Chapter 4

Reasoning in Geometric Modeling

Knowledge that mathematics plays a role in everyday experiences is very important. The ability to use and reason flexibly about mathematics to solve a problem is equally valuable. These two come together in mathematical modeling to solve real-world problems. When a real-world situation calls for visualizing relationships between objects (such as distances between cell phone towers or the shape and dimensions of a hallway through which a sofa needs to be moved), the ability to develop and reason with a geometric model is crucial. Introducing high school students to problem situations within a real-world context and asking them to use their mathematical skills to model and solve the problem opens new avenues for learning mathematics. When students confront such problems, they are more likely to explore a range of solution methods, reason about the strengths of a particular method, search for connections among areas of mathematics, and employ various reasoning habits, such as *monitoring progress* and *reflecting* on the viability of a solution.

This chapter presents only one mathematical investigation—an expanded version of an example in *Focus in High School Mathematics: Reasoning and Sense Making* (NCTM 2009). In that book, Example 17: Clearing the Bridge (pp. 64–70) briefly outlines a sequence of tasks related to a situation frequently encountered in the world of commercial transport: a tractor-trailer truck becomes stuck under a bridge. This chapter's extended version of the original example provides a series of tasks that begin with the basic geometric setup of the situation. The tasks then increase in difficulty as the students explore the mathematics of the situation more deeply, including related trigonometry and function concepts. Each task has its own point of closure, but this juncture also leads naturally to more questions that fuel students' interest and can carry those with sufficient understanding through subsequent tasks. Students have opportunities to *make connections* throughout the discussion. The text specifies other relevant reasoning habits at the end of each task.

In the set of tasks that follow, as in modeling problems generally, students traverse the modeling cycle as they—

- encounter a real-world situation;
- build a mathematical model of the situation that includes relevant mathematical descriptors and assumptions—some of which will be simplifying assumptions;
- derive conclusions within the mathematical model;
- interpret these results in the real-world context.

The modeling process is often repeated as the model is adjusted, additional assumptions are made, or simplifying assumptions are removed.

The Problem: Clearing the Bridge

Mathematician Henry Pollak (2004) wondered why tractor trailers often got stuck under a certain underpass when the maximum clearance—the height from the roadway to the bridge—was clearly labeled on a sign. This is the situation that this chapter's investigation explores. Here, the bridge is level and located just at the base of a descending roadway, as illustrated in figure 4.1.

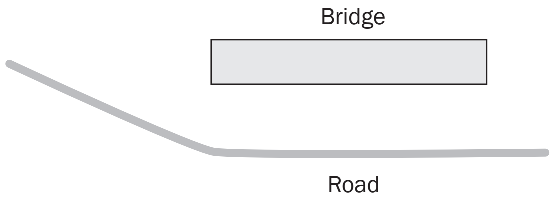


Fig. 4.1. A road descending a grade and passing under a bridge

Suppose that the truck driver knows the trailer height—that is, the distance from the top of the trailer to the road when the trailer sits on a flat road. Why might the trailer get stuck even if its height is less than the maximum clearance indicated on a sign such as that in figure 4.2? The next few tasks guide students in analyzing the situation through the use of a mathematical model.

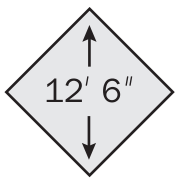


Fig. 4.2. A sign indicating maximum clearance

Many aspects of the problem can be approached on several levels of geometric thinking. For example, task 1, which calls for an initial model of the situation, could begin with a discussion of how to build a model of a truck trailer. This would involve students in geometric thinking at van Hiele levels 1 and 2 (Burger and Shaughnessy 1986). Level 1 entails the ability to reason about shapes and other geometric configurations according to their appearance as visual wholes. At level 2, students explicitly attend to, conceptualize, and specify shapes by describing their parts and spatial relationships among the parts. So, work on the Clearing the Bridge problem could start with a level 1 activity in which students draw a picture of a truck and represent it with a rectangular box sitting on the wheels. Students can use drawings created on paper or with interactive geometry programs to develop initial visual models of the situation.

As the tasks progress, they allow more advanced students to consider the range of variables and make appropriate adjustments to enhance the model. In addition, some aspects of the problem can be used as a springboard for some new topics (e.g., asymptotes, inverse functions, or parametric equations), as well as a context for applying students' previous knowledge. Although all the tasks presented in this chapter are not appropriate for all students, the mathematics involved in the first few tasks offers a rich introduction to geometric modeling and numerous possibilities for supporting geometric reasoning. (The "Field of Vision" problem in chapter 1 offers another example of the role that geometric modeling can play in problem solving.)

It is sometimes difficult in commercially available interactive drawing programs to alter lengths or move geometric objects slightly so that measurements are whole numbers. Such was the case in selecting an example of road grade measurement in degrees for this investigation. Although the selected measurement—25.7°—could be set to round to the nearest whole number, some greater precision is desirable in work on the problem. Thus, on the one hand, this number is a reasonable one for students to use; on the other hand, a grade of 25.7° is not realistic. However, the grade has been deliberately exaggerated to make useful geometric configurations more visible. The final task involves researching and using more realistic measurements for trailer length and road grade.

Task 1: Building an initial model

As is typical in mathematical modeling, students begin the modeling process with a simplified model. In this case, they represent the trailer with a two-dimensional view that pictures the trailer box as a rectangle. For the drawing, students assume that the trailer box is attached to front and back wheels at axle level, as shown in figure 4.3.

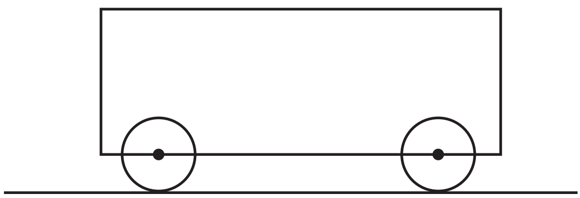


Fig. 4.3. A truck trailer attached to front and rear wheels at axle level

The students also make some assumptions about the wheels. (In this investigation, the term *wheel* includes both the rim and the tire.) First, they assume that the wheels are the same size. Second, they assume that the wheels do not flatten out on the bottom as the trailer sits on the roadway. That is, they assume that the wheels retain their circular shape while the truck sits on the roadway (see fig. 4.3). This is a simplifying assumption, since in reality tires that are correctly inflated leave a small, slightly extended footprint.

These assumptions about the trailer box and the wheels ensure that the top and bottom of the trailer box are parallel to the flat road. Having a trailer box level with the flat road is important for many reasons, including steering control, wear on tires, and load balance. However, reasoning from the assumptions to this result is not immediate, and teachers should explore it with students. Why do these assumptions lead to the fact that the top and bottom of the trailer box are parallel to the flat road?

In whatever format students encounter task 1 (class discussion, group work, or individual work), they should recognize that the assumption that the trailer wheels can be represented in the two-dimensional model by circles means that the wheels in the drawing would be tangent to the road. This means that the radius of each wheel would be perpendicular to the flat road, as indicated in figure 4.4.

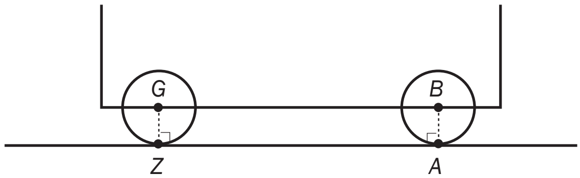


Fig. 4.4. With circles representing the wheels, the wheels are tangent to the roadway.

Because students assume that the wheels are the same size, the radii of the wheels would be the same length. That is, in figure 4.4, $AB = ZG$. (For simplicity, this investigation denotes the length of a line segment by the segment's "unmodified" letter symbols; for example, *AB,* without other specification, denotes "the length of the line segment *AB.*") The students' attention now turns to confirming that line *BG* is parallel to line *AZ*.

In Euclidean geometry, there are many ways to prove that lines *BG* and *AZ* are parallel. Students might focus on the quadrilateral *ABGZ*, first constructing diagonal *BZ*, as in figure 4.5, and reasoning as below:

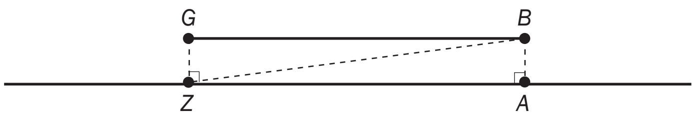


Fig. 4.5. Constructing $\overline{BZ}$ to aid in proving $\overline{BG} \parallel \overline{AZ}$

> In addition to being equal in length, sides *AB* and *ZG* are parallel because they both are perpendicular to line *AZ*. Hence, the alternate interior angles *GZB* and *ABZ* are congruent. Then it is easy to see that triangles *GZB* and *ABZ* are congruent by the side-angle-side postulate for triangle congruence. Hence, $\angle ZGB$ is a right angle. Then, since segments *BG* and *AZ* are both perpendicular to line *GZ*, they are parallel. So, line *BG* is parallel to line *AZ*. It also follows that $\angle GBA$ is a right angle and, thus, quadrilateral *ABGZ* is a rectangle.

Table 4.1 summarizes the key elements and reasoning habits (NCTM 2009, pp. 9–10, 55–56) that are illustrated by the students' work in task 1.

Table 4.1
Key Elements and Reasoning Habits Illustrated in Task 1

Key Elements of Reasoning and Sense Making with Geometry

- **Construction and evaluation of geometric arguments**
- **Geometric connections and modeling**

Reasoning Habits

- **Analyzing a problem**
  - *Seeking patterns and relationships*
  - *Looking for hidden structure*
- **Implementing a strategy**
  - *Making purposeful use of procedures*
  - *Making logical deductions*

Table 4.1—*Continued*

Reflecting on a solution
Justifying or validating a solution

Task 2: Identifying variables

Students now construct a two-dimensional representation of the situation that shows the danger as the trailer passes under the bridge. They make a list of the measurement items related to their representation that would be important to consider in analyzing whether or not the trailer will clear the bridge. Clearly, the height of the trailer is one, but only one, of the measurements that are important.

By reflecting and creating a visual model, students should easily see that if part of the road slopes, one set of trailer wheels is jacked up on the sloping part as the truck passes under the edge of the bridge. This causes a portion of the trailer to be raised higher than it would be on a flat surface and creates the danger that the trailer will get stuck. So the real question is, *How much higher has the trailer been raised at the point at which it passes under the bridge?* The representation that students (or groups of students) produce is likely to include at least some of the characteristics shown in the diagram in figure 4.6.

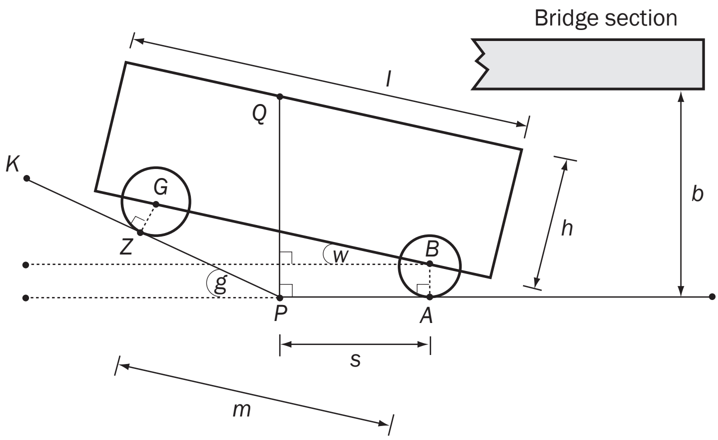


Fig. 4.6. Variables in the situation of the trailer passing under the bridge

Teachers can guide the class in reaching a consensus about a geometric model and the set of items to be measured. The tasks that follow assume that students have decided on the model in figure 4.6. Associated items to be measured include the following:

- The trailer's height, as measured from the flat ground to the top of the trailer (not labeled in fig. 4.6)
- The height, h, from the top of the trailer to the wheel axles. (This is the height of the trailer box in fig. 4.6.)
- The distance, r, from the axles to the ground. (In fig. 4.6, $r = AB$.)
- The distance, m, between the wheel axles. This distance affects the tilt of the trailer. (Students may suggest looking at the total length of the trailer, l, in addition to or in place of m at this point; task 3 addresses this issue.)
- The distance that the rightmost axle extends under the bridge. (This is denoted by s in fig. 4.6.)

- The "dangerous height" of the trailer. (This height is PQ in fig. 4.6.)
- The grade, g, of the sloping part of the road. (This is indicated by an angle measure in fig. 4.6; see the discussion of the grade of the road in the note on p. 83.)
- The tilt of the truck, w, as indicated by an angle measure in figure 4.6. (This item will play a role in some subsequent tasks. Students may or may not suggest it at this time, and the teacher can decide whether or not to discuss it now. One possibility would be to wait until task 4.)
- The height, b, of the bridge as measured from the flat part of the road to the bottom of the bridge

Discussion of students' work on task 2

For the initial modeling in task 2, some students may prefer to visualize the situation mentally or by making a few draft sketches. Others may prefer making (or using) a physical representation of a truck and a road. Still others may prefer to use an interactive drawing utility and create a representation such as the one in figure 4.8. Students can make use of such a utility in a number of tasks at an early level of geometric reasoning. For example, given both the wheel radius and the distance between axles, they might be able to figure out how to construct a drawing like that in figure 4.4. To do so, they could reason that since the quadrilateral in figure 4.5 is a rectangle, and the opposite sides of a rectangle are equal in length, the distance between the axles is the same as the distance between the points where the wheels touch the roadway. So, to construct their drawing, they might develop the following procedure:

1. Create a segment marking the axle separation along the line representing the roadway
2. Construct a perpendicular line at one end of the axle segment constructed in step 1
3. Shorten the line to a segment that represents the wheel radius
4. Construct a congruent segment at the other end of the axle separation segment, either by repeating steps 2 and 3 or by translating the segment along the axle segment
5. Connect the endpoints of the radii with a segment to represent the bottom of the trailer

Task 2 calls for a drawing that represents the danger for the truck in the situation of passing under the bridge. Not all students would represent the road with two straight line segments that meet at a vertex. This is a simplifying assumption. In reality, there would be a curving transition from the sloping portion of the road to the level portion. However, in all likelihood this portion of the road would be far away from the onset of dangerous trailer elevation. Students can discuss the effects of a gentler transition after completing their analysis of this simpler case.

Students might suggest including additional measurements, such as the distance from the flat roadway to the sloping wheel of the truck (PZ) or the maximum height of the truck. These could be included in the list. In any case, classroom discussion should make the points that students might (a) list more measurements than they actually need, or (b) discover that they have omitted some measurements that they need to add later as they go deeper into the problem. Adding measurements that turn out to be essential is simply part of adjusting a mathematical model to fit the situation.

The assumption that the trailer sits on the wheel axles may also be a point of discussion. Again, this is a simplifying assumption. Subsequent iterations of the model could consider more complicated wheel and trailer configurations, including more wheels. Other angles might be suggested as measures of tilt. An alternative choice would be $\angle ZAP$. However, angle w, as originally defined, will be used subsequently in this set of tasks.

Task 2 and several subsequent tasks are related to the theme of reasoning with number and measurement elaborated in *Focus on High School Mathematics: Reasoning and Sense Making*. The idea of *what* to measure is a central task in the analysis of this problem. For example, the issue is not the precision (or round-off error) of the clearance sign that indicates the height from the horizontal portion of the road to the bridge. Rather, the maximum length of the line segment *PQ,* as compared to the bridge height, is what is important. As students later discover, this length is not the same for every truck and, in fact, varies with a number of the parameters in the bulleted list above. Although the total height of the trailer from the level road is not easy to label in the graphic, students who give a little thought to it at this point can easily see that it is the sum of the height of the trailer box, *h*, and the axle height, *AB*.

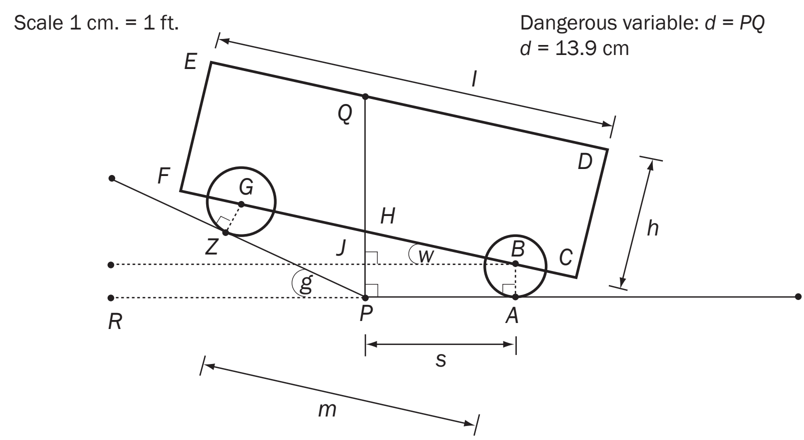
A second measurement decision involves making a reasonable choice of measurement units. In the real-world context of the problem, this issue arises in the decision about how to measure the grade (slope) of the road. Road grade is a measure of the road incline from the horizontal. In this sequence of activities, the road grade, *g*, is the degree measure of the acute angle formed by the sloping road segment with the extended horizontal road segment. In typical situations, road grade is measured as a percentage, as in a 9% grade (see fig. 4.7). That is, the typical grade measurement identified on highway signs is equivalent to tan (*g*) converted to a percentage.

Fig. 4.7. The road grade is typically given as a percentage.

The choice of unit of measure for the road grade is up to the teacher. However, the selection of degrees for the measure of grade angle provides consistency with how students would measure other angles in the geometric context of the problem. In particular, students compare the tilt angle, *w*, and the grade angle, *g*, in a later activity. So the tasks use degrees throughout the investigation. It is worth stating the (obvious) assumptions in the model that the grade of the road, *g*, is always nonnegative and will never be as great as 90°. The teacher might lead a discussion of grade as a percentage measure at this point or soon afterward.

A central goal of the analysis in subsequent tasks is to uncover how components of the model interact with each other. Although versions of these tasks can be implemented without technology, an interactive visual model can bring the mathematical inquiry to life and help to reveal several aspects of the effects of changes in parameters on the "dangerous height." As presented, the tasks assume that students have available an interactive model such as that captured statically in figure 4.8.

Using an interactive model such as the one depicted in the figure, students can adjust the distance, *m,* between the axles by moving a slider. The effect is to draw the elevated truck wheel up the incline without moving the wheel that is on the level surface. The size of the wheels can also be adjusted with a slider that changes axle height, *r,* from the road. Moving this slider adjusts both wheels equally. Without changing the trailer's dimensions, students can move it further along the level portion of the road (in either direction) by dragging the point *A* with the mouse. They can adjust the grade, *g,* of the sloping portion of the road by dragging the point *K.* Any of these four adjustments changes the tilt angle, *w*. Moving a slider adjusts the height, *h,* of the trailer box by raising the roof of the box. A slider for trailer overhang, *v* (not shown explicitly in fig. 4.8), lengthens the trailer extension beyond the wheels. (In the interactive model pictured, this adjustment affects both wheel overhangs equally.)



slide Axle separation: $m = BG$ (slider moves only back wheel)	m = 19.1 cm
slide Trailer overhang: $v = BC = GF$	v = 4.2 cm
slide Axle height: $r = AB$	r = 2.3 cm
slide Trailer height: $h = CD$	h = 9.4 cm
Trailer length: $l = DE$; adjusted by changing axle separation or trailer overhang	l = 27.6 cm
Distance of trailer under bridge: $s = AP$; adjusted by dragging point A (moves both wheels and whole trailer)	s = 13.3 cm
Grade angle: $g = m\angle RPK$; adjusted by dragging point K	g = 25.7°
Tilt angle: $w = m\angle JBH$	w = 3.9°

Fig. 4.8. A possible model produced with an interactive drawing utility

Through exploration of the model, students notice that the dangerous height is unaffected by changes in one of these parameters, trailer overhang (v), though it is altered in various ways by changes in others. Tasks 3–5 progress through different levels of analysis that investigate the change in the dangerous height, d, which results from a change in the distance, m, between the axles. Table 4.2 summarizes the key elements and reasoning habits (NCTM 2009, pp. 9–10, 55–56) that are illustrated by the students' work in task 2.

Table 4.2
Key Elements and Reasoning Habits Illustrated in Task 2

Key Elements of Reasoning and Sense Making with Geometry

Conjecturing about geometric objects

Analyzing configurations

Making conjectures about relationships

Geometric connections and modeling