity is very significant is in the definition of the six initial trigonometric functions: sine, cosine, tangent, cotangent, secant, and cosecant. The very word *trigonometry* suggests the "-metry" (measurement) of "trigons" ("three-angles"), a plausible name for triangles and one that is more consistent with the common naming scheme for most other plane shapes.

Sine and cosine of a given angle are usually specified first, in terms of ratios of side lengths of a right triangle containing the particular angle. So, if the angle is itself the starting place (see fig. 2.1), an infinite number of right triangles contain the desired angle. Why does it not matter which one we use?

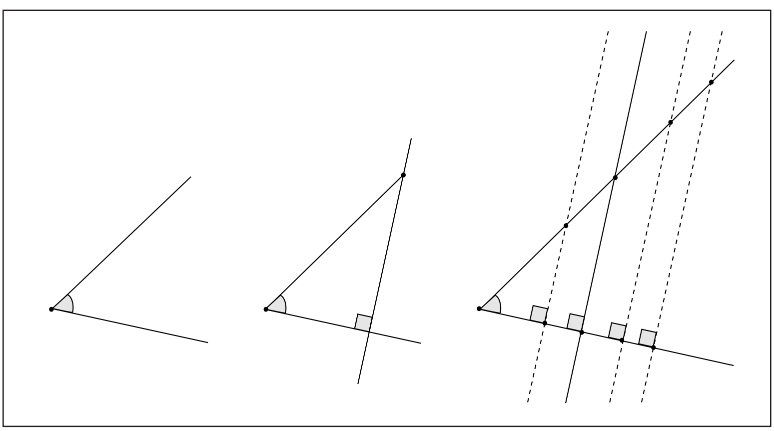

Fig. 2.1. An angle and an associated family of right triangles

Because these triangles are all similar, one to another, the ratios of corresponding sides are always the same. Because all six of the trigonometric functions are defined as side-length ratios, it does not matter which right triangle we use. So the whole of trigonometry rests on similar triangles and their properties.

Additional discussion of trigonometric functions and their applications appears in *Developing Essential Understanding of Functions for Teaching Mathematics in Grades 9–12* (Cooney, Beckmann, and Lloyd 2010).

An alternative name for the family of trigonometric functions is the *circular functions*, because deriving them from circles is relatively straightforward. There is an important right triangle that is related to the circle as shown in figure 2.2, and it explains where two of the six triangular functions' names come from. The Latin verb *tango* (from which the Argentinian dance derives its name) means "I touch," and a tangent is a "touch-line" of the circle—the term that Robert Record proposed. As mentioned earlier, the Latin verb *seco* means "I cut" (and is the source of the name of the scissor-like gardening tool *secateurs*, as well as the geometric term *sector*—the region of a circle cut off by a *secant* line). So a secant is a "cut-line" of the circle. If θ names the angle in the right triangle at the center of the circle, *and if the radius of the circle is 1 unit*, then the length of the segment shown in color on the tangent line in figure 2.2 is $\tan(\theta)$, and the distance along the secant line from the center of the circle to the tangent line is $\sec(\theta)$ (see the dashed segment shown in color in the figure). Notice that these are definitions by genesis.

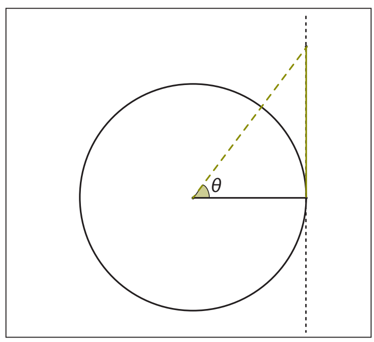

Fig. 2.2. The right triangle that defines the secant and the tangent, using a circle of radius 1

Inside any circle sits a second right triangle that is similar to the first one, as shown in figure 2.3a. This second triangle is more commonly drawn in textbooks and is often displayed without the circle from which it was generated, as in fig. 2.3c. Applying Pythagoras's theorem to each of these two triangles in turn (for the same reason, if the radius is chosen to be 1 unit, $\sin(\theta)$ and $\cos(\theta)$ are the lengths of the two non-radial sides of the smaller triangle) produces two of the common trigonometric identities:

$$1 + [\tan(\theta)]^2 = [\sec(\theta)]^2 \quad \text{and} \quad [\cos(\theta)]^2 + [\sin(\theta)]^2 = 1$$

Being familiar with similar triangles—and the length ratio properties that are invariant across them—is fundamental to high school trigonometry.

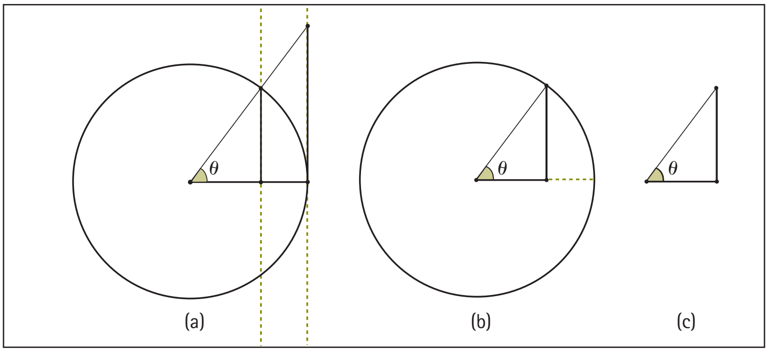

Fig. 2.3. The similar triangles that give rise to the trigonometric definitions

Combining the two perspectives discussed briefly in this section, the unit circle centered at the origin gives ($\cos(\theta)$, $\sin(\theta)$) as the coordinates of one vertex of the smaller right triangle in figure 2.3a and (1, $\tan(\theta)$) as the coordinates of the vertex of the larger right triangle in figure 2.3a. Using similar triangles, it is possible to derive the more common initial definition of $\tan(\theta)$ as the ratio of $\sin(\theta)$ to $\cos(\theta)$ *as a theorem*. Or alternatively, this argument shows the equivalence of the two definitions of $\tan(\theta)$: (1) the length of the segment of the tangent line that goes from the point of tangency (of the unit circle and the tangent line) to the intersection of the other secant line with the tangent line, and (2) the ratio of $\sin(\theta)$ to $\cos(\theta)$.

Looking Vertically at School Geometry

We use a different case—that of transformations in grades 6–8 and in postsecondary mathematics—to illustrate the "vertical" development of geometric thinking in mathematics.

Transformations in grades 6–8

The middle school geometry curriculum, as articulated in NCTM's (2000) Content Standards, provides opportunities for students to work with transformations. Students in grades 6–8 often describe their work colloquially, using terms such as "flips," "turns," "slides," and "zooming." They describe the new sizes, orientations, and positions of shapes under these transformations. Such descriptions are, for the most part, qualitative, and they remain in the visual register. In other words, students in the middle grades work with the fact that a "turn" does not change the orientation of a shape or its size by using a purely visual apprehension of the shape. If asked to identify the transformation linking one shape to its image, students draw on visual strategies. For example, given a shape and its rotated image, as in figure 2.4, students can *see* that one shape is the rotated image of the other, and they may be able to explain some of the reasons for this: the shapes are congruent, and the orientation has not changed. But these are property-based arguments and not definition-based ones.

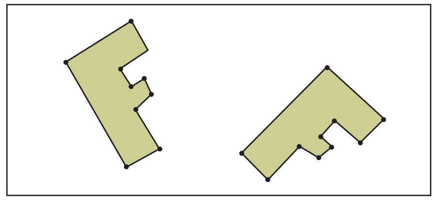

Fig. 2.4. A shape and its rotated image

Furthermore, given a shape, students may be able to sketch its reflected or rotated image. But this can be difficult, since it involves working very closely with the properties of the given transformation. This is particularly true for non-canonical configurations. So, for example, students are likely to find drawing the reflected image in figure 2.5a much easier then doing so in figure 2.5b, since they can use visual approximation to do the former task, but the latter requires them to give more attention to the properties of reflection (unless they turn the piece of paper around, making the line of reflection vertical!).

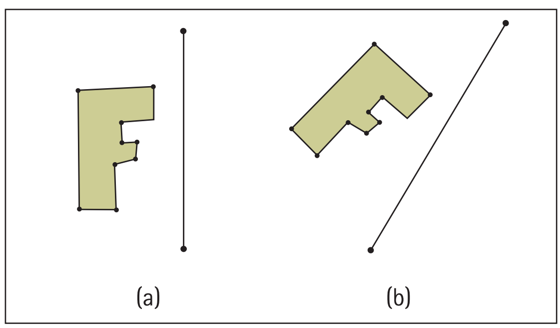

Fig. 2.5. Reflecting a shape across a line of reflection

Similarly, students might be able to identify a line of reflectional symmetry that a shape such as a rectangle or a heart has, but they will probably find it much more challenging to construct, say, the center and angle of rotation in figure 2.4 or the line of reflection in figure 2.6 (especially since it is oblique).

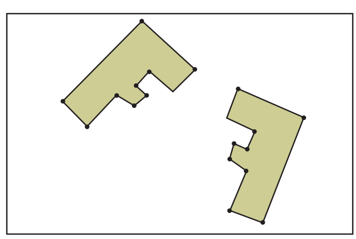

Fig. 2.6. An oblique line of reflection

Constructing the line of reflection involves working backward from the properties—that is, going from the fact that a point and its image have to be equidistant to a line to the idea that the line will therefore pass through the midpoint of the segment connecting the point and its image. Reflect 2.1 challenges you to identify the center of rotation in figure 2.4.

Reflect 2.1

Specifying the particular rotation shown in figure 2.4 requires identifying both the center of rotation and the angle of rotation. Try estimating each first. Which one is easier to identify? What properties of rotation are you using when you identify each one?

You might begin by recognizing that any point on the pre-image shape and its corresponding image point must lie on the same circle. But there are infinitely many circles whose centers pass through the perpendicular bisector of a pre-image point and its corresponding image point. So the center must be where all the perpendicular bisectors of a pre-image point and its corresponding image intersect. Once that point is identified, the angle comes easily. Can you find a way of identifying first the angle and then the center?

The goal of this discussion is to point to the ways in which middle school work on transformations focuses mainly on visual apprehension and on identifying properties. However, since these properties are grasped visually, they are not always available for use in constructing images or symmetries.

Because students are familiar with these transformations when they enter high school, it is important to offer tasks for which visual apprehension will not suffice. It may also be necessary for students to identify and describe these properties as invariances across a range of examples. So, for example, in a dynamic geometry environment, students can become aware of the properties of reflection by observing how a shape and its image behave as they drag the shape or the line of reflection on the screen. Measurements of lengths and angles can help students turn these observations into more precise statements (for instance, "The line of reflection is the same distance from A as it is from A'"). Construction challenges, using either straightedge and compass or a DGE, can enable students to act on these observations: if the line of reflection is the same distance from A as it is from A', then to find the line of reflection, they need to find the midpoint of the segment AA'. Construction demands a discursive interaction, and this is why it is a crucial part of developing a geometric discourse.

In high school, the focus is on bringing other means of representation to work with transformations (such as coordinates, vectors, functions, and matrices). If students begin this work without having developed a language-based understanding of transformations, then they will be more and more challenged by tasks that are resolvable only through an application of definitions.

For further discussion of transformations, see *Developing Essential Understanding of Geometry for Teaching Mathematics in Grades 6–8* (Sinclair, Pimm, and Skelin 2012).

In middle school, transformations are largely seen as processes that turn one shape into another, as opposed to a function—a mathematical object. Repeated attention to shape A turning into shape B casts A as the initial object and B as the final one; there is a directionality—an arrow of time "before" and "after"—and even the language of pre-image and (after-)image reinforces this. Although perhaps surprising, this means that it can be difficult to think of B as an initial object.

Additional discussion of the equals sign appears in *Developing Essential Understanding of Equations, Expressions, and Functions for Teaching Mathematics in Grades 6–8* (Lloyd, Herbel-Eisenmann, and Star 2011).

This need to recognize reversibility is similar to the situation encountered in the elementary grades with regard to addition. Repeated exposure to statements such as $3 + 4 = \square$ ends up casting the right-hand side of the equation as the endpoint of the operation on the left. This makes it difficult for students to know how to handle statements such as $\square = 3 + 4$, or $3 + 4 = \square + 2$.

In the case of transformations, the difficulty that students often have in thinking of B as an initial object becomes evident when they are asked to work with compositions of transformations. Because B was the reflection of A, B can be hard for students to see now as the object to be reflected again into C—not to mention trying to forget B in order to relate A to C! In fact, when working on the composition of transformations, the focus is the transformation itself and not the shape being transformed. In other words, whereas middle school geometry draws attention to the process of turning A into B, high school geometry turns the process into an object ("reflecting" into "a reflection") and then *does things* with that object (in this case, composes it). This is a very important shift, which is often invisible, since students are quite able to use the words correctly (such as *reflecting* and *reflection*), without using them in the same way as the textbook or the teacher.

Teachers need to support this objectification process—namely, the process of turning a process into an object—in several ways for students to work successfully both with composition and with other representations of transformations. One way, which we have already mentioned, is to help students develop a more language-based understanding of the transformations through work with constructions. Another way is to change the roles of A and B (so that A is not always turning into B) by asking questions such as, "Now that you have reflected A to find B, what would happen if you reflected B, using the same line of reflection?" or, "B is the result of reflecting A across this line of reflection, but A was erased; can you recover it?"

Similarly, even before beginning work on composition, which focuses on transformations as objects, teachers might allow students to undertake a chain of transformations, such as reflecting A to obtain B, then rotating B to get C, and so on. Here, the focus would still be on the process of transforming, but the chain of transformations would provide instances of B also being an initial shape instead of always being just an ending shape.

One final way of supporting the process of objectification might involve having students work with the collection of symmetries of certain shapes. By determining whether—and by how much—a given shape can be rotated and superimposed on itself, for example, students shift their attention away from the process of rotating to the number of symmetries that the shape has. They will find that the square (see fig. 2.7a) rotates by 90, 180, 270, and, of course, 360 degrees, thus yielding rotational symmetry of order 4. They will discover that the rectangle appears to have rotational symmetry only of order 2 (see fig. 2.7b). Reflect 2.2 invites you to explore the rotational symmetries of other shapes.

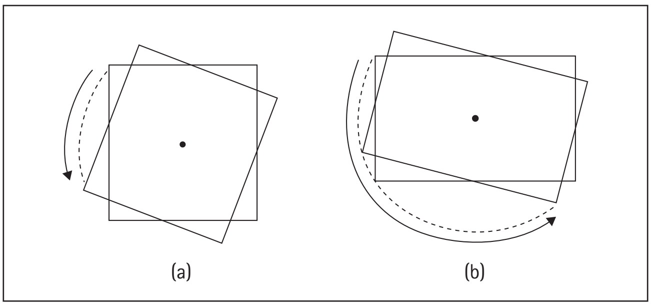

Fig. 2.7. Actions illustrating rotational symmetries of a square and a rectangle

Reflect 2.2

Explore the rotational symmetry of shapes other than the square and the rectangle. Can you find any shapes that have rotational symmetry of order 3? What about rotational symmetry of order 5?

We made an important choice here in working with the order of rotational symmetry—namely, that every shape has at least order 1 rotational symmetry, given that every shape overlaps itself after a rotation of 360 degrees. It would be natural to think a shape that has no rotational symmetry should have a rotational symmetry of order 0. But since it is convenient to define rotational symmetry of order n as rotations by an angle of $360°/n$ without changing the shape of the object, n cannot be allowed to be 0. This provides another small instance of where a definition is affected by a desire.

Yet more transformations: Linear algebra in postsecondary education

There are two main approaches to linear algebra: one is called "coordinate-free," and, we suppose, the other might be called "coordinate-full" (or "coordinate-expensive"). Students' first encounter with linear algebra is usually coordinate-full and full of matrix manipulations as well; a second encounter (usually in an abstract algebra course or possibly an honors linear algebra section) involves the study of vector spaces—and linear transformations of them—and matrices play a considerably smaller role.

Both types of linear algebra courses can be greatly enhanced by the use of geometry, especially in two and three dimensions. Earlier work in secondary school with transformations and isometries comes into play, providing both motivation and important imagery for what can otherwise, at times, degenerate into a mass of specific and uninformative calculations.

Shearing is a transformation that preserves area and parallelism, but not length, angle, or perpendicularity. See *Developing Essential Understanding of Geometry for Teaching Mathematics in Grades 6–8* (Sinclair, Pimm, and Skelin 2012).

Many intricate relations exist among the following related sets of ideas: isometries of the plane, 2×2 matrices, linear transformations of the plane, dilations, shears, and affine transformations of the plane. Unfortunately, in the short space we have here, we can only hint at some of the connections among these ideas and attempt to link them back to earlier points in this book and in *Developing Essential Understanding of Geometry for Teaching Mathematics in Grades 6–8* (Sinclair, Pimm, and Skelin 2012).

In what follows, think of a linear transformation as a transformation of the coordinate plane that preserves lines (i.e., collinearity relations) and, in particular, the origin. Examples of these transformations include rotating the plane around the origin, reflecting it across the line $y = x$, shearing parallel to the x- or y-axis, or dilating the plane with the center of dilation at the origin.

If we imagine the coordinate plane, a unit square has four key points that form the corners: (0, 0), (1, 0), (0, 1), and (1, 1). Under a general linear transformation, these points get mapped to (0, 0), (a, b), (c, d), and $(a + c, b + d)$. Notice that the origin always stays fixed, as every linear transformation maps the origin to itself. This means that neither non-identity translations nor glide reflections can be linear transformations, since they have no fixed points. The particular cases of rotations about the origin or reflections across a line that passes through the origin are linear transformations. Translations and glide reflections, rotations about a point that is not the origin, and reflections about a line that does not pass through the origin are all closely related to linear transformations and are examples of what are called *affine transformations*. An affine transformation is a linear transformation followed by a translation: in the case of a pure translation, the linear transformation involved is the identity transformation, which sends every point to itself. (As we mentioned in our discussion of Big Idea 4 in chapter 1, the identity transformation can be seen as a zero rotation or a zero translation, depending on definitions. But our preference is to see it as a zero rotation, not least because it is a linear transformation and has a simple matrix representation).

Big Idea 4

A written proof is the endpoint of the process of proving.

Notice that while in high school, transformations are typically categorized according to whether or not the transformation preserves size and shape; here, however, the categories of linear and affine transformations draw on very different properties. In other words, classification of isometries as linear or nonlinear transformations provides a different way to sort isometries from our usual categories of translations, reflections, rotations, and glide reflections. This alternative sorting is similar to, for example, sorting polygons into regular or non-regular ones instead of by means of our usual names for them (triangles, quadrilaterals, pentagons, and so on). It also makes one wonder whether there are other ways to categorize these transformations.

Under linear transformations in three dimensions, the unit cube is transformed to a parallelopiped or compressed or collapsed onto a plane through the origin, onto a line through the origin, or onto the origin itself.

Under any linear transformation, the origin is always mapped to the origin, lines through the origin remain lines through the origin, and, in general, the unit square is transformed to a parallelogram (see fig. 2.8). We add "in general" to our claim to take care of some "monster" examples. Some linear transformations compress the whole plane onto a line through the origin, and, in one extreme case, the linear transformation whose matrix has all zero entries compresses the whole plane onto the origin itself.

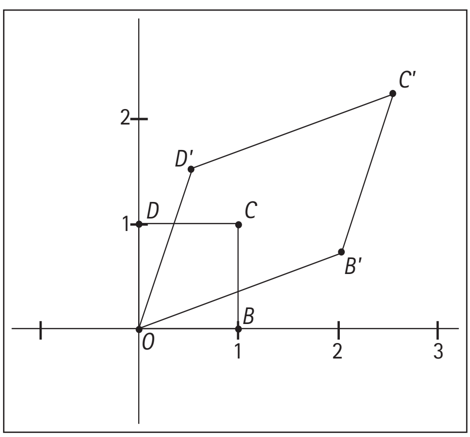

Fig. 2.8. Pre-image (unit square *OBCD*) and image (parallelogram *OB'C'D'*) under a linear transformation

Every linear transformation of the plane (equipped with the usual rectangular axes) can be associated with a 2×2 matrix, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and the diagram in figure 2.8 indicates that if we know