Chapter

2

Connections: Looking Back and Ahead in Learning

In chapter 1, we explored five big ideas related to teachers' understanding of expressions, equations, and functions at the middle school level. These areas of algebra are also important elements of students' learning in the upper elementary grades and high school. How do the big ideas that we have developed connect to mathematics that students encounter before and after middle school? A response to this question is the focus of chapter 2.

Connections in Grades 3–5

In grades 3–5, students work with general properties of addition and multiplication. They begin to use variables to represent numbers in statements of commutativity and associativity as well as to stand for quantities in story problems. They explore a variety of patterns and develop rudimentary ideas about functions. This work lays a strong foundation for the ideas about expressions, equations, and functions that students encounter in grades 6–8.

General properties of addition and multiplication

In grades 3–5, students develop and use properties of addition and multiplication of real numbers, including the associative and commutative properties, identity and inverse properties, and the distributive property of multiplication over addition. Students also develop understandings of addition and subtraction, and multiplication and division, as inverse relationships. When teachers help students in grades 3–5 to develop an understanding of general properties of arithmetic, they are also preparing students to apply those properties later to generate equivalent expressions, as we explored in our discussion of symbolic transformations that preserve the equivalence of expressions on both sides of an equals sign (Essential Understanding 1*c*).

Essential Understanding 1c
A relatively small number of symbolic transformations can be applied to expressions to yield equivalent expressions.

Students can gain insight into the meaning of multiplication by connecting it with the idea of the area of a rectangle and working with area models of multiplication. For example, they might be taught that 4 × 5 can be represented with a rectangle with side lengths of 4 and 5. Teachers recognize that this representation is helpful for illustrating some other properties of multiplication.

In the elementary grades, teachers often ask students to use the distributive property to compose and decompose numbers in ways that allow the students to solve problems with understanding. For instance, to calculate the value of 5 × 23, students discover that it makes sense to multiply 5 × 20 and 5 × 3 and add these partial products:

$$5 \times 23 = 5 \times (20 + 3) = (5 \times 20) + (5 \times 3)$$

In figure 2.1, the purple area represents 5 × 20. There are 5 rows of 20 purple squares. The gray area represents 5 × 3 with 5 rows of 3 gray squares. The large rectangle consists of 5 rows of 23 squares and represents the product 5 × 23.

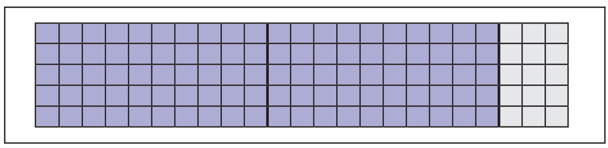


Fig. 2.1. A representation of 5 × 23 as (5 × 20) + (5 × 3)

For a longer discussion of models and properties of multiplication, see *Developing Essential Understanding of Multiplication and Division for Teaching Mathematics in Grades 3–5* (Otto et al. 2011).

This same property—the distributive property of multiplication over addition for real numbers—allows us to reason about equivalent algebraic expressions. For example, an area model of multiplication, like the one used above, can help us "see" that $3(3x + 2)$ and $9x + 6$ are equivalent expressions. In figure 2.2, the purple area represents the product $3 \times 3x$, and the gray area represents 3×2.

$$3(3x + 2) = 3 \times 3x + 3 \times 2 = 9x + 6$$

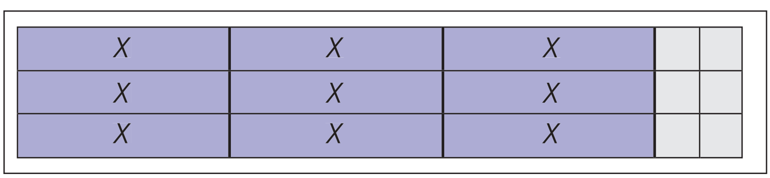


Fig. 2.2. A representation of 3(3x + 2) as 3 × 3x + 3 × 2

Essential Understanding 1c
A relatively small number of symbolic transformations can be applied to expressions to yield equivalent expressions.

Understanding the distributive property can help us in both "putting together" and "taking apart" expressions to find equivalent expressions (Essential Understanding 1*c*). Figure 2.2 illustrates both putting together (or composing, in this case by multiplying) the factors 3 and $3x + 2$ to obtain the expression $9x + 6$, and taking apart (or decomposing, by factoring) $9x + 6$ into the factors 3 and $3x + 2$.

As teachers in the elementary grades engage students in working with more sophisticated multiplication and division problems and strategies, they continue to help their students build a foundation for future work with expressions involving variables. For example, students in grades 3–5 come to understand that to calculate the value of 25×23, it is insufficient to add 20×20 and 5×3. Additional partial products are involved—namely, 5×20 and 20×3, the light purple and darker purple regions, respectively, of the rectangle in figure 2.3. As we discussed in connection with Essential Understanding 1*c*, this understanding provides the basis for analyzing and determining the equivalence of algebraic expressions. Reflect 2.1 invites you to consider how the multiplication of two linear expressions is related to the product represented in figure 2.3.

An extended discussion of properties of number and operations as a basis for developing algebraic ideas appears in *Developing Essential Understanding of Algebraic Thinking for Teaching Mathematics in Grades 3–5* (Blanton et al. 2011).

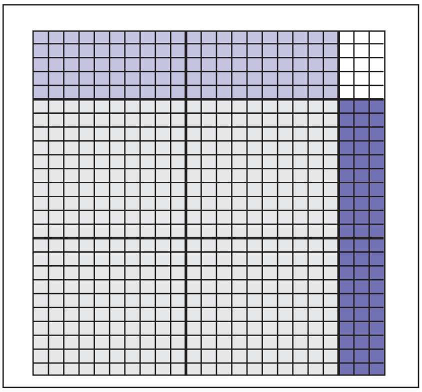


Fig. 2.3. An illustration of the partial products for 25×23

Reflect 2.1

How might you use algebra tiles and an area model to determine whether $(x + 3)(3x + 2)$ and $3x^2 + 6$ are equivalent expressions? How does your understanding of the distributive property, and other general properties, come into play as you consider this question?

Figure 2.4 shows a rectangle with height $x + 3$ and width $3x + 2$, constructed by using algebra tiles. The area of this rectangle is $(x + 3)(3x + 2)$, which is equivalent to $3x^2 + 11x + 6$, as shown in the figure. So, $(x + 3)(3x + 2)$ and $3x^2 + 6$ are not equivalent expressions. A representation like the one in the figure can be helpful for encouraging students to develop understanding of their work in algebra.

Variables, expressions, and equations

When we view the properties of addition and multiplication discussed above as general properties that work for all real numbers, we encounter *variables*. Letters are customarily used to represent

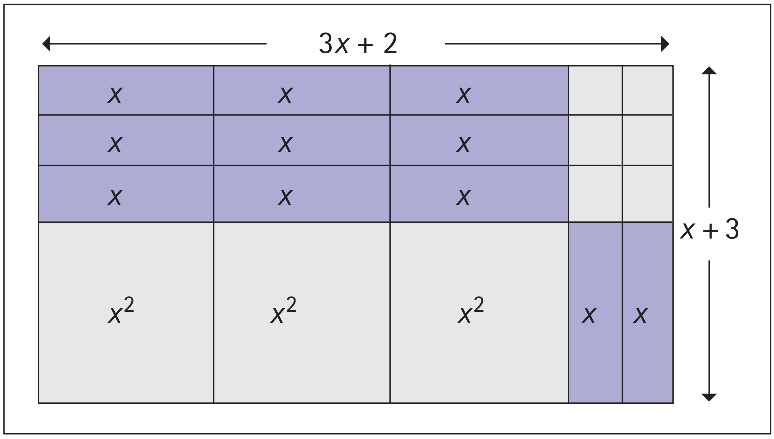


Fig. 2.4. Algebra tiles illustrating $(x + 3)(3x + 2)$ is not equivalent to $3x^2 + 6$

real numbers in these general properties. Developing an understanding of these properties often gives elementary students their first exposure to variables. For example, the commutative property of addition can be expressed as follows:

$$a + b = b + a, \text{ for any real numbers } a \text{ and } b.$$

Students in grades 3–5 gain further experiences with variables as they represent and solve story problems and describe patterns in terms of symbolic expressions and equations. Consider the following situation:

> The service club is planning a soccer game (Maroon Team versus Gray Team) to raise money. The students plan to charge \$2 per ticket for entry to the game. If a group of friends spends \$14 on tickets, how many tickets did they purchase?

A student might use the variable n to describe the cost of buying n tickets with the expression $2 \times n$. The student can then write an equation representing the equivalence of the quantity 14 and the quantity $2 \times n$ as $14 = 2 \times n$. Another student might write the following equation: $14/n = 2$. It is important for the teacher to recognize that these equations are equivalent (assuming that the friends purchased at least one ticket and n is not equal to 0) and to understand why they are equivalent.

In both of these equations, the variable n represents an unknown quantity of tickets. Symbolically, solving either equation demands that students have an understanding of the inverse relationship between multiplication and division, a key idea in the elementary mathematics curriculum. In the examples, variables are used in different ways, to express different things. The need for teachers to help students navigate among these different uses of variable underscores the importance of Essential Understanding 2*a*.

As the big ideas emphasize, expressions and equations are powerful tools for representing and analyzing relationships. To analyze the soccer game situation described above, it is helpful to develop a general rule that relates the variables. If the service club members are thinking about how much income ticket sales will generate, they might develop the following equation:

$I = 2 \times n$, where I represents income and n represents tickets sold.

This equation describes a proportional relationship between two variables.

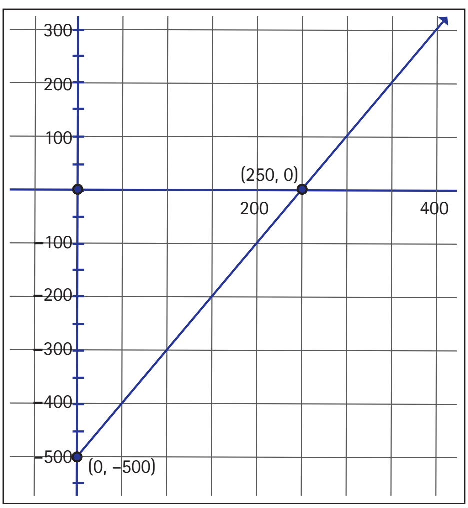
Over time, as students in grades 6–8 become increasingly comfortable in using mathematical notation to represent their thinking, teachers can encourage them to explore more complex relationships between variables. For instance, suppose the club has $500 worth of expenses for the soccer game. If n represents the number of tickets sold, and P represents the profit for the club, the students can describe the relationship between n and P in the following equation:

$$P = 2 \times n - 500$$

Thinking about the two variables, n and P, as related in this way can help students to recognize that the service club would need to sell 250 tickets to break even. This break-even point can be seen on the graph in figure 2.5 as the point at which the profit line crosses the horizontal axis (when $P = 0$). In the table of values that the figure also shows, the break-even point appears in the row in

Essential Understanding 2*a*
Variables have many different meanings, depending on context and purpose.

For a discussion of the inverse relationship between multiplication and division, see *Developing Essential Understanding of Multiplication and Division for Teaching Mathematics in Grades 3–5* (Otto et al. 2011).



Number of tickets sold (n)	Profit (P)
0	–500
50	–400
100	–300
150	–200
200	–100
250	0
300	100
350	200
400	300

Fig. 2.5. A graph and a table of values representing profit from a soccer game for which organizers sell tickets for $2 each and have $500 of expenses

which profit is zero and the number of tickets sold is 250. This more complex relationship, which might be examined by students in the upper elementary grades, is very similar to the profit situation that you explored in Reflect 1.21 and we then discussed in depth.

Patterns and functions

In the elementary grades, students explore a variety of patterns and represent them in words, tables, graphs, and symbols. It is important for teachers to understand the role that multiple representations play in analyses of patterns of change. This use of different representations is expressed in Essential Understanding 4*b*. Middle school teachers who have developed this understanding can appreciate the value of the work that elementary school students do in making predictions about what comes next in a pattern or sequence and developing generalizations about relationships in patterns. Consider the growing pattern in figure 2.6.

Essential Understanding 4*b*
Functions can be represented in multiple ways—in algebraic symbols, situations, graphs, verbal descriptions, tables, and so on—and these representations, and the links among them, are useful in analyzing patterns of change.

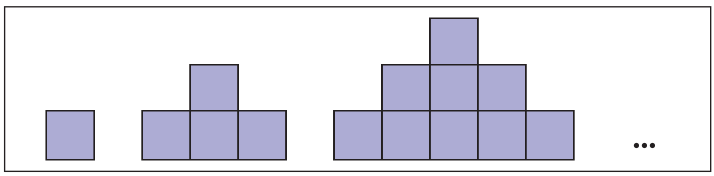


Fig. 2.6. A pattern of tiles

Students in grades 3–5 might make sense of and figure out how to continue the pattern by drawing or arranging tiles. They are able to describe this pattern with words and numbers. Students might describe the shapes as being made up of a number of unit squares that are sums of consecutive odd numbers: 1 square, 1 + 3 squares, and 1 + 3 + 5 squares. Recognition of the pattern allows students to identify the next element in the pattern, a shape made up of 1 + 3 + 5 + 7 squares. Some students in grades 3–5 recognize that by counting the number of unit squares that compose the shape, they are determining the area of the shape. Reflect 2.2 encourages you to investigate *area* in this pattern.

Reflect 2.2

Represent the relationship between a shape's position in the pattern in figure 2.6 and its area by using a table, graph, and equation. Think about what essential understandings you are drawing on as you consider this pattern.

We can use a table such as that in figure 2.7 to represent the relationship between the shape's position in the pattern and its area. This table draws attention to the idea that the pattern suggests a squaring function. If n represents the position in the pattern, then