



CHAPTER

1

Making meaning for whole number addition and subtraction

CASE 1

Insects and spiders and counting on

Dan

Grade 1, September

CASE 2

Going up and down with numbers

Wendy

Kindergarten, March

CASE 3

Complexities of counting back

Denise

Grade 2, October

CASE 4

Missing part versus missing change

Kina

Grade 1, January

CASE 5

Valentine stickers

Jody

Grades 1 and 2, January

As you read the following word problems, consider for a moment which of the operations—addition, subtraction, multiplication, or division—you would use to find the answer to each one.

There are 13 insects in front of us on the paper and 5 spiders behind my back. What is the total number of insects and spiders?

There are 10 girls in our class, and 8 of them are here today. How many girls are out today?

We can easily identify the operations that would give us the answers: addition ($13 + 5 = 18$) and subtraction ($10 - 8 = 2$). But what is it that children must understand to solve these story problems? What happens when such problems are posed to children who have not yet learned symbols for the operations and have not yet learned their arithmetic facts? How do children find answers to such problems when they haven't yet learned to add or subtract? The first three cases of this chapter allow us to examine these questions. As you read these cases, take notes on the following questions:

CASE 6

Finding the difference: Should I add or subtract?

Machiko

Grade 4, November

CASE 7

Number line models for subtraction

Carl

Grade 7, June

- What do we learn about the operations while following the counting strategies that young children use to solve problems and by pondering the confusions that arise for them?
- What do we learn about the ideas that children must put together in order for them to develop an understanding of the operations?

In the remaining cases in this chapter (cases 4 through 7), we meet students who have developed some ideas about addition and subtraction and can solve problems using these operations. However, the questions they ask, the con-

fusions they hold, and the insights they offer reveal the greater complexity underlying what may seem to be very basic concepts. One set of questions involves mathematical issues for you:

- Can a single situation be modeled by different operations?
- Can a single operation model different kinds of actions?
- What can we see about the operations through different representations such as cubes and number lines?

A second set of questions prompts us to consider the children's processes of putting ideas together:

- In these samples from grades 1–7, what can be observed about the ways in which students' ideas develop?
- What appear to be some of the difficulties students face as they learn about addition and subtraction?
- How can different representations support students' understanding of addition and subtraction?

As you read the cases, take notes on these questions. Then, after reading the cases, review the questions again.

case 1**Insects and spiders and counting on****Dan****GRADE 1, SEPTEMBER**

As part of our science study of insects and spiders, the children sorted small plastic models onto two large sheets of paper labeled “Insects” and “Spiders.” As they placed their figures, they had to tell what attributes they were using to make their decisions. After we sorted the models and confirmed that each was in the correct set, I asked, “How many insects are there?”

The insects were positioned randomly on the sheet. I called on Mike. As he counted the insects, his finger jumped around from one to another, making it difficult to keep track of which ones he had counted and which he had missed. He ended up miscounting by one, coming up with 14. Some children said they disagreed with his count. 5

I next asked Annie to count the insects. Although she didn’t count them in an easily discernible order (e.g., left to right, or top to bottom, or by moving them as she counted), she was clearly able to keep track because she did not recount or leave out any insects. She came up with the correct answer of 13. 10

I asked one more child, Renaldo, to count to see whether his answer would agree with either of the previous answers or would be completely different. He counted the insects in an organized way: as he said each number, he moved one item into a group of already counted insects. He came up with 13, just as Annie did. 15

I asked the group, “Why did we get different answers, 14 and 13?” Renaldo quickly replied that Mike had “counted too many.” Nods from many of the children seemed to suggest agreement. I think he meant that Mike had counted some insect or insects more than once. Because the children seemed content with Renaldo’s response, I assumed they agreed that 13 was the answer and did not pursue this question any further. 20

The children were quickly able to see that there were only 5 spiders on the other sheet. Because we had not counted the total number of insects and spiders, I decided to find out how the children would determine how many there were altogether if I removed the 5 spiders from view. How would they solve the problem? How would they represent items that were absent and keep them in the count? Would they start at 13 or at 1 to get the new count? 25

After I put the spiders behind my back I said, “If we know that there are 13 insects and that there are 5 spiders to be added, how many would there be altogether?”

Lindsey immediately raised her hand and answered 18. She explained that she thought of 13 insects in her head and counted on 5 more spiders in her head. 30

Then I called on Mike. He recounted the 13 insects very carefully, one by one (in contrast to his earlier somewhat careless, random counting), and then touched and counted 5 imaginary

spiders with one hand while raising 5 fingers, one at a time, on his other hand. Like Lindsey, he also came up with 18.

Liliana took a different approach. She counted the 13 insects and then separated 5 from that group and counted them on from the 13, also ending up with 18. 35

I brought the 5 spiders into view and together we all counted the 13 insects and 5 spiders to confirm that there really were 18.

These three students seem to represent a developmental sequence that children may go through in becoming comfortable with adding quantities together. Liliana started from 1, counted the 13 visible insects, and then separated 5 from that group as a concrete way to keep track of the additional 5 that she was counting. Mike seemed to be less dependent on the concrete objects. He also started at 1 with the 13 insects, but then counted the additional 5 by touching imaginary spiders with one hand while keeping track with 5 fingers on his other hand. He seemed to have internalized that there are a total of 5 fingers on one of his hands as he did not count them out before counting them on. It did not seem to be a problem for him that his fingers could be labeled “1, 2, 3, 4, 5” in one situation and “14, 15, 16, 17, 18” in another situation. Liliana and Mike either did not recognize or did not trust that the 13 previously counted insects would still be 13, or they could not comfortably start counting in the middle of the number sequence. Lindsey, by contrast, clearly trusted that the new number included the original 13 and didn’t feel a need to start counting from 1. She seemed able to visualize a group that she could label 13 plus 5 more individual spiders that she could easily count on in her head. 40 45 50

I wondered if the children had a sense of the commutative property of addition, and decided to investigate. Would they recognize that we would get the same answer whether we began with the 5 and added on 13 or started with 13 and added on 5? 55

I put the 13 insects behind my back, showed the children the 5 spiders, and asked how many there would be altogether if I were to add in the missing 13. Many of the children immediately said 18. I raised the question, “How do you know it is the same answer?” Although I can’t recall a clear answer from anyone, they gave variations on knowing that there were still 13 insects and we had already counted 5 spiders. I asked if there was any way that we could count them all up by first using the 5 spiders that we could see. No one responded. 60

Did these children really understand the commutative property for addition? Or was the answer easy for them because there were no new variables—that is, there were still 13 insects and 5 spiders? Even though I had changed the placement of each, the children may not have seen this as a different situation. Would they have said so quickly that the answer was still 18 if I had changed or eliminated a defining variable, for example, if the 13 and the 5 were all insects? 65

I tried a problem with different numbers to see how they would deal with adding on a missing number of items that was more than 10: “What if I told you there were 11 insects to add to the 5 spiders we already have in front of us?” Liliana counted the 5 spiders that she saw. Then, without counting out ten fingers beforehand, she continued counting on her fingers one at a time to 15, and said she needed 1 more to make 16. 70

I found it interesting that Liliana was not daunted by the fact that she had only ten fingers. She seemed to have internalized that all her fingers always make a group called “10” and 1 more finger makes another group called “11.” In addition, Liliana seemed comfortable with the fact that her ten fingers could be called “1, 2, 3, 4, 5, 6, 7, 8, 9, 10” in one situation and “6, 7, 8, 9, 10, 11, 12, 13, 14, 15” in another. 75

To verify that the children understood the importance of one-to-one correspondence in counting, I added 11 insects to the group of 5 spiders. I counted the 5 carefully, and then I continued to count the extra 11 insects randomly until I was way into the 20s. Many of the children shouted “No!” and said I could only count each insect once. Renaldo recounted for the group, carefully moving and placing each one into a new group of already-counted insects. He got 16, confirming Liliana’s answer. 80

Many of the children seem to understand the need for one-to-one correspondence and for only counting each item one time so that you get a specific answer that is repeatable. Many may have some sense of the commutative property for addition, but I cannot be sure of this without further investigation. Some of them clearly understand that a group of items has a constant value even when another group is added to them, and that it is not necessary to start over from 1 when adding the new items on one by one; others do not have this awareness or trust yet. Some interesting questions come up: 85

- When do children move from merely counting to actually understanding addition? 90
- Is it when they can count on from a number rather than having to start again from 1?
- Or can children who have to start at 1 every time still understand that they are adding and not just counting? 93

case 2

Going up and down with numbers

Wendy

KINDERGARTEN, MARCH

As we start each day in kindergarten, we assemble at the rug area for our opening meeting. One 95
of the things we discuss is the number of children in school. This morning, as usual, we counted
the number of boys, which today totaled 9, and then the number of girls, which totaled 8. I
wrote on the board:

9 boys
8 girls 100

I then asked, “How many boys are out today?”

Natalie raised her hand and answered, “When all the boys are here, we have 10 boys, so
today we have 9 boys, so 1 boy is out.”

“How many girls are out today?”

Peter raised his hand and said, “It’s 2.” He knew that there were 10 girls in the class when 105
all were present.

I asked, “How did you figure it out?”

Peter replied, “8 and 1 and . . .” He seemed to be stuck in his thinking. I feel he did figure
it out correctly, but then had trouble explaining.

I then asked, “Can someone else tell me how many girls are out today?” 110

Denisha’s hand went up. She said, “It’s 2, because 8 is 2 numbers down from 10.”

“What does that mean?” I asked Denisha.

“When all the girls are here, it’s 10,” she told us. “You go down from 10 by 1, that’s 9, and
down 1 more, that’s 8, so that’s 2.”

We usually count around the circle to find the total number in class for the day. I decided 115
instead to see if the children could come up with the total just by looking at the numbers.

9 boys
8 girls

“How many kids here today?” I asked. Daniel answered, “17.”

I said, “Daniel, if you look at the numbers of boys and girls, can you tell me how you got 120
17?” He didn’t reply. I think that Daniel counted around the circle and didn’t want to tell me
that.

Rachelle offered, “8 and 9 makes a 1 and a 7.” When I asked Rachelle how she knew that,
she answered, “I counted in my mind.”

“How did you count in your mind? Say it out loud.” 125

She began, “1, 2, 3, 4, 5, 6, 7, 8, 9 . . .” But she didn’t go on. She wasn’t using her fingers, so I’m not sure how she was going to explain the answer.

“Did anyone else figure it out in a different way?” I wondered. Tamara’s hand went up and she said, “It’s 17, 8 and 9. Take 1 away from the 9 that makes it an 8, and 8 and 8 makes 16. So you put the 9 back. You go up one more from 8, you make a 9 and you have the 8, you get 17. You’re going one more higher than 16.” 130

I was amazed at how several of the children could see the relationship that numbers have to each other. They used the phrase “going up or down” to describe this. Tamara “went down” one from the 9 to make 8. It seemed easier for her to add 8 and 8, yet she knew that 9 was one more than 8 so the 16 had to “go up” by one. I feel that Peter was thinking about “going up” when he said 8 and 1, but couldn’t say the one more to make 10. Denisha described “going down” from 10 by one, and by one again to get 8. She then knew 2 girls were absent. Being able to go up and down with numbers seems to be a key to understanding number differences. 135 138