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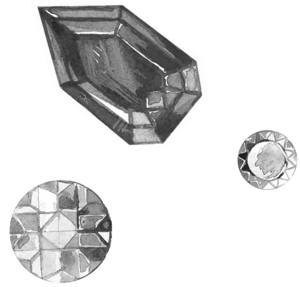


Reasoning about Sums of Consecutive Numbers

OVERVIEW

During the primary and elementary grades, major emphasis is placed on learning numbers and developing proficiency with the basic arithmetic operations. As the curriculum in the middle grades expands to include more algebra, geometry, and other topics, there is a danger that numbers and operations may be overshadowed on the mistaken assumption that students have by now mastered these topics and should focus elsewhere. But, in fact, number and operations remain central themes as students expand their knowledge to include integers and rational numbers and as they develop facility working with fractions, decimals, and proportional reasoning.

Derived from "Reasoning about Sums of Consecutive Numbers" in *Navigating through Problem Solving and Reasoning in Grades 6–8* (NCTM 2009b, pp. 12–16).

In the middle grades, students have numerous opportunities to develop and use number concepts in a variety of contexts and applications. They are able to focus more directly on properties of numbers than they were in earlier grades, and they can explore interesting number patterns and relationships. This activity invites students to reason and think analytically about sums of consecutive numbers, to note patterns, and to investigate whether and why those patterns make sense. Such exploration of numbers and their relationships promotes the development of number sense. Because this activity and the questions it raises will probably be new for most students, you are encouraged to work through it yourself before presenting it to the students.

"Reasoning and proof should be a consistent part of students' mathematical experience in prekindergarten through grade 12. Reasoning mathematically is a habit of mind, and like all habits, it must be developed through consistent use in many contexts." (NCTM 2000, p. 56)

GOALS

- Solve problems using numerical and algebraic expressions and equations.
- Work with concepts related to consecutive addends, divisibility, and powers of 2.
- Make conjectures, determine their validity, and support or contradict them by making mathematical arguments.
- Explain problem-solving strategies and justify solutions.

MATERIALS NEEDED

Copies of the activity sheet Sum-thing about Consecutive Numbers, one for each student

Copies of the Recording Sheet for Sum-thing about Consecutive Numbers, one for each student

Engage

In this activity, students will investigate the counting numbers from 1 to 35 to determine which of those numbers can be written as a sum of consecutive counting numbers. As they do, they will look for patterns, make conjectures, and present explanations and arguments to support or refute those conjectures.

Introduce the activity with a brief discussion of the meaning of *consecutive numbers*: two or more integers for which the *absolute difference* between any two of them is 1. Relate this to the number line and the fact that the distance between any two consecutive integers is 1. We refer to this as "absolute difference" because the distance is not concerned about the direction of movement from one integer to its neighbor—that is, the absolute difference between 4 and 5 is 1, regardless of whether one is moving from 4 to 5 or from 5 to 4 [$|4 - 5| = |5 - 4| = 1$]. Also tell the students that in the initial activity they will be working with only the *counting numbers,* that is, the set {1, 2, 3 . . .}.

Show the students the following three examples of sums of consecutive counting numbers:

$$3 + 4 = 7$$

$$2 + 3 + 4 = 9$$

$$4 + 5 + 6 + 7 = 22$$

Here 7, 9, and 22 are the sums of two, three, and four consecutive numbers, respectively. Ask the students to each write one or two more examples of sums of consecutive counting numbers, and call on several students to share their examples. Be sure that the students demonstrate understanding of the task and of the meaning of consecutive numbers before proceeding with the activity.

Explore

Give each student a copy of the activity Sum-thing about Consecutive Numbers and pose the following challenge: Consider the counting numbers from 1 through 35. We want to see how many of them can be written as a sum of two or more consecutive counting numbers. Use the chart in question 1 on the activity sheet to record all the consecutive sums that you can find. Some of the numbers may be written as sums in more than one way. For example, 9 is shown earlier as a sum of three consecutive numbers (2 + 3 + 4), but 9 is also the sum of two consecutive numbers (4 + 5). In the chart, record *all* the ways you can find for the numbers from 1 to 35.

Let the students work individually for 5 or 10 minutes, then arrange them in twos or threes so they can share and discuss their findings. Allow time for the teams to continue working together to complete the chart in question 1 for the numbers 1 through 35.

The next step in the investigation is to ask the students to examine the sums they have recorded earlier to see if they can discover any patterns or relationships. To facilitate that exploration, give each student a copy of the Recording Sheet for Sum-thing about Consecutive Numbers and have them organize their earlier work in a table where for each number from 1 through 35 (the rows), the possible sums are recorded in the columns according to the number of addends in each case. Table 1.1 shows a copy of a completed table.

Table 1.1. Numbers 1–35 as Sums of Consecutive Numbers

Number	Two Addends	Three Addends	Four Addends	Five Addends	Six Addends	Seven Addends
1						
2						
3	1 + 2					
4						
5	2 + 3					
6		1 + 2 + 3				
7	3 + 4					
8						
9	4 + 5	2 + 3 + 4				
10			1 + 2 + 3 + 4			
11	5 + 6					
12		3 + 4 + 5				
13	6 + 7					
14			2 + 3 + 4 + 5			
15	7 + 8	4 + 5 + 6		1 + 2 + 3 + 4 + 5		
16						

Table 1.1. Numbers 1–35 as Sums of Consecutive Numbers *(continued)*

17	8 + 9					
18		5 + 6 + 7	3 + 4 + 5 + 6			
19	9 + 10					
20				2 + 3 + 4 + 5 + 6		
21	10 + 11	6 + 7 + 8			1 + 2 + 3 + 4 + 5 + 6	
22			4 + 5 + 6 + 7			
23	11 + 12					
24		7 + 8 + 9				
25	12 + 13			3 + 4 + 5 + 6 + 7		
26			5 + 6 + 7 + 8			
27	13 + 14	8 + 9 + 10			2 + 3 + 4 + 5 + 6 + 7	
28						1 + 2 + 3 + 4 + 5 + 6 + 7
29	14 + 15					
30		9 + 10 + 11	6 + 7 + 8 + 9	4 + 5 + 6 + 7 + 8		
31	15 + 16					
32						
33	16 + 17	10 + 11 + 12			3 + 4 + 5 + 6 + 7 + 8	
34			7 + 8 + 9 + 10			
35	17 + 18			5 + 6 + 7 + 8 + 9		2 + 3 + 4 + 5 + 6 + 7 + 8

When the students have finished organizing their sums in the previous step, direct them to examine their tables and write descriptions of any patterns that they observe. Have the students share their observations and collect them in a classroom display, but do not delve into them yet. Right now, we want to collect as many observed patterns and conjectures as we can; later we will examine them to see what is happening and why. Here are some of the observations that students have reported:

> ***"People who reason and think analytically tend to note patterns, structure, or regularities in both real-world and symbolic objects; they ask if those patterns are accidental or if they occur for a reason; and they conjecture and prove."*** (NCTM 2000, p. 56)

1. We noticed that all the odd numbers from 3 on can be written as a sum of two consecutive numbers. And we also noticed that every time you go from one odd number to the next, the two addends increase by one. For example, 3 = 1 + 2, 5 = 2 + 3, 7 = 3 + 4, and so on.

2. We noticed that, starting with 3, every second number can be written as a sum of two consecutive numbers; starting with 6, every third number (6, 9, 12 . . .) can be written as a sum of three consecutive numbers; starting with 10, every fourth number (10, 14, 18, 22 . . .) can be written as a sum of four consecutive numbers. A similar pattern continues for all the columns.
3. We saw something that links the first two patterns. You showed us that for the sums of two consecutive numbers, every time you go to the next odd number, the addends start one number higher. The same kind of pattern is true down all the columns. For instance, $6 = 1 + 2 + 3$, and for the next numbers in the column, $9 = 2 + 3 + 4$, then $12 = 3 + 4 + 5$, and so on. You can always get the next case just by starting the addends one number higher. And that works for four or five or any number of consecutive addends. You could keep going like that forever.
4. We saw that all the numbers that are sums of three consecutive addends are divisible by 3: 6, 9, 12, 15, and so on. And all the numbers that are sums of five addends are divisible by 5: 15, 20, 25, 30, 35. We only have two examples of numbers with seven addends (28 and 35), but they're both divisible by 7.
5. We noticed something else about some of the odd numbers. Look at 15: $15 = 3 \times 5$, and 15 is a sum of three consecutive numbers and of five consecutive numbers. Same thing for 35: $35 = 5 \times 7$, and 35 is a sum of both five and seven consecutive numbers. But sometimes it doesn't seem to work that way, like for 21—it's a sum of three consecutive numbers, but not of seven—and we wonder why. Maybe we just don't have enough information.
6. But there's more to what you just said. Some of the odd numbers, like those you mentioned, are sums in more than one way. But we noticed that prime numbers on the list, from 3 on, are sums of only two consecutive numbers and never more than that.
7. We couldn't find any consecutive numbers that give sums of 1, 2, 4, 8, 16, or 32. Oh, wait! Those numbers are all powers of 2: 2^0, 2^1, 2^2, 2^3, 2^4, 2^5. Something must be going on here. We wonder what.

Congratulate the students for their observations. They are beginning to think like mathematicians, who continually look for patterns and for reasons to explain them. Depending on their background and prior experiences, the students will vary in their ability to note and articulate patterns, and you may need to assist them in formulating their observations. For example, some may not immediately recognize the powers of 2, or they may be unfamiliar with using exponential notation to represent powers; that can be an opportunity for you to introduce those ideas to the class.

"The role of the teacher is to engage students in tasks that promote reasoning and problem solving and facilitate discourse that moves students toward shared understanding of mathematics." (NCTM 2014, p. 11)

Before you delve further into the reasons underlying the students' observations, ask them to look at question 3 on the Sum-thing about Consecutive Numbers sheet, which challenges them to apply the patterns they have observed to five other numbers—45, 57, 62, 75, and 80—and use their observations to write those numbers as sums of consecutive addends in as many ways as they can. Give them sufficient time to work, individually or in teams, to discover the sums and to write which pattern they used to determine each case; then have them share their discoveries with the class. Figure 1.1 shows the possible sums. Note that one of the numbers, 75, can also be written as a sum of 10 consecutive numbers, which exceeds the number of columns on the students' chart and does not reflect any of the observed patterns reported earlier. Do not be surprised if the students do not find that case; if they do not, you can use it as an opportunity to challenge them to "look for one more solution for 75."

FIG. 1.1

Sums of consecutive addends for five numbers

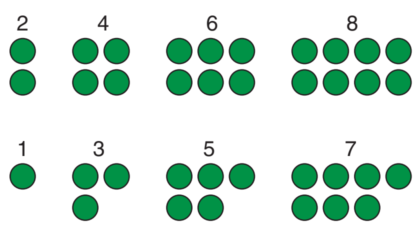
45 is the sum of two, three, five, and six consecutive addends:
(22 + 23), (14 + 15 + 16), (7 + 8 + 9 + 10 + 11),
(5 + 6 + 7 + 8 + 9 + 10)
57 is the sum of two, three, and six consecutive addends:
(28 + 29), (18 + 19 + 20), (7 + 8 + 9 + 10 + 11 + 12)
62 is the sum of four consecutive addends:
(14 + 15 + 16 + 17)
75 is the sum of two, three, five, six, and ten consecutive addends:
(37 + 38), (24 + 25 + 26), (13 + 14 + 15 + 16 + 17),
(10 + 11 + 12 + 13 + 14 + 15),
(3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12)
80 is the sum of five consecutive addends:
(14 + 15 + 16 + 17 + 18)

Now that the students have discovered a number of patterns and shown that they can apply those patterns in new situations, it is time to look deeper into what is happening, to ask whether their observations will always apply, and why they are (or perhaps are not) true. You might choose to look at the first observed pattern together to see why it makes sense and to model an approach for the students before you ask them to examine other patterns on their own.

The first observation noted that all odd numbers, starting with 3, can be written as a sum of two consecutive numbers. Why is that true? Let's begin by recalling what we know about even and odd numbers: We know that all even numbers are multiples of 2 and that all odd numbers are one more (or one less) than a multiple of 2. We can represent evens and odds visually (see figure 1.2), and we can express the same properties algebraically as $N_{even} = 2n$ and $N_{odd} = 2n + 1$ for any integer value of n. Since we are always adding two consecutive numbers, one of them will be even and the other will be odd, and the sum of an even number and an odd number is always an odd number. To show that algebraically, we have $(2n) + (2n + 1) = (4n + 1)$. To reverse the process, if we start with any odd number, N, and divide it by 2, we will always have a remainder of 1. For example, 35 ÷ 2 = 17, remainder 1. The two addends are 17 and (17 + 1), which gives 35 = 17 + 18.

FIG. 1.2

Visual representation of even and odd numbers



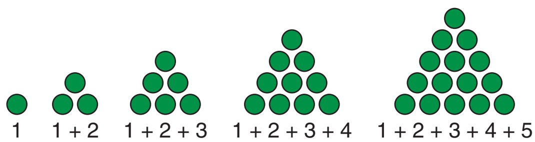
The goals throughout this activity have been to encourage the students to look for patterns, to make conjectures based on the observed relationships, and to develop explanations and arguments to either support or contradict their conjectures. We saw earlier an example of making sense of the observation about the sums of two consecutive numbers. One way to approach the challenge of analyzing the rest of the discoveries that the students reported is to assign each small group the task of examining one of the posted patterns and preparing a presentation to the class. Allow ample time for the groups to discuss their assigned task and to prepare their reports. As you circulate throughout the classroom, be prepared to answer questions the students may have for you and to make helpful suggestions, if appropriate. Following are summaries of some of the reasoning and explanations that students have reported about sums of consecutive numbers.

"Effective teaching of mathematics consistently provides students, individually and collectively, with opportunities and supports to engage in productive struggle as they grapple with mathematical ideas and relationships." (NCTM 2014, p. 10)

Some students looked at the second observed pattern—that, starting with 3, every second number is a sum of two consecutive addends; starting with 6, every third number is a sum of three consecutive addends; starting with 10, every fourth number is a sum of four consecutive addends, and so on. They focused first on the starting number in each column—3, 6, 10, 15, 21, and 28—and recognized them as belonging to the sequence of triangular numbers (see figure 1.3). [If your students are unfamiliar with the triangular numbers, take the time to introduce them and explore some of their properties, because those numbers will arise often and figure prominently in problems the students will encounter throughout their mathematical studies.]

FIG. 1.3

The first five triangular numbers



Once the students recognized the appearance of the triangular numbers, they were able to make sense of the rest of the pattern by using the visual representation in figure 1.4. They then used figure 1.5 to demonstrate why the four-addend column started with 10 and why every fourth number after 10 could also be written as a sum of four consecutive addends. Their representation also allowed them to explain the third pattern posted earlier: You can always get the next case just by starting the addends one number higher.

FIG. 1.4

Visualizing patterns observed in the table of sums

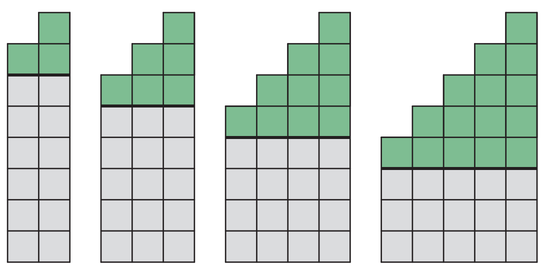
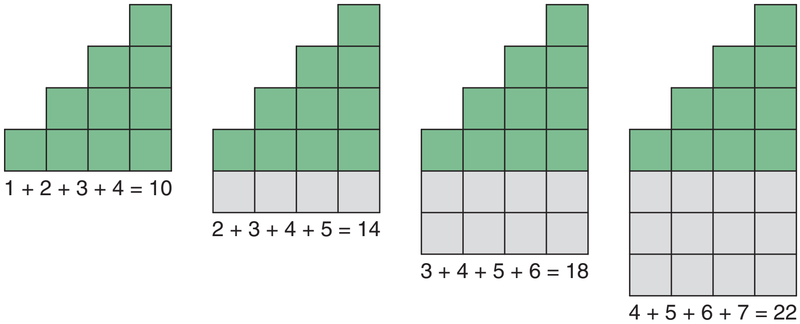


FIG. 1.5

Representing sums of four consecutive numbers

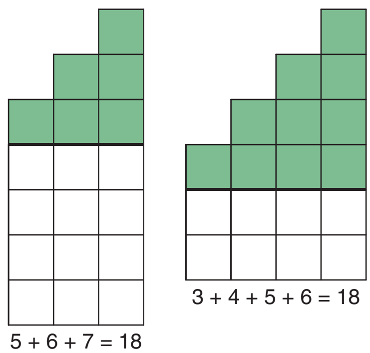


> ***"The role of the student is to be actively involved in making sense of mathematics tasks by using varied strategies and representations, justifying solutions, making connections to prior knowledge or familiar contexts and experiences, and considering the reasoning of others."*** **(NCTM 2014, p. 11)**

After the students understood what was happening in the previous example, they also showed how they could reverse the relationship to decide whether a given number could be a sum of four consecutive addends: Subtract 10 from the number in question and see if the remainder is a multiple of 4. They illustrated their strategy with the earlier problem of expressing 62 as a sum of four consecutive addends: $62 - 10 = 52$ and $52 \div 4 = 13$. Hence, following the pattern shown in figure 1.5, add 13 gray rows below the triangular array to get $14 + 15 + 16 + 17 = 62$. The strategy is adaptable to any number of addends: To determine if a given number, N, can be written as a sum of n consecutive numbers, subtract from N the nth triangular number, then see if the remainder is a multiple of n. It also allows one to determine if a number can be written as a consecutive sum in more than one way, as illustrated in figure 1.6 [$18 - 6 = 12$, and $12 \div 3 = 4$; $18 - 10 = 8$, and $8 \div 4 = 2$].

FIG. 1.6

Eighteen as a sum of three and of four consecutive addends



Another group of students examined the fourth observation—that if a number, N, is a sum of an odd number, n, of consecutive addends, then N is divisible by n. They began with $15 = 3 \times 5$, and recognizing that 15 could be a sum of three 5s or of five 3s, they developed the "balancing act" illustrated in figure 1.7.

FIG. 1.7

Fifteen as a sum of three and of five consecutive addends

$5 + 5 + 5 = 15$ $\quad$ $3 + 3 + 3 + 3 + 3 = 15$

$4 + 5 + 6 = 15$ $\quad$ $1 + 2 + 3 + 4 + 5 = 15$

Because there are an odd number of addends, there is always a middle term that functions as a pivot point in the expression. After locating the middle term, the addends to the left and right are adjusted in pairs, each time decreasing the term on the left while increasing the corresponding term on the right, as shown in figure 1.7.

A few students even noticed that if the middle addend is itself an odd number, they could use the balancing-act strategy a second time to yield another sum of consecutive numbers. They demonstrated with the number 45 from problem 3 on the activity sheet:

$$14 + 15 + 16 = 45$$
$$5 + 6 + 7 + 8 + 9 + 10 = 45$$

Here the 15 in the center position of the first expression is decomposed into (7 + 8), and the continued balancing results in the pairs (6 + 9) and (5 + 10), or three 15s. It was the application of that strategy to the 5-addend expression for 75 that led to the discovery of the 10-addend expression in figure 1.1.

Extend

Many of the patterns that the students observed were explored and explained in the previous discussion, but a few unanswered questions remain. One was posed in observation 5, when the students noted that "21 is a sum of three consecutive numbers, but not of seven, and we wonder why." An answer to their question can be found if we extend the set of possible addends from the counting numbers to the integers. Using the balancing-act strategy on the number 21 and allowing the consecutive addends to be integers leads to the following:

$$3 + 3 + 3 + 3 + 3 + 3 + 3 = 21$$
$$0 + 1 + 2 + 3 + 4 + 5 + 6 = 21$$

Note that the previous line is equivalent to the expression from the six-addend column on the chart. A more graphic example is shown if we consider the number $27 = 3 \times 9$:

$$3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 = 27$$
$$-1 + 0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 = 27$$

Here we see that the first three terms sum to zero, so the expression we are left with is the one already noted for six consecutive counting numbers. The students can try other examples. In each case the expression they get using integers reduces to an expression with the same or fewer counting-number addends. The important point is that their pattern, or "rule," which seemed to fall short in some situations does indeed hold true if they apply it using the integers instead of only counting numbers.

The extension to integers also provides insight into the sixth observation earlier, that the prime numbers are sums of only two, and never more than two, consecutive numbers.

Consider, for example, the prime number 11; it has only two factors and can only be written as 11×1. Thus, we get

$$1+1+1+1+1+1+1+1+1+1+1$$
$$-4+-3+-2+-1+0+1+2+3+4+5+6,$$

all of which simplifies to $5 + 6 = 11$. Again, the students can check other examples for reinforcement.

The remaining unanswered questions concern numbers that are sums of an even number of consecutive addends. Consider first any number, N, that is not a power of 2, such as 26 or 27. When those numbers are written as a sum of an even number of consecutive addends, there are two middle terms, as shown here:

$$5 + (6 + 7) + 8 = 26 \qquad 2 + 3 + (4 + 5) + 6 + 7 = 27$$

As these examples show, the sum of the two middle terms is a divisor of N ($6 + 7 = 13 = 26 \div 2$; $4 + 5 = 9 = 27 \div 3$). This gives a clue to the relationship between the number, N, and the number of addends, n: 26 is a sum of four addends, and $26 \div 4 = 6\frac{1}{2}$, whereas 27 is a sum of six addends and $27 \div 6 = 4\frac{1}{2}$. When the quotient of $N \div n = q\frac{1}{2}$, that quotient is an "imaginary" pivot point for the sum [$6\frac{1}{2}$ for the pair $(6 + 7)$; $4\frac{1}{2}$ for the pair $(4 + 5)$ in the two cases]. The rest of the expression parallels the balancing-act relationship observed for an odd number of addends. It also explains why 57, in the numbers from question 3 of the activity sheet, could be a sum of six addends but not of four: $57 \div 6 = 9\frac{1}{2}$ (and the two middle terms are $9 + 10$), but $57 \div 4 = 14.25$, so the four-addend case does not exist. The students should apply this approach to verify some of the other entries in their table.

Finally, there is the case of the powers of 2, for which no sums of consecutive addends were found. Why not? The answer to this question requires algebra that may be beyond the mathematical background of some of your students, and for them you can opt to defer the question at this time. But for students with sufficient algebraic knowledge, we can use the triangular-number approach discussed earlier to answer the question.

First, we need to find an expression for the nth triangular number. Figure 1.8 shows one way to determine this by writing the nth triangular number first as $T_n = 1 + 2 + 3 + \cdots + (n - 1) + n$ and again as a sum with the terms reversed. Adding the two expressions term by term yields n partial sums of $(n + 1)$ as the value of $2T_n$, from which we derive the expression for T_n. To help the students understand the generalization, you might want to have them apply it to a few specific cases. For example:

FIG. 1.8

Finding an expression for the *n*th triangular number

Determine the value of the *n*th triangular number as follows:

$$\begin{aligned} T_n &= 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n \\ T_n &= n + (n-1) + (n-2) + \cdots + 3 + 2 + 1 \\ 2T_n &= (n+1) + (n+1) + \cdots + (n+1) + (n+1) \\ &= n(n+1) \\ T_n &= \frac{n(n+1)}{2} \end{aligned}$$

$$
\begin{aligned}
T_5 &= 1 + 2 + 3 + 4 + 5 \\
T_5 &= 5 + 4 + 3 + 2 + 1 \\
2T_5 &= 6 + 6 + 6 + 6 + 6 \\
&= 5 \times 6 \\
T_5 &= \frac{5 \times 6}{2} = 15
\end{aligned}
$$

Using the triangular-number strategy developed earlier in this activity to see if a power of 2 (call it 2^k) can be written as a sum of n consecutive addends, 2^k must equal the nth triangular number plus a multiple of n, or

$$
\begin{aligned}
2^k &= T_n + n \cdot a \\
&= \frac{n(n+1)}{2} + n \cdot a \\
2^{k+1} &= n(n+1) + 2na \\
&= n[(n+1) + 2a]
\end{aligned}
$$

Now, if n is an even number, then $(n + 1)$ and $[(n + 1) + 2a]$ are both odd; and, if n is odd, then $(n + 1)$ and $[(n + 1) + 2a]$ are both even. In either case, we have (even) × (odd) = odd = $2^{(k+1)}$, which is impossible.

> ***"An effective teacher provides students with appropriate challenge, encourages perseverance in solving problems, and supports productive struggle in learning mathematics."*** (NCTM 2014, p. 11)

Not all of your students may be ready for all of the explanations developed here in the Extend section. That will depend on their algebraic background, and you can decide how far to pursue the inquiry at any given time.

SUMMARY

This activity helps build students' number sense by returning them to the familiar operation of addition and encouraging them to explore more deeply, going beyond "getting the answer" to looking for patterns and relationships, then reasoning and making sense of the connections that they observe. The insights the students develop as a result of formulating conjectures and then explaining and justifying those conjectures help build their confidence and promote their ability to reason mathematically. Repeated opportunities such as this to explore relationships, make conjectures, and justify results help students see and expect that mathematics makes sense and that they can understand and use it.