Chapter 3

Challenges: Learning, Teaching, and Assessing

Geometry's roots can be seen in children's everyday experiences. Children can build on these experiences and engage in thinking from a mathematical point of view when they develop skill in using mathematical practices (Common Core State Standards Initiative [CCSSI] 2010) and mathematical habits of mind, such as defining, conjecturing, experimenting, and proving (Goldenberg, Cuoco, and Mark 1998) to build mathematical understanding.

This chapter addresses the challenges of bridging everyday experiences and mathematical experiences of geometry and measurement through a series of scenarios and tasks. The tasks involve the concept of symmetry. Recall that a symmetry is an isometry that maps a figure onto itself. Although this is just one topic in the grades 3–5 geometry curricula, our intent is to model how students might be engaged in mathematical practices and might develop mathematical habits of mind through investigations that draw on the teacher's ownership of the big ideas and essential understandings discussed in chapter 1.

Essential mathematical practices have been articulated in various lists and versions. Among these are the Process Standards elaborated in *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics [NCTM] 2000), the strands for mathematical proficiency outlined by Kilpatrick, Swafford, and Findell (National Research Council 2001), and the Standards for Mathematical Practice presented in the Common Core State Standards for Mathematics (CCSSI 2010). This chapter focuses on the practices of defining, conjecturing, arguing and refuting, looking for commonalities and generalizing, and reasoning mathematically. This last, more general practice involves students in explaining and proving as they use all these practices to develop concepts of symmetry and isometry. Although we have named the practices as though they are distinct, they are interconnected. We separate them only to highlight each one in our discussion.

Building Definitions and Using Counterexamples

Big Idea 3

A classification scheme specifies the properties of objects that are relevant to particular goals and intentions.

Defining is an important activity in mathematics and the primary constituent of classification, which is the focus of Big Idea 3. Defining is a dynamic practice, as suggested by Essential Understandings 3*a* and 3*c*, and it is closely connected to other mathematical practices, such as conjecturing, solving, explaining, and proving. Defining involves students in collaboratively building concepts through their mathematical experiences, negotiating meanings, and clarifying their understanding in their discussions with others.

It is important for children to have the opportunity to engage actively in these aspects of defining. Branford wrote forcefully on this point in 1908 (quoted by Griffiths and Howson [1974, pp. 216–17]):

Essential Understanding 3*a*

Classification schemes and associated defining properties depend on the purposes and contexts envisioned for mathematical investigation, and multiple classification schemes are often possible.

> To me it appears a radically vicious method, certainly in geometry, if not in other subjects, to supply a child with ready-made definitions, to be subsequently memorized after being more or less carefully explained. To do this is surely to throw away deliberately one of the most valuable agents of intellectual discipline. The evolving of a workable definition by the child's own activity, stimulated by appropriate questions, is both interesting and highly educational.

Branford's comments imply that defining in geometry does not always produce agreement or immediate clarity. Many people are surprised to find that there is not universal agreement about all definitions. For example, are parallelograms best defined as quadrilaterals with one rotational symmetry or as quadrilaterals with two pairs of opposite sides parallel? The answer depends on the nature of the mathematical system that one is trying to develop.

Essential Understanding 3*c*

Classification leads to investigation of criteria for particular classes of shapes, and such investigation can lead in turn to the identification of new properties and relationships among objects in the class.

When such situations arise, the best way to help young students conceptualize an idea may not be immediately clear. In chapter 1, the discussion of Essential Understanding 3*c* considered a geometric definition of "straight" not as an a priori axiom but instead as successive redescriptions of "straight" across contexts of everyday movement in space, such as walking without changing direction, inscribing walks on paper, and considering the meaning of "straight" on different geometric surfaces. Recall the question of what happens to a line drawn on a piece of paper if the paper is rolled into a tube (see Reflect 1.30). Instead of telling students to accept definitions, teachers can involve students in the practice of defining and developing important foundational concepts of isometry and symmetry through experience.

Beginning with familiar occurrences gives students the opportunity to develop more robust conceptions anchored in experience but extended in generality and precision by thinking mathematically. For example, drawing on the earlier investigation of the meaning of “straight,” imagine a class of fourth graders investigating parallelism. Suppose that the students are looking for examples around the classroom, and recognizing that the lines on a ruled piece of notebook paper are parallel, they suggest a criterion of equal distance. But then the teacher might ask them about the stripes on the flag hanging in the corner: “Can we describe them as parallel lines? What if we placed the flag flat on a table? How about when the flag is waving in the wind? What if we draped the flag on a cylinder or sphere?” If young students are to appreciate the importance of defining terms and the challenge of reaching agreement, they must be allowed to formulate their own definitions, experiencing the need to clarify proposed definitions, modify them in light of feedback, and improve on their initial statements.

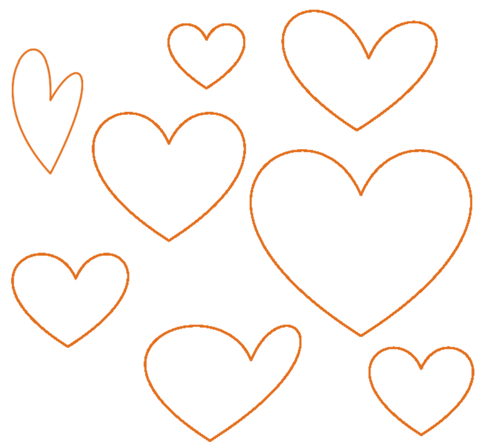
This process-based approach to defining suggests the need to change the common practice of beginning a new topic of study with vocabulary lists and definitions. A naked definition supplied at the start of a lesson has no context and thus little or no meaning for many students. By contrast, a well-designed problem can lead students to play with ideas to see how geometric objects work, to experiment by taking various actions on those objects, and to define as a way of clarifying the nature of the objects and actions arising during the course of investigation.

Introducing and Working with Reflection Symmetry

The problem in figure 3.1 is designed to introduce reflection symmetry. Students’ prior experience may have included working with shapes containing parallel sides, but they have not developed a definition or solved problems.

In the statement of the problem, the phrase “look right” is deliberately ambiguous, prompting students to explore their own and others’ assumptions and interpretations as they solve the problem. The dialogue that appears after figure 3.1 shows a discussion of students’ solutions to this problem in Ms. Fuji’s fourth-grade class:

Derrick was making a card for his mom's birthday. He wanted to put lots of heart stickers on it. Derrick looked at the stickers and said, "Some of these don't look right!"



Which do you suppose were the heart stickers that Derrick thought didn't "look right"? What could Derrick have been noticing when he made the comment?

Fig. 3.1. Problem introducing reflection symmetry

Wendy: I think Derrick was looking at the hearts that aren't shaped right.

Ms. Fuji: Does everyone know which hearts Wendy means?

Tyler: Yes, the ones with bumps.

Keisha: Do you mean bulges?

Wendy: I mean the ones that aren't the same on both sides. Hearts have to be the same on both sides.

Ms. Fuji: Could someone show us where Wendy and Tyler might be looking to decide that some hearts aren't the same on both sides?

Keisha: [*Pointing to the two leftmost, the bottom middle, and the top rightmost hearts*] Here?

[*Wendy and Tyler nod yes.*]

Ms. Fuji: How did you decide which hearts aren't the same on both sides?

Tyler: One side looks too fat or too skinny.

Ms. Fuji: How did you decide about this one? [*Points to the lower of the two leftmost hearts, an asymmetrical heart that might appear to some to be symmetrical.*] Can anyone think of a way we could be sure that this heart isn't the same on both sides?

Jamal: We could fold it in half. It should match up if it's the same.

The discussion continued with Ms. Fuji asking Jamal to demonstrate his method for determining whether the sides match. The students used Jamal's cut-and-fold method in carrying out the investigation and sorting the hearts into two groups. Ms. Fuji helped the students relate the action of folding the heart along the vertical axis to their sorting decisions. Before the period was over, the students decided that by hearts that didn't "look right," Derrick must have meant the ones that didn't match up when folded.

Although neither *symmetry* nor *symmetrical* was mentioned by name in the discussion, the students gained a lot of experience with the concepts. Ms. Fuji's implementation of the task suggests that she knew that the need to classify the hearts as those that "look right" and those that do not involves classification and relies on properties of symmetry (Essential Understanding 3*c*).

As the class continued to work with the topic in the next days, they would be ready to understand the term *symmetry* when Ms. Fuji introduced it. This vignette illustrates the power of context and experience in the introduction of new terms and ideas. Moreover, because Ms. Fuji used the students' own words early in the discussion to highlight the salient aspects of the experience, the students were more likely to make a connection between their experiences and the process of defining the term.

Essential Understanding 3c

Classification leads to investigation of criteria for particular classes of shapes, and such investigation can lead in turn to identification of new properties and relationships among objects in the class.

Making Conjectures

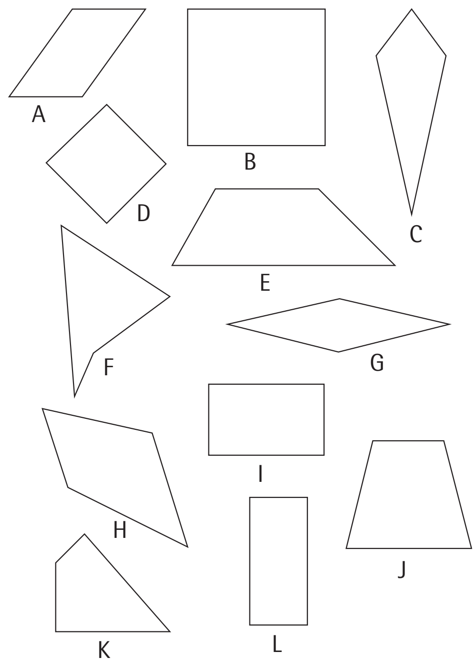
We often form notions of what we consider to be true or how something works before we are ready to apply them to every case. Conjecturing is serious business involving "formulating and producing general statements about patterns and relationships and evaluating their reasonableness" (Lampert 2001, p. 71).

In conjecturing, students articulate conditions or assumptions under which they will evaluate a supposition in light of a finding. After creating the conjecture, they revise it on the basis of mathematical evidence. In classrooms where revising conjectures is accepted as a normal practice, rather than seen as something that only the less capable students need to do, everyone's assertions are regarded as worthy of consideration and subject to question.

Using reflection symmetry in classifying

Conjecturing can be used to initiate explorations or can result from explorations. Study the task for students shown in figure 3.2 about lines of symmetry in quadrilaterals.

Sort the quadrilaterals below into categories, using as many categories as you think necessary. Describe each of your categories.



Find any lines of symmetry and any centers of rotation symmetry in the quadrilaterals.

Make observations about the relationship between your categories and the lines of symmetry in the quadrilaterals in that category.

Fig. 3.2. Sorting quadrilaterals task

Without invariance in properties there is no reason to name a classification or define it, as Sinclair, Pimm, and Skelin observe in *Developing Essential Understanding of Geometry for Teaching Mathematics in Grades 6–8* (2012a).

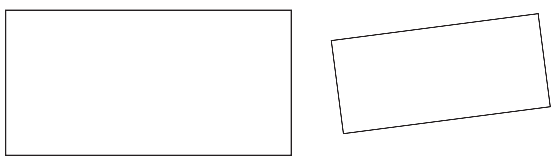
The intent of this task is for students to form conjectures on the basis of their observations about the relationships among the categories that they formed and the symmetries of the quadrilaterals in each category. If students place quadrilaterals I and L in a category, do they see that both figures have a line of symmetry through the midpoints of each pair of opposite sides? Which figures have four lines of symmetry? What kinds of figures have two lines of symmetry? What kinds of figures have only one line of symmetry? Questions like these help students make conjectures about the properties of the quadrilaterals.

To extend students' thinking, you might ask them to add a figure that would belong in each of their categories. What kinds of problems prompt students to make or test conjectures? Generally, problems that generate or involve lots of examples give students more opportunities to distinguish between features that vary and features that are invariant. When students become aware of an invariance, they can begin to describe it, and this leads to formulating a conjecture.

Sinclair, Pimm, and Skelin (2012b) connect invariance to refining conjectures and formulating new theorems.

Using symmetry in exploring conjectures

The task in figure 3.3 invites students to investigate a given conjecture and to use symmetry in their reasoning. Examine the task, and then consider how Mr. Jackson guided his fifth-grade students in thinking about it, as shown in the vignette that follows.



After solving several problems calling for finding the perimeter of a rectangle, Monique made the following conjecture, "I think the measure of the perimeter of a rectangle will always be an even number."

Use your experience with symmetry to make a judgment about Monique's conjecture. Write an explanation of why you agree or disagree with Monique's conjecture.

Fig. 3.3. Investigating perimeter conjecture task

The students in Mr. Jackson's fifth-grade class began sketching rectangles and assigning values to the lengths of the sides. In a short time, most students decided that they agreed with Monique's conjecture. However, they had difficulty in using symmetry to make their judgments. Mr. Jackson initiated a discussion:

Mr. Jackson: So most of you have decided you agree with Monique.

Tasha: See, every time I got an even number.

Mr. Jackson: How about others?

[*Students indicate that they too have gotten even numbers for the perimeters.*]

Mr. Jackson: Do you think that will *always* be true?

[*Students call out yes.*]

Mr. Jackson: What kinds of values did you give the lengths of the sides?

[*Students call out various odd and even whole numbers.*]

Mr. Jackson: Think about the numbers you are hearing us name and what kind of numbers they are.

Alicia: Whole ones.

Mr. Jackson: Did anyone use any other kinds of numbers?

[*No one responds. Mr. Jackson waits. Several students look down at their papers; some start writing.*]

Molly: Wait, what if we use $5\frac{1}{2}$ units for the length?

Mr. Jackson: What do you want the width to be?

Sam: Three.

The students then found the perimeter of a 3-unit by $5\frac{1}{2}$-unit rectangle and revised Monique's conjecture to include the condition that the lengths of the sides have to be natural numbers. Although this is not the only condition under which the conjecture is true, it was a step in the development of conjecturing for the students.

However, the students still had not used symmetry to formulate a rationale for why the conjecture is true under the right conditions. Mr. Jackson decided to raise the question again when the students had more experience with symmetry. Drawing on Essential Understanding 1*b*, he also noted that symmetry offers a geometric rationale for the conjecture that a rectangle whose dimensions are natural numbers of units always has a perimeter that is an even number of units—an idea to which he planned to return in the future.

Essential Understanding 1*b*

Transformation offers a means by which to explain geometrical phenomena in ways that build on spatial intuitions.

Mr. Jackson listened carefully to his students' statements and made an assessment of their development in conjecturing. He judged that the students were more focused on the numbers than on the symmetries of the rectangles. He used what they were noticing to advance their thinking about a necessary condition for the conjecture to be true.

Arguing, Refuting, Explaining, Generalizing

Mathematical practices are interconnected; the distinctions among them are indefinite. Arguing, explaining, and generalizing are different forms of mathematical reasoning. They have a structure that follows the students' process and logic. In mathematics, arguing is more than having a different opinion. When one argues, one is trying to convince others that something is true. Arguments are grounded in contested claims but go beyond conjecture. An argument begins with the belief that under the stated conditions, a particular statement will *always* be true or a solution will *always* work.

Most students in grades 3–5 tend to reason inductively, relying on the examples they find to show that something is true (Ellis et al. 2012). We can help students go beyond merely producing examples by asking them to explain why a particular example is relevant to their argument or how the example shows that the argument makes sense. The key to prompting viable arguments is to put students in the position of explaining why they believe something to be true. Positioning students to explain *why* can result in fruitful investigations of the structure of mathematical systems.

Finding lines of reflection symmetry

Tasks that elicit common misunderstandings that students have can help them reconsider important mathematical ideas. One such common misunderstanding is the belief that rectangles have four reflection symmetries, one through each pair of opposite sides and one through each diagonal, as shown in figure 3.4.

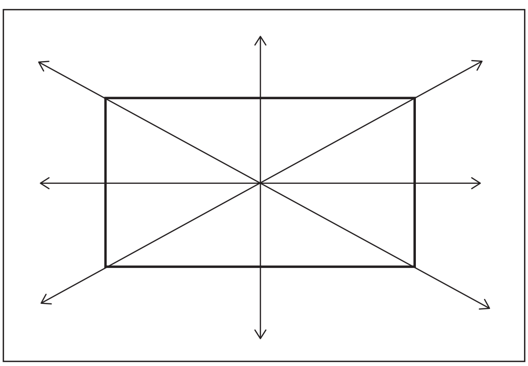


Fig. 3.4. A misrepresentation of the lines of symmetry in a rectangle

When asked to justify their claim, students might say that a line of symmetry "cuts the rectangle in half." This may be true in some sense, but cutting a rectangle in half is not a sufficient requirement for a line of symmetry. Experimenting with reflections by using a Mira (see fig. 3.5) or a mirror to test lines of symmetry may help children use spatial reasoning in powerful ways to make arguments. Eventually, considering other types of isometries and thinking about interesting points such as the point of intersection of the diagonals could lead to new opportunities to consider symmetry.

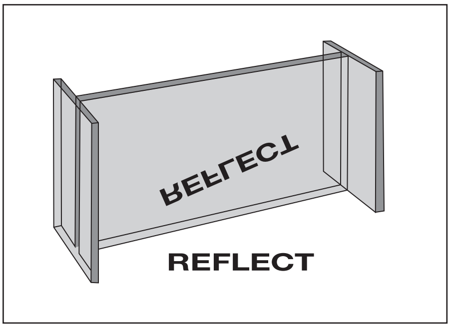
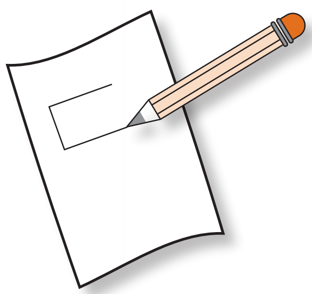


Fig. 3.5. Creating a reflection with a Mira

Reasoning and explaining why things are as they are

The final task in this sequence on conjecturing and symmetry is one that can be used to prompt students to reflect on concepts embodied in the previous tasks or can be offered as an assessment problem. Figure 3.6 shows the task.



For each of the following, draw a four-sided figure that has—

a. exactly four lines of symmetry;
b. exactly three lines of symmetry;
c. exactly two lines of symmetry;
d. exactly one line of symmetry;
e. zero lines of symmetry.

Describe each four-sided figure that you sketch, and explain why it has exactly that number of lines of symmetry.

Fig. 3.6. Symmetry task

Not only does this problem address much of the content related to reflection symmetry in grades 3–5 but it also embodies many of the practices highlighted in this chapter. Students can experiment, if they need to, and test their conjectures about what type of quadrilateral will suit each specification, and they must describe (define) their solutions and explain why they fit the conditions.

Students will not be able to find a four-sided figure with exactly three lines of symmetry, as specified in part (b). This characteristic is not possible to satisfy, but its presence in the list requires students to reason. Giving students a problem that does not have a solution can be frustrating if they have not had experience in devising and revising conjectures, making and refuting arguments, justifying findings, and explaining their reasoning. At the same time, students who have been engaging in these practices develop the confidence to look at each new problem critically to assess whether the problem situation is plausible.

Proving by Systematic Search

Often, ideas about symmetries and isometries are put to use in problems and contexts that initially may seem far removed from either topic. For example, chapter 1 referred to third-grade students who were trying to determine all the possible nets of a cube and were using isometries to identify equivalent nets. The task was not presented as one focused on finding isometries, but isometries turned out to be useful tools for considering equivalence. For these third-grade students, knowing when to stop generating candidate nets was difficult. If many people found the same nets, would that mean that all possibilities had been found? Note that the need for explanation arose from the activity of the students. It did not originate with an instruction to "prove that...."

Instead, the problem prompted the students to generate examples, and often, as noted earlier, having numerous examples provides rich opportunities to notice similarities and differences. However, the generation of these examples did not resolve the issue of knowing when to stop. This dilemma inspired several students to abandon a generate-and-test method in favor of a "system" determined by the number of squares in a column, or "backbone," of the net.

This method of exhaustive search convinced them that there were only eleven different nets. In figure 3.6, the count (the circled numbers) is for each unique net with columns of four, three, and two squares, so that there are six possibilities for a column of four squares, four for a column of three squares, and only one for a columns of two squares. The method constitutes an informal proof in that it explains the necessity for eleven nets, given the students' definitions of equivalent nets based on isometries and the kinds of configurations of squares that constitute nets—and the teacher's awareness of Essential Understanding 1*a*. This example, along with an emerging body of research, suggests that the kernel of thinking about proof as explaining why can be cultivated in the elementary grades (Stylianou, Blanton, and Knuth 2009; Lehrer and Lesh 2012).

Essential Understanding 1*a*

Transformation supplies a dynamic basis for analyzing and describing a variety of situations and relationships.

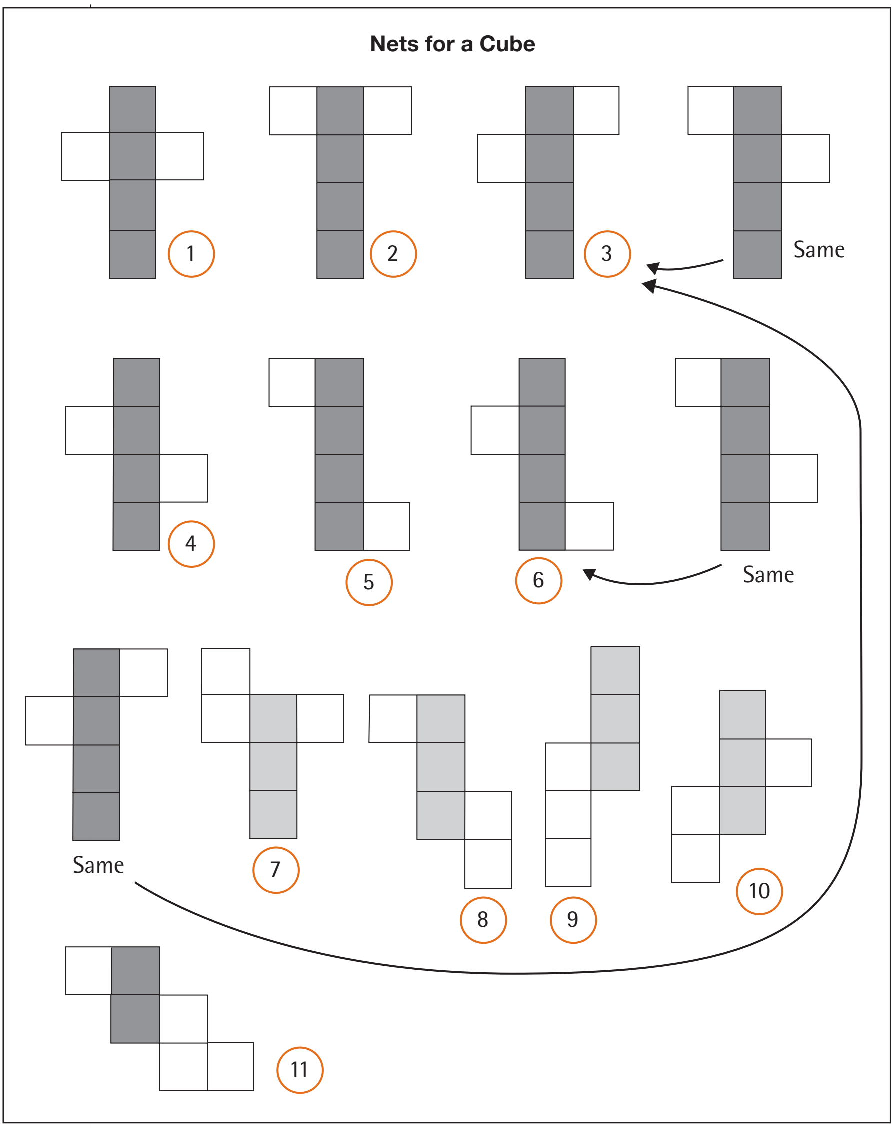


Fig. 3.6. Diagrams and counts used in student-generated informal proof of the necessity of eleven nets of a cube

Conclusion

Geometry and measurement are areas of mathematics with tangible forms that originate in everyday experiences of space. These commonplace experiences are useful for nurturing mathematical practices, such as defining, conjecturing, experimenting, and explaining. Emphasizing these mathematical practices while you are teaching can give your students an opportunity to "reinvent" geometry (van Hiele 1986). Observing and developing visualizing skills, exploring and experimenting with properties of shapes, formulating definitions of geometric objects after having gained experience with them, and constructing arguments and explanations are all part of the reinventing process.

The big ideas focusing on transformation, measurement, and classification from chapter 1 transcend the content of geometry in grades 3–5. Teachers familiar with them can draw on the essential understandings to guide their work as they help students learn about nets and a variety of geometric topics. They also can use them to develop, explain, and anticipate new approaches to other topics in their mathematics curriculum.