CHAPTER 1

Benefits of Teaching through Problem Solving

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STUDENTS in a fifth-grade class discuss their ideas about the following problem:

> Suppose 39 students want to share 5 candy bars fairly. How much can each student get?[1]

Leo: That's 5 divided by 39, and we decided last year that you can't divide a bigger number into a smaller number.

Anthony: I think that 39 ÷ 5 will be 7 remainder 4, but I think that 5 ÷ 39 will make a decimal number.

Jackson: I think that you will end up with a fraction of a number because, well, because 5 and 39—you can't divide 5 by 39 equally. I think it's going to be a number below 0.

After some further discussion about which notation (39 ÷ 5 or 5 ÷ 39) actually represents the situation in this problem and what sorts of numbers might be possible answers (e.g., fractions, decimals, remainders, "smaller numbers"), Mitchell chooses the correct notation and proposes that the answer will be pieces of candy bars.

Mitchell: So if each kid was going to get equal shares, they would have to cut the five candy bars into little equal pieces.

Teacher (MaryAnn): Can you name those equal pieces?

Mitchell: They might be candy bars.

Teacher: Can you name the fraction that they might be?

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1 This scenario is adapted, with permission, from *Making Meaning for Operations* (Schifter, Bastable, and Russell 1999, pp. 77–82).

Mitchell: [After a long pause] They wouldn't be able to do it.

The teacher stops to take a class poll.

Teacher: How many people think that you can do the problem 5 ÷ 39? How many think no, you can't?

The results are yes, 13; no, 15.

After a pause, Leo says that he wants to change his no to a yes. On the board, he draws five rectangles for the five candy bars and shows that partitioning two candy bars into sevenths will yield fourteen pieces. Then, without making lines, he indicates that partitioning four candy bars will produce twenty-eight pieces. He pauses when he realizes that the fifth candy bar will give him a total of thirty-five pieces. He then draws in lines to show that he has cut the last bar into eleven pieces. Now he is satisfied because he has a total of thirty-nine pieces. (See fig. 1.1.)

Cynthia quickly responds that Leo's representation cannot be

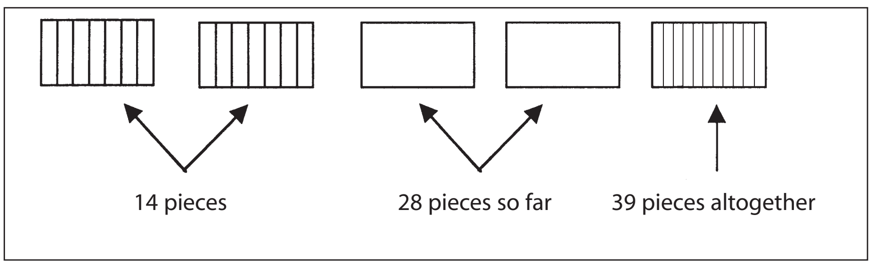

Fig. 1.1 Leo's solution to the candy-bar problem

correct because it does not yield equal shares. She seems sure of her statement. Four rectangles are divided into sevenths and only one is divided into elevenths. "That's a problem," she says. Eventually, Laila makes a suggestion.

Laila: If I cut each of the five candy bars into thirty-nine pieces and then give each kid one piece from each candy bar, you could have each kid have five-thirty-ninths of a candy bar.

After further discussion, most of the class seems convinced that Laila has proposed a valid solution to the problem, although, as will be seen subsequently, the discussion begins again when Leo and Cynthia propose an entirely different way of thinking about dividing the candy bars.

As this extended discussion of the candy-bar problem in MaryAnn's class illustrates, engaging children in problem-solving situations can give them opportunities for exploring, discussing, experimenting with, and attempting to make sense of mathematical ideas. MaryAnn did not tell her students how to solve the candy-bar problem, choosing instead to support them in figuring it out on their own. In the process, her students revealed and worked through a number of confusions and misconceptions about division and fractions. Several days after the problem had first been posed and solved, the students were still debating and discussing mathematical ideas that would, no doubt, reappear again and again and be dealt with in more and more depth in the weeks and months to come. I revisit this vignette later in this chapter because it serves as a useful example of teaching mathematics through problem solving.

What Does Understanding Mathematical Ideas Mean?

One can think about a model of learning mathematics in which understanding is represented by an increasingly connected and complex web of mathematical knowledge. According to this model, students develop understanding when they figure out how each new idea is related to other things they already know (Brownell 1947; Hiebert and Carpenter 1992). Understanding grows as a student's own personal web of connections becomes more and more complex. Indeed, the web might be imagined as a "hammock-like structure in which knots are joined to other knots in an intricate webbing. Even if one knot comes undone, the structure does not collapse, but still bears weight—as opposed to what might happen if each individual rope was strung only from one point to another, with no interweaving" (Russell 1999, p. 4).

As an illustration of how mathematical ideas are connected in complex, weblike ways, consider figure 1.2, which shows a concept map about fractions made by two fourth-grade boys. As I sat watching them construct their map, the boys took turns thinking about what else they knew about fractions, adding nodes to their map, and drawing lines to link each new node with other nodes. Their map reveals many of the connections between fractions and decimals and whole-number operations that make sense to them.

I quizzed the boys about the meaning of the node that says "go further" and its connection with the node "complex [fraction]" and was surprised by the depth of understanding revealed by their response. As one boy explained, "Well, you know that you can change any fraction to other [equivalent] fractions by multiplying or dividing the top and bottom by the same thing, right? Like 4/6

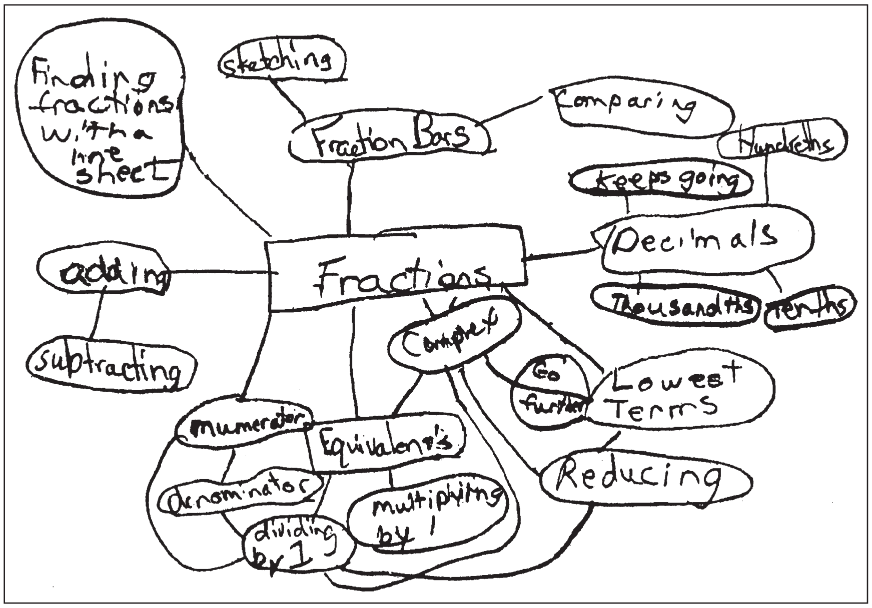

Fig. 1.2. Fractions concept map prepared by two fourth graders

can be 8/12 or 2/3?" I agreed, so he continued his explanation. "What's really neat is that you could go *further* with multiplying or dividing, if you wanted to. Like, you could divide by 2 over and over, like 4/6, then 2/3, and then 1 over 1 1/2, then 1/2 over 3/4, and so on. And the fractions might get messier and messier. They're called *complex fractions* because it's a fraction over another fraction. Isn't that neat?" I readily agreed that the idea of an infinite family of messier and messier equivalent fractions was certainly "neat." I was impressed not only with the enthusiasm these boys showed for thinking about fractions but also with their ability to communicate about the connections they recognized among many different mathematical ideas.

Problem Solving and Understanding—Mutual Support Systems

This book's authors believe that the primary goals of mathematics learning are understanding and problem solving, and that these goals are inextricably related because learning mathematics with understanding is best supported by engaging in problem solving.

The connection between solving problems and deepening understanding is symbiotic. Teachers want students to be able to solve problems—in mathematics and in the real world. After all, if they cannot solve problems with the mathematics they learn, what good is it? To be able to solve problems, one must have deep, conceptual understanding of the mathematics involved; otherwise, one will be able to solve only routine problems. So to become a good problem solver, a student must truly understand the inherent concepts. Thus, *understanding enhances problem solving.*

A problem is, by definition, a situation that causes disequilibrium and perplexity. A primary tenet of teaching through problem solving is that individuals confronted with honest-to-goodness problems are forced into a state of needing to connect what they know with the problem at hand. Therefore, *learning through problem solving develops understanding.* Students' mental webs of ideas grow more complex and more robust when the students solve problems that force them to think deeply and to connect, extend, and elaborate on their prior knowledge.

The next section of this chapter discusses the benefits of learning with understanding. This section could just as well be called "The Benefits of Learning through Problem Solving" because these two activities go hand in hand. To validate this link, after reading through this section, read through it again, substituting *problem solving* wherever the word *understanding* appears. Does the text still make sense?

Benefits of Learning with Understanding

Learning with understanding—making sense of new ideas by connecting them with existing knowledge in coherent ways—is, admittedly, often harder to accomplish and takes more time than simply memorizing or mimicking, yet the benefits of learning with understanding outweigh the challenges. Hiebert (this volume) discusses several reasons why understanding is essential, but much more can be said. Indeed, mathematics educators have identified at least six reasons why students should learn mathematics with understanding (Hiebert and Carpenter 1992; Van de Walle 2001).

Understanding Is Motivating

Nothing is more rewarding than the confident feeling that ideas make sense; and nothing is more frustrating than not understanding. Students who do not understand an idea often feel so discouraged and defeated that they give up even trying to learn. Such students must be motivated to learn by outside rewards (e.g., threat of a test, money for good grades, a gold star on a classroom chart, or desire to please a parent or teacher). By contrast, to understand something is a very motivating and intellectually satisfying feeling. When ideas make sense to students, they are prompted to learn by their desire for even deeper understanding. They want to learn more because feeling successful in connecting new ideas with old is an exhilarating experience. As Hiebert and his colleagues (1997) so aptly stated, "Understanding breeds confidence and engagement; not understanding leads to disillusionment and disengagement" (p. 2).

Understanding Promotes More Understanding

Another important benefit of understanding is that it promotes even more understanding. People make sense of their world by trying to use whatever ideas or procedures they have available to them. When confronted with unfamiliar mathematical problems, they attempt to use ideas or computational methods that they have used before. If those ideas or methods are poorly understood, they may be inappropriate for the situation at hand or may be incorrectly applied; thus, wrong answers are likely to result. However, this outcome is less likely to occur if students have made sense of the mathematics they have learned instead of having learned it without meaning. For example, Cauley (1988) observed that third graders often incorrectly perform multidigit subtraction by subtracting the top, smaller digit from the bottom, larger digit. However, he also found that such errors were more likely to be committed by students who lacked conceptual understanding of the multidigit subtraction procedure. When subtraction truly makes sense to students, they are more likely to recognize the nonsense answers that result from incorrect subtraction methods. Another research study, with fourth graders, revealed that those who had learned to connect decimal numerals with physical representations of decimal quantities were more likely to invent appropriate procedures for dealing with problems they had not encountered before, such as ordering decimals by size and converting between decimal and common fraction forms, than were those who had not previously made these connections (Wearne and Hiebert 1988).

In other words, when students' attempts to solve novel problems grow out of well-connected networks of mathematics understanding, the resulting mathematics is more likely to make sense and to be productive. "Inventions that operate on understandings can generate new understandings, suggesting a kind of snowball effect. As networks grow and become more structured, they increase the potential for invention" (Hiebert and Carpenter 1992, p. 74).

Understanding Helps Memory

When ideas are disconnected, they are hard to remember. Most adults can recall memorizing seemingly endless lists of disconnected facts in school—in various school subjects, not just mathematics. Those same adults often admit that many of the items on those lists were quickly forgotten after the class or the test was over. By contrast, when individual ideas make sense because they are connected with one another in the solver's web of understanding, much less information needs to be remembered. For example, memorizing different rules for the placement of the decimal point in addition, subtraction, multiplication, and division problems involving decimals is unnecessary if a solver simply understands one concept: how place value is represented in decimal notation. By using a fundamental understanding of place value, the solver can easily figure out where the decimal point belongs in any answer. A single big concept—which is actually a richly connected network of smaller concepts—is easier to remember than many small, unrelated concepts.

Understanding Enhances Transfer

Transfer is perhaps the single biggest challenge in all of education. It is one thing to be able to perform well on a test after having learned that very same test material in recent school lessons. It is another thing to be able to apply that learning in new, unanticipated situations, in or out of school. Yet, obviously, transfer should be the goal of education, because school prepares students for a world outside the classroom, for solving future problems that one may not even imagine today.

Many teachers have heard the plaintive cry of students who, faced with a mathematics word problem, plead, "Just tell me what to do (Add? Subtract? Multiply? Divide?), and then I'll be able to solve it just fine." These students have learned certain mathematical ideas or procedures without making sense of them and thus have difficulty transferring these ideas to the novel situation of a problem in context. By contrast, ideas or procedures that make sense to students are much more easily extended and applied.

For example, many of the concepts that students find very difficult when learning to work with fractions are actually just extensions of fundamental concepts of whole-number arithmetic. On the one hand, the process of adding fractions, which may require finding common denominators so that like quantities can be combined, can be better understood if students already recognize that adding whole numbers involves combining like quantities—adding hundreds to hundreds, tens to tens, and ones to ones (which is why we line up the digits according to place value in the standard algorithm before beginning to add). On the other hand, multiplying fractions does not require first obtaining common denominators, for much the same reason that we generally do not need to line up the decimal points before multiplying decimal numerals.

Understanding Influences Attitudes and Beliefs

Understanding leads students to see mathematics in a positive light—as a subject that makes sense because it is logical and connected. When students appreciate the underlying structures of mathematics, they see it as a reasonable, approachable subject; as a result, their self-confidence with mathematics soars and they are generally more willing to tackle challenging problems. By contrast, students who have learned mathematics without understanding are often successful only with solving problems similar to those they have already seen. Unable to see how mathematical ideas are related or useful, these students often see the subject itself in a negative light, viewing it as arbitrary and mysterious—a subject that only “geniuses” can master.

Understanding Promotes the Development of Autonomous Learners

A major goal of school mathematics instructional programs must be to support students in becoming autonomous learners. As described in *Everybody Counts,* “to understand what they learn, [students] must enact for themselves verbs that permeate the mathematics curriculum: examine, represent, transform, solve, apply, prove, communicate” (National Research Council 1989, pp. 58–59). Students learn more and better when they are helped to take control of their own learning by defining learning goals and monitoring their own progress in achieving them.

Children engage in problem solving and sense making quite naturally when they are very young. Preschoolers often drive their parents crazy asking “Why . . .?” as they take things apart, try to put them back together, and just generally attempt to make sense of the world around them. Unfortunately, traditional United States teaching practices often encourage children to suspend their curiosity and to turn off their intuitive ways of thinking, prompting them instead to move—at least in school—toward just trying to imitate whatever the teacher or the textbook tells them to do. Students of all ages bring many ideas with them to school. These ideas include both informal intuitions about the world around them and prior school-based knowledge, and all these ideas can