CHAPTER 1

Setting the Stage

Imagine walking into a high school classroom where students are working on a statistics unit in which they are fitting a function to data and then using the function they created to solve a problem. As the class begins, the teacher asks the class what they know about bungee jumping. Students indicate that it involves jumping off something high, like a bridge, while connected to an elastic cord. As one student explains, “You jump off and you fall, but then the cord springs you back up again and again.” The teacher then asks, “What happens if the cord is too short or too long?” Students respond that if the cord is too short, it might not be much fun because you wouldn’t fall very far and then you wouldn’t spring back much. But if the cord is too long, you could crash into the ground. The teacher then shows a YouTube video of a bungee jump at Victoria Falls (https://www.youtube.com/watch?v=UQFMy9Tz8dY), which captivates students’ attention and leaves many exclaiming, “Cool. I want to try that!” (This lesson is adapted from NCTM Illuminations, https://illuminations.nctm.org/Lesson.aspx?id=2157.)

The teacher then explains that they are going to model a bungee jump using Barbie dolls and rubber bands: “You will conduct an experiment, collect data, and then use the data to predict the maximum number of rubber bands that should be used to give Barbie a safe jump from 400 cm.” She provides each group of students with a Barbie and 20 rubber bands and indicates that other supplies they need (e.g., a large piece of paper, measuring tool, tape) can be found on the resource table at the back of the room. She then asks the class: “What is it you need to figure out?” Students respond that they need to figure out how far Barbie will fall as the number of rubber bands increases. The teacher then demonstrates how to attach the rubber band to Barbie’s feet and how to attach one rubber band to the next so that they all do it the same way.

As students begin their work, the teacher monitors the activity, intervening as needed to ensure that they are constructing the bungee cord correctly, using measuring tools appropriately, and keeping track of the data as they continue to add rubber bands to the bungee cord. As students conclude their data collection, the teacher reminds them that they need to create a scatterplot of the data and determine a line of best fit, which they could check using a web-based applet. (See http://illuminations.nctm.org/Activity.aspx?id=4186 for an applet that can support this investigation.) She explains that once they have their line of best fit, they need to predict the maximum number of rubber bands they will need for Barbie's 400 cm jump.

When predictions have been finalized, the teacher explains that they are going to reconvene on the second floor stairwell, where she has already marked a height of 400 cm. She explains that they will test their conjectures with the number of rubber bands they predicted and determine how close they come to 400 cm.

The class ends with students returning to the classroom and discussing as a group how accurate their predictions were, why some lines of best fit might have been more accurate than others, and what the slope and *y*-intercept of the equations actually mean in the bungee Barbie context.

A Vision for Students as Mathematics Learners and Doers

The lesson portrayed in this opening scenario exemplifies the vision of school mathematics that the National Council of Teachers of Mathematics (NCTM) has been advocating for in a series of policy documents over the last 25 years (1989, 2000, 2006, 2009a). In this vision as in the scenario, students are active learners, constructing their knowledge of mathematics through exploration, discussion, and reflection. The tasks in which students engage are both challenging and interesting and cannot be answered quickly by applying a known rule or procedure. Students must reason about and make sense of a situation and persevere when a pathway is not immediately evident. Students use a range of tools to support their thinking and collaborate with their peers to test and refine their ideas. A whole-class discussion provides a forum for students to share ideas and clarify understandings, develop convincing arguments, and learn to see things from other students' perspectives.

In the "bungee Barbie" scenario, students were faced with a problem, and they needed to collect and analyze data in order to solve it. All students could enter the problem by creating bungees of different lengths and dropping Barbie to see how far she fell, measuring the length of each jump, recording data, and constructing a scatterplot. Students were able to make a guess at the line of best fit and then check their guess through the use of the applet. During the discussion, students reported on the accuracy of their predictions, reflected on why some predictions were better than others, and were pressed to consider what the line of best fit equation meant in the context of the bungee Barbie task. When the issue of how confident they should be about their equation came up, the teacher could then introduce and discuss the meaning of the correlation coefficient (which was generated by the applet).

The vision for student learning advocated for by NCTM, and represented in our opening scenario, has gained growing support over the past decade as states and provinces have put into place world-class standards (e.g., National Governors Association Center for Best Practices and Council of Chief State School Officers [NGA Center and CCSSO] 2010). These standards focus on developing conceptual understanding of key mathematical ideas, flexible use of procedures, and the ability to engage in a set of mathematical practices that include reasoning, problem solving, and communicating mathematically.

A Vision for Teachers as Facilitators of Student Learning

Meeting the demands of world-class standards for student learning will require teachers to engage in what has been referred to as "ambitious teaching." Ambitious teaching stands in sharp contrast to the well-documented routine found in many classrooms that consists of homework review and teacher lecture and demonstration, followed by individual practice (e.g., Hiebert et al. 2003). This routine has been translated into the "gradual release model": I Do (tell students what to do); We Do (practice doing it with students); and You Do (practice doing it on your own) (Santos 2011). In instruction that uses this approach, the focus is on learning and practicing procedures with limited connection to meaning. Students have limited opportunities to reason and problem-solve. While they may learn the procedure as intended, they often do not understand why it works and apply the procedure in situations where it is not appropriate. According to W. Gary Martin (2009, p. 165), "Mechanical execution of procedures without understanding their mathematical basis often leads to bizarre results"—that is, at times students get answers that make no sense, yet they have no idea how to judge correctness because they are mindlessly applying a procedure they do not really understand.

In ambitious teaching, the teacher engages students in challenging tasks and then observes and listens while they work so that he or she can provide an appropriate level of support to diverse learners. The goal is to ensure that each and every student succeeds in doing high-quality academic work, not simply executing procedures with speed and accuracy. In our opening scenario, we see a teacher who is engaging students in meaningful mathematics learning. She has selected an authentic task for students to work on, provided resources to support their work (e.g., a method for measuring and recording data, use of an applet for investigating line of best fit, partners with whom to exchange ideas), monitored students while they worked and provided support as needed, and orchestrated a discussion in which students' contributions were key. However, what we don't see in this brief scenario is exactly *how* the teacher is eliciting thinking and responding to students so that every student is supported in his or her learning. According to Lampert and her colleagues (Lampert et al. 2010, p. 130):

> Deliberately responsive and discipline-connected instruction greatly complicates the intellectual and social load of the interactions in which teachers need to engage, making ambitious teaching particularly challenging.

This book is intended to support teachers in meeting the challenge of ambitious teaching by describing and illustrating a set of teaching practices that will facilitate the type of "responsive and discipline-connected instruction" that is at the heart of ambitious teaching.

Support for Ambitious Teaching

Principles to Actions: Ensuring Mathematical Success for All (NCTM 2014) provides guidance on what it will take to make ambitious teaching, and the rigorous content standards it targets, a reality in classrooms, schools, and districts in order to support mathematical success for each and every student. At the heart of this book, *Taking Action: Implementing Effective Mathematics Teaching Practices in Grades 9–12*, is a set of eight teaching practices that provide a framework for strengthening the teaching and learning of mathematics (see fig. 1.1). These teaching practices describe intentional and purposeful actions taken by teachers to support the engagement and learning of each and every student. These practices, based on knowledge of mathematics teaching and learning accumulated over more than two decades, represent "a core set of high-leverage practices and essential teaching skills necessary to promote deep learning of mathematics" (NCTM 2014, p. 9). Each of these teaching practices is examined in more depth through illustrations and discussions in the subsequent chapters of this book.

Establish mathematics goals to focus learning. Effective teaching of mathematics establishes clear goals for the mathematics that students are learning, situates goals within learning progressions, and uses the goals to guide instructional decisions.

Implement tasks that promote reasoning and problem solving. Effective teaching of mathematics engages students in solving and discussing tasks that promote mathematical reasoning and problem solving and allow multiple entry points and varied solution strategies.

Use and connect mathematical representations. Effective teaching of mathematics engages students in making connections among mathematical representations to deepen understanding of mathematics concepts and procedures and as tools for problem solving.

Facilitate meaningful mathematical discourse. Effective teaching of mathematics facilitates discourse among students to build shared understanding of mathematical ideas by analyzing and comparing student approaches and arguments.

Pose purposeful questions. Effective teaching of mathematics uses purposeful questions to assess and advance students' reasoning and sense making about important mathematical ideas and relationships.

Build procedural fluency from conceptual understanding. Effective teaching of mathematics builds fluency with procedures on a foundation of conceptual understanding so that students, over time, become skillful in using procedures flexibly as they solve contextual and mathematical problems.

Support productive struggle in learning mathematics. Effective teaching of mathematics consistently provides students, individually and collectively, with opportunities and supports to engage in productive struggle as they grapple with mathematical ideas and relationships.

Elicit and use evidence of student thinking. Effective teaching of mathematics uses evidence of student thinking to assess progress toward mathematical understanding and to adjust instruction continually in ways that support and extend learning.

Fig. 1.1. The Eight Effective Mathematics Teaching Practices (NCTM 2014, p. 10)

Ambitious mathematics teaching must be equitable. Driscoll and his colleagues (Driscoll, Nikula, and DePiper 2016, pp. ix–x) acknowledge that defining equity can be elusive but argue that equity is really about fairness in terms of access—"providing each learner with alternative ways to achieve, no matter the obstacles they face"—and potential—"as in potential shown by students to do challenging mathematical reasoning and problem solving." Hence, teachers need to pay attention to the instructional opportunities that are provided to students, particularly to historically underserved and/or marginalized youth (i.e., students who are Black, Latina/

Latino, American Indian, low income) (Gutierrez 2013, p. 7). Every student must participate substantially in all phases of a mathematics lesson (e.g., individual work, small-group work, whole-class discussion) although not necessarily in the same ways (Jackson and Cobb 2010).

Toward this end, throughout this book we will relate the eight effective teaching practices to specific equity-based practices that have been shown to strengthen mathematical learning and cultivate positive student mathematical identities (Aguirre, Mayfield-Ingram, and Martin 2013). Figure 1.2 provides a list of five equity-based instructional practices, along with brief descriptions.

Go deep with mathematics. Develop students' conceptual understanding, procedural fluency, and problem solving and reasoning.

Leverage multiple mathematical competencies. Use students' different mathematical strengths as a resource for learning.

Affirm mathematics learners' identities. Promote student participation and value different ways of contributing.

Challenge spaces of marginality. Embrace student competencies, value multiple mathematical contributions, and position students as sources of expertise.

Draw on multiple resources of knowledge (mathematics, language, culture, family). Tap students' knowledge and experiences as resources for mathematics learning.

Fig. 1.2. The Five Equity-Based Mathematics Teaching Practices
(Adapted from Aguirre, Mayfield-Ingram, and Martin 2013, p. 43)

Central to ambitious teaching, and at the core of the five equity-based practices, is helping each student develop an identity as a doer of mathematics. Aguirre and her colleagues (Aguirre, Mayfield-Ingram, and Martin 2013, p. 14) define mathematical identities as

> the dispositions and deeply held beliefs that students develop about their ability to participate and perform effectively in mathematical contexts and to use mathematics in powerful ways across the contexts of their lives.

Many students see themselves as "not good at math" and approach math with fear and lack of confidence. Their identity, developed through earlier years of schooling, has the potential to affect their school and career choices. Anthony and Walshaw (2009, p. 8) argue:

> Teachers are the single most important resource for developing students' mathematical identities. By attending to the differing needs that derive from home environments, languages, capabilities, and perspectives, teachers allow students to develop a positive attitude to mathematics. A positive attitude raises comfort levels and gives students greater confidence in their capacity to learn and to make sense of mathematics.

The effective teaching practices discussed and illustrated in this book are intended to help teachers meet the needs of each and every student so that all students develop confidence and competence as learners of mathematics.

Contents of This Book

This book is written primarily for teachers and teacher educators who are committed to ambitious teaching practice that provides their students with increased opportunities to experience mathematics as meaningful, challenging, and worthwhile. It is likely, however, that education professionals working with teachers would also benefit from the illustrations and discussions of the effective teaching practices.

This book can be used in several different ways. Teachers can read through the book on their own, stopping to engage in the activities as suggested or trying things out in their own classroom. Alternatively, and perhaps more powerfully, teachers can work their way through the book with colleagues in professional learning communities, in department meetings, or when time permits. We feel that there is considerable value added by being able to exchange ideas with one's peers. Teacher educators or professional developers could use this book in college or university education courses for practicing or preservice teachers or in professional development workshops during the summer or school year. The book might be a good choice for a book study for any group of mathematics teachers interested in improving their instructional practices.

In this book we provide a rationale for and discussion of each of the eight effective teaching practices and connect them to the equity-based teaching practices when appropriate. We provide examples and activities intended to help high school teachers develop their understanding of each practice, how it can be enacted in the classroom and how it can promote equity. Toward this end, we invite the reader to actively engage in two types of activities that are presented throughout the book: *Analyzing Teaching and Learning (ATL)* and *Taking Action in Your Classroom*. Analyzing Teaching and Learning activities invite the reader to actively engage with specific artifacts of classroom practice (e.g., mathematics tasks, narrative cases of classroom instruction, video clips, student work samples). Taking Action in Your Classroom provides specific suggestions regarding how a teacher can begin to explore specific teaching practices in her or his classroom. The ATLs are drawn, in part, from activities found in the *Principles to Actions Professional Learning Toolkit* (http://www.nctm.org/PtAToolkit/). Additional activities beyond what can be found in the toolkit have been included to provide a more extensive investigation of each of the eight effective mathematics teaching practices.

The video clips, featured in the Analyzing Teaching and Learning activities, show teachers who are endeavoring to engage in ambitious instruction in their urban classrooms and students who are persevering in solving mathematical tasks that require reasoning and problem solving. The videos, made available by the Institute for Learning at the University of Pittsburgh, provide images of aspects of effective teaching. As such they are examples to be analyzed rather than models to be copied. (You can access and download the videos and their transcripts by visiting NCTM's More4U website [nctm.org/more4u]. The access code can be found on the title page of this book.)

As you read this book and engage with both types of activities, we encourage you to keep a journal or notebook in which you record your responses to questions that are posed, as well as make note of issues and new ideas that emerge. These written records can serve as the basis for your own personal reflections, informal conversations with other teachers, or planned discussions with colleagues.

Each of the next eight chapters focuses explicitly on one of the eight effective teaching practices. We have arranged the chapters in an order that makes it possible to highlight the ways in which the effective teaching practices are interrelated. (Note that this order differs from the one shown in fig. 1.1 and in *Principles to Actions* [NCTM 2014]).

Chapter 2: Establish Mathematics Goals to Focus Learning
Chapter 3: Implement Tasks That Promote Reasoning and Problem Solving
Chapter 4: Build Procedural Fluency from Conceptual Understanding
Chapter 5: Pose Purposeful Questions
Chapter 6: Use and Connect Mathematical Representations
Chapter 7: Facilitate Meaningful Mathematical Discourse
Chapter 8: Elicit and Use Evidence of Student Thinking
Chapter 9: Support Productive Struggle in Learning Mathematics

Each of these chapters follows a similar structure. We begin a chapter by asking the reader to engage in an Analyzing Teaching and Learning (ATL) activity that sets the stage for a discussion of the focal teaching practice. We then relate the opening activity to the focal teaching practice and highlight the key features of the teaching practice for teachers and students. Each chapter also highlights key research findings related to the focal teaching practice, describes how the focal teaching practice supports access and equity for all students, and includes additional ATL activities and related analysis as needed to provide sufficient grounding in the focal teaching practice. Each chapter concludes with a summary of the key points and a Taking Action in Your Classroom activity in which the reader is encouraged to purposefully relate the teaching practice being examined to her or his own classroom instruction.

While we are presenting each of the effective teaching practices in a separate chapter, within each chapter we highlight other effective teaching practices that support the focal practice. In the final chapter of the book (chapter 10: Pulling It All Together), we consider how the set of eight effective teaching practices are related and how they work in concert to support student learning. In chapter 10, we also consider the importance of thoughtful and thorough planning in advance of a lesson and evidence-based reflection following a lesson as critical components of the teaching cycle and necessary for successful use of the effective teaching practices.

An Exploration of Teaching and Learning

We close the chapter with the first Analyzing Teaching and Learning activity, the Case of Vanessa Culver, which takes you into Ms. Culver's classroom where algebra 1 students are exploring exponential relationships. The case presents an excerpt from a lesson in which Ms. Culver and her students are discussing and analyzing the various strategies students used to solve the Pay It Forward task. (Note: This case, written by Margaret Smith [University of Pittsburgh], is based on a lesson planned and taught by Michael Betler, a student completing his secondary mathematics certification and MAT degree at the University of Pittsburgh during the 2013–2014 school year.)

When new teaching practices are introduced in chapters 2–9, we relate the new practice to some aspect of the Case of Vanessa Culver. In so doing, we are using the case as a touchstone to which we can relate the new learning in each chapter. The case provides a unifying thread that brings coherence to the book and makes salient the synergy of the effective teaching practices (i.e., the combined effect of the practices is greater than the impact of any individual practice).

Analyzing Teaching and Learning 1.1

Investigating Teaching and Learning in an Algebra Classroom

As you read the Case of Vanessa Culver, consider the following questions and record your observations in your journal or notebook so that you can revisit them when we refer to the Pay It Forward task or lesson in subsequent chapters:

- What does Vanessa Culver do during the lesson to support her students' engagement in and learning of mathematics?
- What aspects of Vanessa Culver's teaching are similar to or different from what you do?
- Which practices would you want to incorporate into your own teaching practices?

Exploring Exponential Relationships: The Case of Vanessa Culver

Ms. Culver wanted her students to understand that exponential functions grow by equal factors over equal intervals and that, in the general equation $y = b^x$, the exponent (x) tells you how many times to use the base (b) as a factor. She also wanted students to see the different ways the function could be represented and connected. She selected the Pay It Forward task because it provided a context that would help students in making sense of the situation, it could be modeled in several ways (i.e., diagram, table, graph, and equation), and it would challenge students to think and reason.

The Pay It Forward Task

In the movie *Pay It Forward*, a student, Trevor, comes up with an idea that he thinks could change the world. He decides to do a good deed for three people, and then each of the three people would do a good deed for three more people and so on. He believes that before long there would be good things happening to billions of people. At stage 1 of the process, Trevor completes three good deeds. How does the number of good deeds grow from stage to stage? How many good deeds would be completed at stage 5? Describe a function that would model the Pay It Forward process at *any* stage.

Ms. Culver began the lesson by telling students to find a function that models the Pay It Forward process by any means necessary and that they could use any of the tools that were available in the classroom (e.g., graph paper, chart paper, colored pencils, markers, rulers, graphing calculators). As students began working in their groups, Ms. Culver walked around the room stopping at different groups to listen in on their conversations and to ask questions as needed (e.g., How did you get that? How do the number of good deeds increase at each stage? How do you know?). When students struggled to figure out what to do, she encouraged them to try to visually represent what was happening at the first few stages and then to look for a pattern to see if there was a way to predict the way in which the number of deeds would increase in subsequent stages.

As she made her way around the room, Ms. Culver also made note of the strategies students were using (see fig. 1.3) so she could decide which groups she wanted to have present their work. She decided to have the strategies presented in the following sequence. Each presenting group would be expected to explain what they did and why and to answer questions posed by their peers. Group 4 would present their work first since their diagram accurately modeled the situation and would be accessible to all students. Group 3 would go next because their table summarized numerically what the diagram showed visually and made explicit the stage number, the number of deeds, and the fact that each stage involved multiplying by another 3. Groups 1 and 2 would then present their equations one after the other. At this point Ms. Culver decided that she would give students 5 minutes to consider the two equations and decide which one they thought best modeled the situation and why.

Below is an excerpt from the discussion that took place after students in the class discussed the two equations that had been presented in their small groups.

Ms. C.: So who thinks that the equation $y = 3x$ best models the situation? Who thinks that the equation $y = 3^x$ best models the situation? [*Students raise their hands in response to each question.*]

Ms. C.: Can someone explain why $y = 3x$ is the best choice? Missy, can you explain how you were thinking about this?

Missy: Well, group 1 said that at every stage there are three times as many deeds as the one that came before it. That is what my group (4) found too when we drew the diagram. So the "$3x$" says that it is three times more.

Ms. C.: Does everyone agree with what Missy is saying? [*Lots of heads are shaking back and forth indicating disagreement.*] Darrell, why do you disagree with Missy?

Darrell: I agree that each stage has three times more good deeds than the previous stage, I just don't think that $y = 3x$ says that. If x is the stage number like we said, then the equation says that the number of deeds is three times the stage number—not three times the number of deeds in the previous stage. So the number of deeds is only 3 more, not 3 times more.

Ms. C.: Other comments?

Kara: I agree with Darrell. $y = 3x$ works for stage 1, but it doesn't work for the other stages. If we look at the diagram it shows that stage 2 has 9 good deeds. But if you use the equation, you get 6 not 9. So it can't be right.

Chris: $y = 3x$ is linear. If this function were linear, then the first stage would be 3, the next stage would be 6, then the next stage would be 9. This function can't be linear—it gets really big fast. There isn't a constant rate of change.

Ms. C.: So let's take another look at group 3's poster. Does the middle column help explain what is going on? Devon?

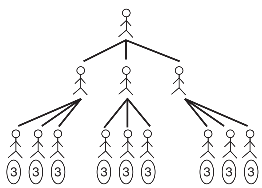
Devon: Yeah. They show that each stage has 3 times more deeds than the previous one. For each stage, there is one more 3 that gets multiplied. That makes the new one three times more than the previous one.

Angela: So that is why I think $y = 3^x$ best models the situation. Stage 1 had 3 good deeds, stage 2 had three people each doing three deeds so that is 3^2, stage 3 had 9 people (3^2) each doing 3 good deeds, so that is 3^3. The x tells how many 3's are being multiplied. So as the stage number increases by 1, the number of deeds gets three times larger.

Ms. C.: If we keep multiplying by another three like Angela described, it is going to get big really fast like Chris said. Chris also said it couldn't be linear, so take a minute and think about what the graph would look like.

At this point Ms. Culver asked group 5 to share their graph and proceeded to engage the class in a discussion of what the domain of the function should be, given the context of the problem. The lesson concluded with Ms. Culver telling the students that the function they had created was called *exponential* and explaining that exponential functions are written in the form of $y = b^x$. She told students that in the 5 minutes that remained in class, they needed to individually explain in writing how the equation related to the diagram, the table, the graph, and the problem context. She thought that this would give her some insight regarding what students understood about exponential functions and the relationship between the different ways the function could be represented.

<table>
<tr><th>Group 1
(equation—incorrect)</th><th>Group 2
(a table like groups 6’s & 7’s and an equation)</th><th>Group 3
(a diagram like group 4’s and a table)</th></tr>
<tr>
<td>$y = 3x$

At every stage there are three times as many good deeds as there were in the previous stage.</td>
<td>$y = 3^x$</td>
<td>
<table>
<tr><th>x
(stages)</th><th></th><th>y
(deeds)</th></tr>
<tr><td>1</td><td>3</td><td>3</td></tr>
<tr><td>2</td><td>3×3</td><td>9</td></tr>
<tr><td>3</td><td>$3 \times 3 \times 3$</td><td>27</td></tr>
<tr><td>4</td><td>$3 \times 3 \times 3 \times 3$</td><td>81</td></tr>
<tr><td>5</td><td>$3 \times 3 \times 3 \times 3 \times 3$</td><td>243</td></tr>
</table>
</td>
</tr>
<tr><th>Group 4
(diagram)</th><th>Group 5
(a table like groups 6’s & 7’s and a graph)</th><th>Groups 6 and 7
(table)</th></tr>
<tr>
<td>

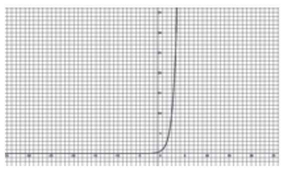
So the next stage will be 3 times the number there in the current stage so 27×3. It is too many to draw. You keep multiplying by 3.</td>
<td></td>
<td>
<table>
<tr><th>x
(stages)</th><th>y
(deeds)</th></tr>
<tr><td>1</td><td>3</td></tr>
<tr><td>2</td><td>9</td></tr>
<tr><td>3</td><td>27</td></tr>
<tr><td>4</td><td>81</td></tr>
<tr><td>5</td><td>243</td></tr>
</table>
</td>
</tr>
</table>

Fig. 1.3. Vanessa Culver’s students’ work

Moving Forward

There are many noteworthy aspects of Ms. Culver's instruction and examples of her use of the effective teaching practices. However, we are not going to provide an analysis of this case here. Rather, as you work your way through chapters 2 through 9, you will revisit the case of Ms. Culver and consider the extent to which she engaged in the focal practice and the impact it appeared to have on student learning and engagement. As you progress through the chapters, you may want to return to the observations you made during your initial reading of the case and consider the extent to which you are now seeing things in the case differently.

As you read the chapters that follow, we encourage you to continue to reflect on your own instruction and how the effective teaching practices can help you improve your teaching practice. The Taking Action in Your Classroom activity at the end of each chapter is intended to support you in this process. Cultivating a habit of systematic and deliberate reflection may hold the key to improving one's teaching as well as sustaining lifelong professional development.