into practice

Chapter 1 The Meaning of Multiplication

Essential Understanding 1*a*
In the multiplicative expression $A \times B$, A can be defined as a *scaling factor*.
Essential Understanding 1*c*
A situation that can be represented by multiplication has an element that represents the scalar and an element that represents the quantity to which the scalar applies.
Essential Understanding 1*d*
A scalar definition of multiplication is useful in representing and solving problems beyond whole number multiplication and division.

Multiplication is a scalar process involving two quantities, with one quantity—the *multiplier*—serving as a scaling factor and specifying how the operation resizes, or rescales, the other quantity—the *multiplicative unit*. The rescaled result is the *product* of the multiplication. Understanding multiplication as a scalar operation on whole numbers as well as other numbers is the foundation of multiplicative thinking and underlies Essential Understandings 1*a*, 1*c*, and 1*d*, presented in *Developing Essential Understanding of Multiplication and Division for Teaching Mathematics in Grades 3–5* (Otto et al. 2011). This chapter focuses on helping students recognize situations that call for multiplicative thinking and assessing their understanding of the elements that make these situations multiplicative.

Working toward Essential Understandings 1*a*, 1*c*, and 1*d*

As Otto and colleagues (2011) discuss, "Multiplication is a fundamental operation that is used to solve everyday problems" (p. 10). Yet, many students struggle to develop a deep understanding of multiplication and the underlying ideas related to it. To help students develop such understanding, teachers need to design, adapt, or select worthwhile mathematical tasks for them to work with, interpret the responses that they give, and make instructional decisions on the basis of the thinking that they reveal. These critical practices require specialized knowledge. To begin exploring students' development of this understanding, analyze the problems in figure 1.1. As you consider these problems, think about the questions posed in Reflect 1.1.

Reflect 1.1

Which of the problems shown in figure 1.1 call for multiplicative reasoning? Which call for additive reasoning?

How do these problems compare with the problems that you use to help your students develop additive and multiplicative reasoning?

What are the benefits of using tasks that require students to compare and contrast situations involving multiplicative and additive reasoning?

1. Phil ran 2 miles. Sally ran 3 times the distance that Phil ran. How many miles did Sally run?
2. Phil ran 2 miles. Sally ran 3 more miles than Phil. How many miles did Sally run?
3. Phil ran $^3/_4$ of a mile. Sally ran $^2/_3$ of the distance that Phil ran. How many miles did Sally run?
4. Phil ran $^3/_4$ of a mile. Sally ran $^2/_3$ of a mile more than Phil ran. How many miles did Sally run?

Fig. 1.1. Contextual problems about the number of miles that Sally ran: additive or multiplicative situations?

The four situations in figure 1.1 have some similarities. For example, they all pose the same question: "How many miles did Sally run?" Yet, students' responses to subtle differences in the situations provide opportunities to assess their multiplicative reasoning. Problems 2 and 4 require students to add the two values to determine the total number of miles that Sally ran. By contrast, problems 1 and 3 require students to think multiplicatively. Draw diagrams to represent problems 1 and 3, and then respond to the questions in Reflect 1.2.

Reflect 1.2

What does reasoning multiplicatively mean?

What does reasoning additively mean?

Clearly, you want your students to do more than just provide answers to multiplication facts—telling you only, for example, that 3×5 is 15. Students need to build an understanding of the meaning of multiplicative situations and learn to reason multiplicatively. These essential understandings and competencies will support their future learning when they encounter topics such as prime and composite numbers, factorization and prime factorization, factor and greatest common factor, multiple, area and volume, proportional reasoning, mean, algebraic expressions, linear functions, and place value (Otto et al. 2011).

One key aspect of your students' understanding that you, as their teacher, need to encourage is the development of multiplicative reasoning that extends beyond a view of multiplication as repeated addition (Jacob and Willis 2001). Students should relate multiplicative reasoning to iterating—that is, to making multiple copies—and partitioning sets of objects as well as to the length, area, and volume of physical quantities. When students see the mathematical expression 3×5, for example, they should be able to view 5 as the multiplicative unit (also called the *multiplicand*) and 3 as the scaling factor (also called the *multiplier*) for that multiplicative unit. Such a perspective not only involves recognizing the multiplicative unit, but also being able to iterate it—make multiple copies of it. The expression 3×5, for example, means 3 copies of 5 or 3 groups of 5. This interpretation of 3×5 reflects a critical meaning of multiplication that students must establish.

Students who reason multiplicatively can "see" a multiplicative unit and create multiple copies of it. An initial view of multiplication should involve understanding what it means to create 1, 2, 3, 5, or more copies of a given unit. Note that some researchers refer to the multiplicative unit as the *composite unit* (e.g., Steffe 1992; Lamon 1994; Tzur et al. 2013).

Eventually, we want students to recognize what it means to create $^1/_2$ of the multiplicative unit $2^1/_2$, obtaining the product $^5/_4$. Students who have a deeper understanding of multiplicative reasoning will begin to make multiplicative comparisons and express them in statements such as, "This is half as much as I had before."

Jacob and Willis (2001, p. 307) emphasized the importance of three aspects of multiplicative situations:

> It was the work of Kouba (1989), Steffe (1992), Mulligan and Mitchelmore (1997) and Mulligan and Watson (1998) that led to the conclusion that children must first come to recognise multiplicative situations as involving three aspects: groups of equal size (a multiplicand), numbers of groups (the multiplier), and a total amount (the product). When they can construct and coordinate these factors in both multiplication and division problems prior to carrying out the count, they are thinking multiplicatively.

Let's consider these three critical aspects—the multiplicand (or multiplicative unit); the multiplier; and the product—in the case of two problem situations:

1. Elizabeth has 3 bags with 5 apples in each bag. How many apples does Elizabeth have?

2. Elizabeth has 3 pieces of ribbon. Each ribbon is 5 inches long. What is the total length of the ribbon that Elizabeth has?

Both of these situations can be expressed as $3 \times 5 = 15$. Depending on the context, students approaching these problems work with objects or units of measure—apples in problem 1, inches in problem 2. They must conceptualize some number of objects or units of measure as the multiplicative unit—5 apples in problem 1, 5 inches of ribbon in problem 2. This number (or quantity) of objects or units of measure is the *multiplicand*—in both of these cases, 5. Figure 1.2 illustrates these multiplicative units of 5 apples and 5 inches of ribbon, configured in problem 2 as a 5-inch segment of ribbon.

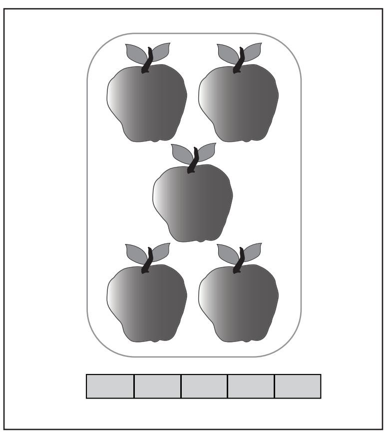

Fig. 1.2. The multiplicative units for the apple and ribbon situations in problems 1 and 2

Students then must iterate the multiplicative unit a number of times. The *multiplier*, or scalar factor, represents the number of iterations. In problems 1 and 2, both of which present situations that can be expressed as 3×5, the multiplier, or scalar factor, is 3, as figure 1.3 illustrates. The total number of objects or units of measure after the operation of the scalar factor is the *product*, or 15, as illustrated in figure 1.4.

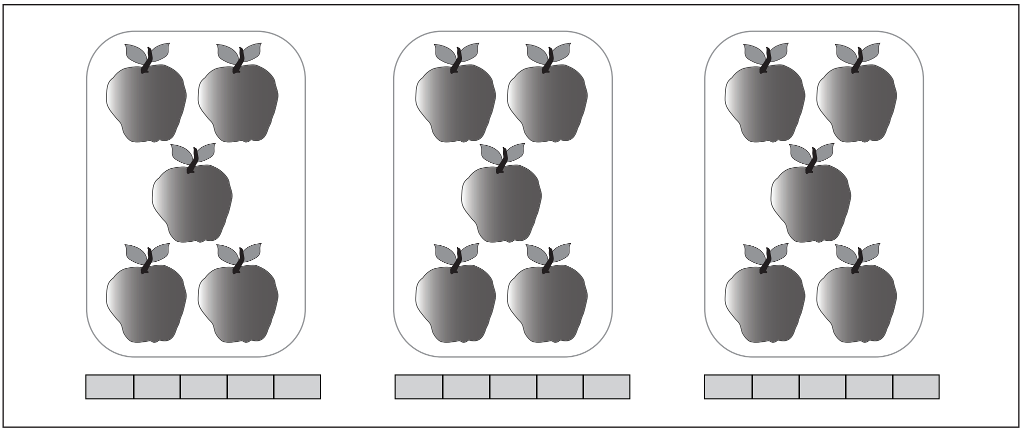

Fig. 1.3. The multiplier, 3, for the apple and ribbon situations in problems 1 and 2

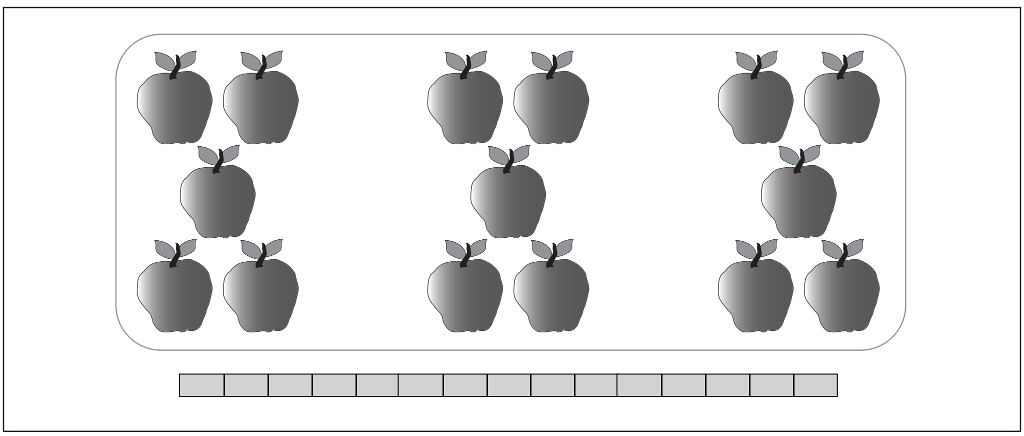

Fig. 1.4. The product

Consider another case, $^2/_3 \times {}^3/_4$, which extends the process of multiplication beyond whole numbers. Figure 1.5 presents a number line representation of $^2/_3 \times {}^3/_4$—a symbolic expression that could represent problem 3 in figure 1.1:

> Phil ran $^3/_4$ of a mile. Sally ran $^2/_3$ of the distance that Phil ran. How many miles did Sally run?

The multiplicand, or multiplicative unit, is $^3/_4$. Therefore, students using a number line representation need to understand that $^3/_4$ of a unit on the number line represents this multiplicative unit. The multiplier (scalar factor) is $^2/_3$, which means that students need to partition the multiplicative unit into 3 equal parts and iterate that part twice. The product is 2 copies of $^1/_3$ of a length of $^3/_4$, or $^2/_3$ of $^3/_4$, which is $^2/_4$ of a unit on the number line.

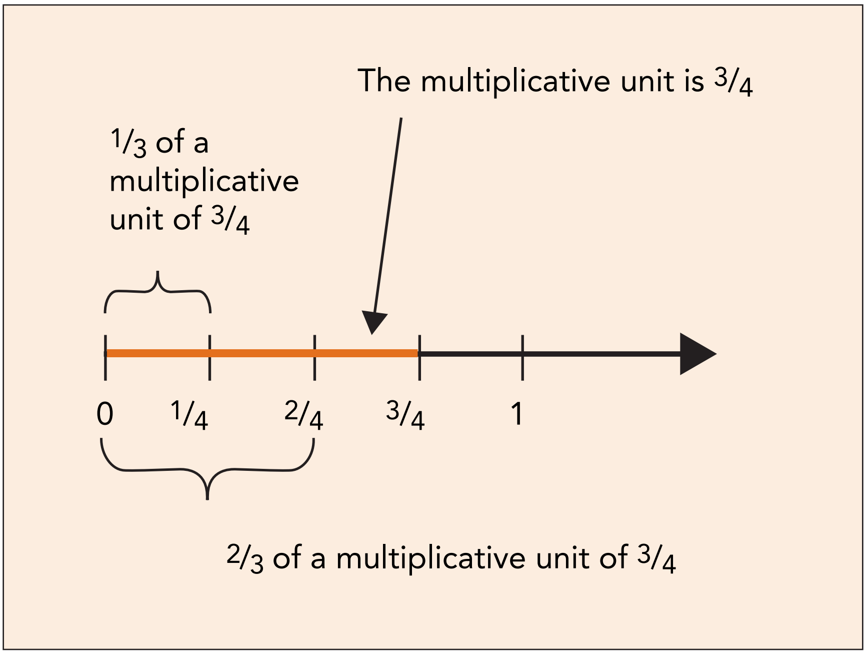

Fig 1.5. Illustrating $2/3 \times 3/4$ on a number line

Initially, students may have little sense of what a multiplicative unit is or that they can quantify a given situation in at least two different ways—one, by counting the number of objects within a multiplicative unit and the number of copies of the multiplicative unit, and the other, by counting the total number of objects in all the copies. For example, students should recognize that a diagram showing 7 circles with 3 smaller circles in each one could be viewed as 7 groups of 3 objects or, at the same time, as 21 objects. They should be able to identify the groups and the objects. However, such a dual view of the number of objects and the number of groups can be difficult for some children to coordinate.

To gain further insight into the different ways that students may view the multiplicative unit and the meaning of multiplication, consider the understanding and misunderstanding that third and fourth graders demonstrated in the work shown in figures 1.6–1.11. The third graders made representations of 7 groups of 3 and 7×3 and gave the product of 7×3, and the fourth graders created representations of 7×3. As you review the samples of student work, consider the questions in Reflect 1.3.