into practice

Chapter 1
Diagrams and Definitions

Essential Understanding 1*a*
A diagram is a sophisticated mathematical device for thinking and communicating.

Essential Understanding 1*b*
A diagram is a "built" geometric artifact, with both a history—a narrative of successive construction—and a purpose.

Essential Understanding 1*c*
A diagram is not a picture. It needs to be interpreted: learning how to read a diagram can be like learning a new language.

Essential Understanding 3*a*
Geometric objects can have different definitions. Some are better than others, and their worth depends both on context and values.

Essential Understanding 3*b*
Definitions in geometry are of two distinct types: definition *by genesis* (how you can create the object) and definition *by property* (how you can characterize the object in terms of certain features).

Essential Understanding 3*c*
Building definitions requires moving back and forth between the verbal and the visual.

The centrality of diagrams and definitions to the study of geometry is reflected by their prominence in Big Ideas 1 and 3 identified in *Developing Essential*

Understanding of Geometry for Teaching Mathematics in Grades 9–12 (Sinclair, Skelin, and Pimm 2012a, pp. 7–8):

> **Big Idea 1.** Working with diagrams is central to geometric thinking.
> **Big Idea 3.** Working *with* and *on* definitions is central to geometry.

To work successfully with diagrams and definitions in geometry, students need experiences that will help them build a strong foundation in the multiple essential understandings associated with these two big ideas.

The challenge of teaching students how to work with and create diagrams and definitions is the focus of this chapter, which examines students' misconceptions and outlines strategies for overcoming them. These strategies can help students revisit their misconceptions and replace them with robust understandings related to Big Ideas 1 and 3. A vignette in which students glimpse the need for precise definitions serves as an entry point to this discussion. This chapter and subsequent ones open by looking directly in on informal interactions among students to illustrate how their awareness of various components of geometry may begin to develop in a school setting.

> Derek and Marta are outside Mr. Ramirez's room, waiting for class to begin. They pass the time by making up jokes and riddles.
>
> "Hey Derek," Marta says. "Mississippi is the hardest state to spell. Spell it."
>
> Derek, a winner of the Cuyahoga County Little Mister Spelling Bee in elementary school, confidently rattles off, "M-I-S-S-I-S-S-I-P-P-I."
>
> "Nope," Marta says.
>
> "Did I miss an *S*?" Derek asks. "Is there another *P* in there somewhere?"
>
> "I'll give you another chance," Marta says.
>
> Derek thinks he knows the problem. "OK, M-I-S-S-I-S-S-S-I-S-S-I-P-P-I."
>
> "Worse," Marta says.
>
> At this point, Derek is becoming red in the face. "Well, *fine*. OK, *you* spell it."
>
> Marta looks smug. "I-T," she says. "'It.' Do you want me to use it in a sentence?"
>
> Derek's jaw drops a little. His brow furrows, and he doesn't say anything as he processes the exchange. Finally, he says, "OK, Marta. You win *this* one. But hey, I bet I can tell you where you got your shoes."

"Gee, Derek, I didn't know you were so interested in clothes," Marta says. "But there's no way you can tell me that."

"Fine. I bet I can tell you where you got your shoes, and I bet I can tell you in what part of town you got your shoes," he says.

"There's still no way," she says.

"OK, let's make this interesting," says Derek. "If I can tell you where you got your shoes and in what part of town you got your shoes, you have to tell Mr. Ramirez that I'm the best math student out there."

"Sure," she says. "Never going to happen."

Derek said, "You got your shoes ... on your feet! And the part of town you got them in is ... in front of Mr. Ramirez's room."

Marta's face now shows the same mix of disbelief and irritation that Derek's showed earlier.

The door opens, and Mr. Ramirez comes out. "Everything OK?" he asks.

Marta walks inside. "Everything's fine, Mr. Ramirez."

From outside, Derek says, "Don't you have something to say to Mr. Ramirez, Marta?"

Marta says, "Sure. You're the best math student *out there*, Derek."

"*Thank* you, Marta," Derek says. "It's kind of you to finally recognize I'm the best."

"But I'm *inside*," Marta says, and she walks to her seat.

Derek and Marta have perplexed and irritated each other with a mixture of imprecision and unexpected precision, or literalism, in their language, and their confusion suggests some of the problems that students face in studying geometry. For decades, researchers have documented the struggles that students have with precise definitions introduced in secondary geometry classes (Burger and Shaughnessy 1986; Confrey 1987; Crowley 1987; Clements and Battista 1992; Gal 2011; Gal and Linchevski 2010; and Özerem 2012). A set of common misconceptions and challenges has emerged from this body of research, and an awareness of these can be useful to classroom teachers as they develop their students' understanding of geometric concepts. These difficulties and misconceptions include the naïve or inexact use of formal terms, definitions, naming conventions, or symbols (Gal and Linchevski 2010; Özerem 2012); an inability to link verbal and pictorial representations (Gal and Linchevski 2010); and underdeveloped spatial reasoning (Clements and Battista 1992; Özerem 2012). Students who are attempting to interpret graphs of functions have a tendency to view the figures as static pictures rather than as representations of dynamic relationships (Monk 2003)—a difficulty that is addressed in *Putting Essential Understanding of Functions into Practice in Grades 9–12* (Ronau, Meyer, and Crites 2014).

Identifying Common Misconceptions and Challenges

This chapter focuses on four misconceptions or challenges related to diagrams and definitions that have emerged from research on learning geometry. Challenges are given prominence because studies have reported that as students learn to work with diagrams and definitions, they exhibit numerous difficulties that are significant but cannot be classified precisely as misconceptions. The four stumbling blocks to students' learning in geometry that are the focus of this chapter are the following:

1. A lack of prerequisite knowledge and skills.
2. A tendency to interpret diagrams as pictures, not as representations of relationships.
3. An inability to link verbal and pictorial representations.
4. A lack of precision in the use of terms, definitions, and symbols.

The chapter addresses each of these difficulties in turn, beginning with a discussion of the usefulness of van Hiele levels to assess students' mastery of prerequisite knowledge and skills.

Evaluating Prerequisite Knowledge and Skills with van Hiele Levels

Students' common lack of prerequisite knowledge and skills for secondary geometry was amply illustrated by Burger and Shaughnessy (1986), who assessed geometry students' levels of geometric thought by using the van Hiele (1980, 1984) model of the development of geometric reasoning. They found that a number of secondary students had incomplete concepts of basic geometric shapes and their properties and demonstrated a rudimentary understanding that put them only at level 0. The van Hiele levels of geometric reasoning have been useful in analyzing students' geometric thinking across the K–12 spectrum (Battista and Clements 1995). This trajectory can also be useful in assessing students' developmental levels in geometric reasoning and, therefore, proof. In the chart shown in figure 1.1, the first column shows the five levels, 0–4 (some authors label the levels 1–5), of the van Hiele model of geometric reasoning. Students must be at level 2 or 3 to be able to work in an axiomatic system in a formal setting. The chart includes all the levels to provide a more complete picture of the model. Because students may enter secondary geometry classes at level 1 or 2, they may be unprepared to launch into formal proof and may struggle with informal proof (Crowley 1987; Burger and Shaughnessy 1986). The chart also includes some strategies that teachers may find useful in helping students who are struggling with the topics and ideas at each stage. Although this chapter provides a cursory introduction to all the levels, it focuses on working with students at levels 1 and 2.

Level of thinking	Characteristics	Examples	Strategies for moving students to the next level
Level 0: Visualization	• Analyzes component parts of figures but cannot explain interrelationships between figures and properties. • Demonstrates thinking that is nonreflective, unsystematic, and illogical.	*Students at level 0—* • recognize basic shapes by their appearance without giving attention to parts, attributes, or properties; • refer to visual prototypes, such as doors, balls, and signs; • are unable to recognize right angles or opposite sides that are parallel in figures.	*Teachers can encourage students at level 0 to—* • examine examples and non-examples; • find "hidden" figures; • rearrange shapes into other shapes (for example, with tangrams and mosaic puzzles).
Level 1: Analytic	• Analyzes and organizes component parts of figures and their attributes. • Understands necessary properties. • Knows that the rotation of a square does not affect its "squareness."	*Students at level 1—* • recognize and name properties but do not understand ordered relationships; • are unable to consider an infinite variety of shapes; • cannot distinguish between necessary and sufficient properties.	*Teachers can encourage students at level 1 to—* • identify relationships by folding, measuring, and looking for symmetry; • fold a sheet of paper with a dot on it; • fold a sheet of paper and cut it, predicting the resulting shape; • sort and draw a variety of shapes.

Fig. 1.1. The van Hiele levels of geometric thinking. Based on Burger and Shaughnessy (1986), Crowley (1987), and Battista and Clements (1995).

Level of thinking	Characteristics	Examples	Strategies for moving students to the next level
Level 2: Informal deduction	• Understands abstract relationships among figures and can follow informal proofs. • Distinguishes between necessary and sufficient properties.	*Students at level 2—* • realize that a square is both a rhombus and a rectangle; • can use deduction to justify observation.	*Teachers can encourage students at level 2 to—* • use informal deductive language (*all*, *some*, *none*, *if-then*, *what if*, ... • use models and drawings (including dynamic geometry software) as tools to look for generalizations and counterexamples; • make and test conjectures; • critique anonymous students' work or reasoning; • use properties to define a shape or determine whether a particular shape is included in a given set.
Level 3: Formal deduction	• Understands, connects, and uses undefined terms, axioms, definitions, and theorems meaningfully.	*Students at level 3—* • reason with undefined terms, definitions, and theorems; • can employ logical argument; • can use axiomatic systems to develop proofs.	*Teachers can encourage students at level 3 to—* • discover the need for proofs; • develop the "idea" of a proof; • explore different types of proofs for the same conjecture; • (as a group) evaluate arguments based on the three components of proof; • (in small groups) evaluate arguments.
Level 4: Rigor	Level 4 comprises advanced geometric thinking beyond the scope of the traditional secondary mathematics classroom. Students at this level can work within and across multiple geometric systems, including Euclidean, hyperbolic, elliptic, and projective.		

Fig. 1.1. *Continued*

Level-to-level progress through experiential phases

The van Hiele levels are hierarchical: A learner must master one level before moving on to the next. The levels are also sequential, since students must proceed from one level to the next in order. Students cannot skip levels, nor can they master levels out of order. The levels are not age dependent; even adults can be at level 0. The primary way to move students from one level to the next is through geometric experiences. Van Hiele focused on the nature of such experiences and proposed five experiential phases that are necessary for moving from one level to the next (Crowley 1987; van Hiele 1980, 1984).

Phase 1: Inquiry/Investigation

Students engage in conversation and conjecture about figures and their properties. This phase serves as an opportunity for the secondary teacher to make an informal assessment of the level of knowledge of the class and to orient the students to the lesson topic.

Phase 2: Directed Orientation

Students investigate the topic under study, attending to properties and guided by materials designed to sequence the experience so that relationships are revealed to them in developmentally appropriate ways.

Phase 3: Explication

Building on their experiences in phase 2, students engage in discussion of the structures that they have observed. Teachers guide this discussion primarily by helping students use more precise and accurate language.

Phase 4: Free Orientation

Students investigate the topic more thoroughly by using more complex tasks. They may engage in an activity such as paper folding, dynamic construction, or transformation, in which they predict a particular outcome, perform the requisite action, determine the result, and reflect on their prediction and the result.

Phase 5: Integration

Students review and summarize their experience, with the goal of forming an overview of the concepts and relationships associated with the topic. At the end of this phase, students should be at a new level of geometric reasoning.

The van Hiele theory of geometric thought posits that students must progress through each phase sequentially to move from one level of geometric reasoning to the next. That is, for students to move from level 1 to level 2, they need sequential experience with level-appropriate activities in each of the five phases.

At level 0, students do not recognize individual components of figures. For example, they do not recognize lines as parallel or perpendicular or angles as corresponding or congruent. They think that a square in an unconventional orientation is a different shape—not a square. Few students in secondary geometry are still at this level, but some are. These students will have difficulty understanding individual properties of figures and will be perplexed by definitions, axioms, and theorems.

Students at level 1 are able to recognize and name properties of figures but are not able to form ideas about ordered relationships of those properties. Secondary geometry students at level 1 will also have a great deal of difficulty with definitions, axioms, and theorems. The concept of proof will be impossible for them.

Students who have reached level 2 are able to reason about abstract relationships within and between figures. At this level, students are able to use informal or empirical proving methods. Many students in secondary geometry are at this level.

At level 3, students are able to conclude that if something is always true, then it must necessarily be true. Students who engage in formal deductive reasoning can operate within a mathematical system. Few students in secondary geometry are at this level.

Van Hiele in the classroom: Hierarchy of Hexagons

Figure 1.2 presents a set of hexagons in an activity called Hierarchy of Hexagons, developed by Danielson (2014). This activity illustrates the pedagogical usefulness of the van Hiele levels in the classroom. In addition to involving students in the act of definition, the activity may help support their ascent from one level to the next. The Hierarchy of Hexagons activity derives from the better-known hierarchy of quadrilaterals. In school geometry, teachers commonly teach, and students attempt to learn, the hierarchy of quadrilaterals (a van Hiele level 2 activity). This hierarchy arranges all the various types of quadrilaterals according to their orders and classes, closely resembling similar hierarchies in the animal and plant kingdoms. Squares exist at the top of the hierarchy, as their definition is the most restricted. Rectangles are directly below, with the square's definition broadening to include noncongruent sides.