Introduction

This book focuses on ideas about algebraic thinking. These are ideas that you need to understand thoroughly and be able to use flexibly to be highly effective in your teaching of mathematics in grades 3–5. The book discusses many mathematical ideas that are common in elementary school curricula, and it assumes that you have had a variety of mathematics experiences that have motivated you to delve into—and move beyond—the mathematics that you expect your students to learn.

The book is designed to engage you with these ideas, helping you to develop an understanding that will guide you in planning and implementing lessons and assessing your students' learning in ways that reflect the full complexity of algebraic thinking. A deep, rich understanding of the breadth of algebraic thinking will enable you to communicate its influence and scope to your students, showing them how this kind of thinking permeates the mathematics that they have encountered—and will continue to encounter—throughout their school mathematics experiences.

The understanding of algebraic thinking that you gain from this focused study thus supports the vision of *Principles and Standards for School Mathematics* (NCTM 2000): "Imagine a classroom, a school, or a school district where all students have access to high-quality, engaging mathematics instruction" (p. 3). This vision depends on classroom teachers who "are continually growing as professionals" (p. 3) and routinely engage their students in meaningful experiences that help them learn mathematics with understanding.

Why Algebraic Thinking?

Like the topics of all the volumes in NCTM's Essential Understanding Series, algebraic thinking is a major area of school mathematics that is crucial for students to learn but challenging for teachers to teach. Students in grades 3–5 need opportunities to think algebraically—that is, to generalize, express, and justify relationships among quantities, as well as reason with generalizations expressed through a variety of representations, if they are to succeed in their subsequent mathematics experiences. Learners often struggle with aspects of algebraic thinking, not because they are incapable of it, but because they need frequent experiences and time to develop important thinking skills. For example, expressing the solution to an equation such as $37 \times 18 = \_\_ \times 37$ is important mathematical work, but so are recognizing and using the structure of the equation to reason about the solution in terms of the commutative property of multiplication. Moreover, in tasks like these, many students understand the equals sign only as a signal to perform a computation, without understanding it as a sign of the equivalence of two quantities. The importance of understanding how computation can provide a context for algebraic thinking and the challenge of understanding how learners generalize fundamental properties of operations and develop an understanding of equivalence make it essential for teachers of grades 3–5 to understand these particular aspects of algebraic thinking extremely well themselves.

Your work as a teacher of mathematics in these grades calls for a solid understanding of the mathematics that you—and your school, your district, and your state curriculum—expect your students to learn about algebraic thinking. Your work also requires you to know how this mathematics relates to other mathematical ideas that your students will encounter in the lesson at hand, the current school year, and beyond. Rich mathematical understanding guides teachers' decisions in much of their work, such as choosing tasks for a lesson, posing questions, selecting materials, ordering topics and ideas over time, assessing the quality of students' work, and devising ways to challenge and support their thinking.

Understanding Algebraic Thinking

Teachers teach mathematics because they want others to understand it in ways that will contribute to success and satisfaction in school, work, and life. Helping your students develop a robust and lasting understanding of algebraic thinking requires that you understand this mathematics deeply. But what does this mean?

It is easy to think that understanding an area of mathematics, such as algebraic thinking, means knowing certain facts, being able to solve particular types of problems, and mastering relevant vocabulary. For example, for the upper elementary grades, you are expected to know such facts as, "The product of any whole number and 1 is that whole number." You are expected to be skillful in solving problems that involve equations with one unknown or in computing with large numbers or numbers expressed in fractional or decimal forms. Your mathematical vocabulary is assumed to include such terms as *additive identity, commutative property, equation, function*, and *solution*.

Obviously, facts, vocabulary, and techniques for solving certain types of problems are not all that you are expected to know about algebraic thinking. For example, in your ongoing work with students, you have undoubtedly discovered that you need not only to know common algorithms for addition, subtraction, multiplication, and division, but also to be able to understand, evaluate, and justify strategies that your students create.

It is also easy to focus on a very long list of mathematical ideas that all teachers of mathematics in grades 3–5 are expected to know and teach about algebraic thinking. Curriculum developers often devise and publish such lists. However important the individual items might be, these lists cannot capture the essence of a rich understanding of the topic. Understanding algebraic thinking deeply requires you not only to know important mathematical ideas but also to recognize how these ideas relate to one another. Your understanding continues to grow with experience and as a result of opportunities to embrace new ideas and find new connections among familiar ones.

Furthermore, your understanding of algebraic thinking should transcend the content intended for your students. Some of the differences between what you need to know and what you expect students to learn are easy to point out. For instance, your understanding of the topic should include a grasp of independent and dependent variables and different types of functions—mathematics that students will encounter later but do not yet understand.

Other differences between the understanding that you need to have and the understanding that you expect your students to acquire are less obvious, but your experiences in the classroom have undoubtedly made you aware of them at some level. For example, how many times have you been grateful to have an understanding of algebraic thinking that enables you to recognize the merit in a student's unanticipated generalization or claim, such as, "When I add two odd numbers, I always get an even number?" How many other times have you wondered whether you could be missing such an opportunity or failing to use it to full advantage because of a gap in your knowledge?

As you have almost certainly discovered, knowing and being able to do familiar mathematics are not enough when you're in the classroom. You also need to be able to identify and justify or refute novel claims. These claims and justifications might draw on ideas or techniques that are beyond the mathematical experiences of your students and current curricular expectations for them. For example, you may need to be able to refute erroneous claims such as, "You can't subtract a larger number from a smaller number, so $5 - 8$ has to be 3." Or you may need to explain to a student using f to represent the number of feet in the length of a field and y to represent the number of yards in that length, why $y = 3f$ is false, despite the fact that people say, "1 yard is 3 feet."

Big Ideas and Essential Understandings

Thinking about the many particular ideas that are part of a rich understanding of algebraic thinking can be an overwhelming task. Articulating all of those mathematical ideas and their connections would require many books. To choose which ideas to include in this book, the authors considered a critical question: What is *essential* for teachers of mathematics in grades 3–5 to know about algebraic thinking to be effective in the classroom? To answer this question, the authors drew on a variety of resources, including research on mathematics learning and teaching, the expertise of colleagues in mathematics and mathematics education, the reactions of reviewers and professional development providers, ideas from curricular materials, and even personal experiences.

As a result, the mathematical content of this book focuses on essential ideas for teachers about algebraic thinking. In particular, chapter 1 is organized around five big ideas related to this important area of mathematics. Each big idea is supported by smaller, more specific mathematical ideas, which the book calls *essential understandings*.

Benefits for Teaching, Learning, and Assessing

Understanding algebraic thinking can help you implement the Teaching Principle enunciated in *Principles and Standards for School Mathematics*. This Principle sets a high standard for instruction: "Effective mathematics teaching requires understanding what students know and need to learn and then challenging and supporting them to learn it well" (NCTM 2000, p. 16). As in teaching about other critical topics in mathematics, teaching about algebraic thinking requires knowledge that goes "beyond what most teachers experience in standard preservice mathematics courses" (p. 17).

Chapter 1 comes into play at this point, offering an overview of algebraic thinking that is intended to be more focused and comprehensive than many discussions of the topic that you are likely to have encountered. This chapter enumerates, expands on, and gives examples of the big ideas and essential understandings related to algebraic thinking, with the goal of supplementing or reinforcing your understanding. Thus, chapter 1 aims to prepare you to implement the Teaching Principle fully as you provide the support and challenge that your students need to develop robust algebraic thinking.

Consolidating your understanding in this way also prepares you to implement the Learning Principle outlined in *Principles and*

Standards: "Students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge" (NCTM 2000, p. 20). To support your efforts to help your students learn about algebraic thinking in this way, chapter 2 builds on the understanding of algebraic thinking that chapter 1 communicates by pointing out specific ways in which the big ideas and essential understandings connect with mathematics that students typically encounter earlier or later in school. This chapter supports the Learning Principle by emphasizing longitudinal connections in students' learning about algebraic thinking. For example, as their mathematical experiences expand, students gradually develop an understanding of connections between arithmetic and algebra and become fluent in reasoning with generalizations about numbers and operations.

The understanding that chapters 1 and 2 convey can strengthen another critical area of teaching. Chapter 3 builds on the first two chapters to show how an understanding of algebraic thinking can help you select and develop appropriate tasks, techniques, and tools for assessing your students' understanding of variables, the equals sign, algebraic ideas embedded in arithmetic, functions, equations, and inequalities. An ownership of the big ideas and essential understandings related to algebraic thinking, reinforced by an understanding of students' past and future experiences with the ideas, can help you ensure that your classroom practice reflects the Process Standards and supports the learning of significant mathematics.

Such assessment satisfies the first requirement of the Assessment Principle set out in *Principles and Standards*: "Assessment should support the learning of important mathematics and furnish useful information to both teachers and students" (NCTM 2000, p. 22). An understanding of algebraic thinking can also help you satisfy the second requirement of the Assessment Principle, by enabling you to develop assessment tasks that give you specific information about what your students are thinking and what they understand. For example, a simple task such as $3 + 4 = \_\_\_ + 5$ can reveal important information about children's understanding of equations and the use of symbols to represent equivalent quantities. The student's response to the question, "Is $3 + 4 = 7 + 5$ a true equation?" can reveal much about how the student understands the equals sign and equations, beyond the use of familiar symbols to express computations.

Ready to Begin

This introduction has painted the background, preparing you for the big ideas and associated essential understandings related to algebraic thinking that you will encounter and explore in chapter 1. Reading the chapters in the order in which they appear can be a very useful way to approach the book. Read chapter 1 in more than one sitting, allowing time for reflection. Absorb the ideas—both big ideas and essential understandings—related to algebraic thinking. Appreciate the connections among these ideas. Carry your new-found or reinforced understanding to chapter 2, which guides you in seeing how the ideas in chapter 1 are connected to the mathematics that your students have encountered earlier or will encounter later in school. Then read about teaching, learning, and assessment issues in chapter 3.

Alternatively, you may want to take a look at chapter 3 before engaging with the mathematical ideas in chapters 1 and 2. Having the challenges of teaching, learning, and assessment issues clearly in mind, along with possible approaches to them, can give you a different perspective on the material in the earlier chapters.

No matter how you read the book, let it serve as a tool to expand your understanding, application, and enjoyment of thinking algebraically.

Chapter

1

Early Algebra: The Big Ideas and Essential Understandings

How would you answer the question, "What is the essential mathematics content that I need to know to prepare my students for success in elementary grades and beyond?" Your first response might not involve algebra. Yet, in recent years, we have come to view algebra in the elementary grades as essential to helping young children become mathematically successful in school in later years and, ultimately, giving them access to a wide array of careers in technical fields that are critical in the twenty-first century. Consider the question in Reflect 1.1.

Reflect 1.1

How would you define *algebra*? Make a list of what you think might be its essential components.

Historically, the general approach to teaching mathematics treated algebra as an isolated topic for high school, or perhaps middle school. Elementary teachers did not teach it, and presumably did not need to know it. But students' difficulties and resulting high failure rates in typical high school algebra courses (Kaput 2008; Kilpatrick, Swafford, and Findell 2001), coupled with algebra's gatekeeping role in school mathematics, led to significant questions about this traditional treatment of algebra. What emerged was the argument that integrating algebra across prekindergarten through grade 12 could provide the coherence and depth that were lacking. Advocating this approach to algebra, Kilpatrick, Swafford, and Findell (2001) explained:

> The study of algebra need not begin with a formal course in the subject. Recent research and development efforts have been

encouraging. By focusing on ways to use the elementary and middle school curriculum to support the development of algebraic reasoning, these efforts attempt to avoid the difficulties many students now experience and to lay a better foundation for secondary school mathematics. (p. 280)

In conjunction with this shift in perspective on algebra, *Principles and Standards for School Mathematics* (National Council for Teachers of Mathematics [NCTM] 2000) provided a road map for a longitudinal approach to teaching and learning algebra. *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics: A Quest for Coherence* (NCTM 2006) and the Common Core State Standards for Mathematics (Common Core State Standards Initiative 2010) more recently reiterated this approach by connecting algebra to the critical ideas that young children need to learn. As a result of these and other efforts, core ideas of algebra have now become an essential part of the knowledge that elementary teachers need to teach mathematics.

In this book, examples with children are taken from classrooms across all elementary grades, and all names are pseudonyms.

Characterizing Early Algebra

What do we mean by *algebra* in the elementary grades, or *early algebra*? How is it like—or different from—its counterpart in secondary grades?

For an discussion of generalizing and its role in mathematical reasoning in the years leading to high school mathematics, see *Developing Essential Understanding of Reasoning for Teaching Mathematics in Prekindergarten–Grade 8* (Lannin, Ellis, and Elliott forthcoming).

Perhaps one of the most significant distinguishing features of early algebra is that it does not resemble the type of algebra that you are likely to have studied in a high school algebra course. It does not focus on the transformational aspects of algebra—that is, techniques or procedures for solving equations or simplifying expressions. Thus, although formal algebra has its historical roots "in the study of general methods for solving equations" (NCTM 2000, p. 37), early algebra brings a more eclectic perspective to the kinds of activities that we might describe as algebra. As we will see, it offers multiple points of entry that draw on arithmetic, functional thinking, mathematical modeling, and quantitative reasoning (Carraher and Schliemann 2007).

Broadly speaking, the heart of early algebra is in generalizing mathematical ideas, representing and justifying generalizations in multiple ways, and reasoning with generalizations (Kaput 2008). Like algebra at any level, as described in NCTM's *Guiding Principles for Mathematics Curriculum and Assessment* (2009), early algebra is a way to "explore, analyze, and represent mathematical concepts and ideas ... and to generalize mathematical ideas and relationships, which apply to a wide variety of mathematical and nonmathematical settings" (p. 4). For example, consider the setting in Reflect 1.2.

stop reflect

See Reflect 1.2 on p. 39.

In answering the question posed in Reflect 1.2, you engaged in a mathematical activity described as generalizing. In particular,

Reflect 1.2

Chair and Leg Problem

Suppose that you have some chairs, and each chair has 4 legs. How would you describe the relationship between the number of chairs and the corresponding number of chair legs?

the task required you to identify a general relationship between an unknown but varying number of chairs and the corresponding number of chair legs. We might describe this relationship in a statement such as, "The number of chair legs is four times the number of chairs." It is a *general* relationship because it characterizes how these two quantities relate for any number of chairs. Generalizing is the process by which we identify structure and relationships in mathematical situations. As in the preceding example, it can refer to identifying relationships between quantities that vary in relation to each other. It can also mean lifting out and expressing arithmetic structure in operations on the basis of repeated, regular observations of how these operations behave. For example, a child learning to add whole numbers might notice that the order in which two numbers are added does not matter—that is, the result of the computation will be the same, regardless of the order. Depending on the task and the types of numbers, the child might also form other generalizations, such as, "Any time you add an even number and an odd number, the result is an odd number."

This book uses literal symbols (letters) to represent variables. Some researchers have found that the use of non-literal symbols, such as Δ, in early algebra can lead to misconceptions. Researchers have also found that most first-grade children are able to use letters as variables.

Moreover, generalizing is never distinct from the language by which the *result* of this activity—a generalization—is expressed or represented. Generalizations may be expressed in a number of ways—through natural language, through algebraic notation using letters as variables (discussed in detail in relation to Big Idea 3), or even through tables and graphs. For example, the commutative property of addition might be expressed in the statement, "You can add two numbers in either order" or, more formally, as $a + b = b + a$, where a and b represent any real numbers.

Big Idea 3

Variables are versatile tools that are used to describe mathematical ideas in succinct ways.

Note that in the Chair and Leg problem in Reflect 1.2, the relationship between the number of chairs and the number of chair legs might be represented as "The number of chair legs is four times the number of chairs." It could also be expressed symbolically, as $l = 4 \times c$, or simply as $l = 4c$, where c represents the number of chairs and l represents the number of legs. It could also be represented in a function table or graph (see fig. 1.1).

In addition to generalizing and expressing generalizations, early algebra involves extending one's thinking beyond producing a generalization to reasoning with generalizations as objects themselves. In this sense, algebraic thinking includes reasoning

c	l
1	4
2	8
3	12
4	16
5	20

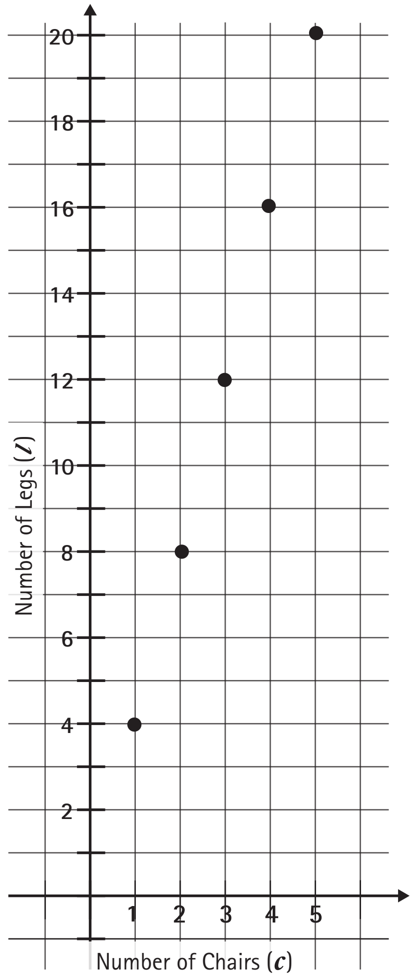

Fig. 1.1. A function table and graph representing the relationship between the number of chairs and the number of chair legs

with structural forms without the need to call particular numbers into play. For example, one child reasoned that $a + b - b = a$ is true because $b - b = 0$ and $a + 0 = a$ (Carpenter, Franke, and Levi 2003). In particular, rather than consider specific computations in the form $a + b - b = a$ (such as $3 + 7 - 7 = 3$), she reasoned with the generalizations $b - b = 0$ and $a + 0 = a$ to argue that $a + b - b = a$ was true. One group of children reasoned that the sum of any three odd numbers is odd by using generalizations that the children had previously established—namely, that the sum of any two odd numbers is even and the sum of any even number and any odd number is odd (Blanton and Kaput 2005). Their argument, "The sum would have to be odd because two odds make an even, and when you add odd plus even, you get odd," was not based on adding sums of three specific odd numbers, but on reasoning with generalizations as objects themselves.

Identifying Algebra Content for Elementary Teachers

NCTM's *Principles and Standards for School Mathematics* (2000, p. 37) identifies four organizing practices in its algebra strand for prekindergarten through grade 12. All students should do the following:

- Understand patterns, relations, and functions
- Represent and analyze mathematical situations and structures using algebraic symbols
- Use mathematical models to represent and understand quantitative relationships
- Analyze change in various contexts

These principles help to detail what we mean by algebra and the particular forms that it might take. Moreover, recent research (see, for example, Kaput, Carraher, and Blanton [2008]) allows us to be more ambitious in this book in defining particular content for elementary teachers.

We have learned that children can think mathematically—algebraically—in even more powerful ways than were envisioned a decade ago. This understanding changes what elementary teachers need to know. As a result, our goal in chapter 1 is to identify essential algebra content that *elementary teachers* need to know, on the basis of our current understanding of how young children reason algebraically. Although some of the ideas discussed in this chapter are those that we would expect children to understand, not all of them are. Where appropriate, we make some of these distinctions explicit. However, it is critical that elementary teachers have a deeper knowledge of particular mathematical ideas so that they can guide students' thinking in appropriate ways and understand how these ideas connect across grades outside their own grade band.

We emphasize that chapter 1 is about algebra content for elementary teachers. We do not intend to develop ideas about how children understand or might learn this content, nor how teachers might teach it. Although we do address some of these ideas briefly in chapters 2 and 3, we encourage you to explore the rich early algebra research base for detailed classroom illustrations about teaching and learning early algebra.

The Big Ideas and Essential Understandings

We do not intend for you to view the big ideas of chapter 1 as mutually exclusive, although we have necessarily presented them that way. Instead, they represent an effort to consolidate core algebra ideas for teachers. The chapter organizes this core knowledge around five big ideas related to different areas: (1) arithmetic as a context for algebraic thinking; (2) equations; (3) variables; (4) quantitative reasoning; and (5) functional thinking. In the remainder of this chapter, we look in more detail at the algebra in each of these areas, and we close the chapter with a brief look at how the big ideas are connected.

Each of the five big ideas that organize this chapter's discussion involves several smaller, more specific "essential understandings." The big ideas and all the associated understandings are identified as a group below to give you a quick overview and for your convenience in referring back to them later. Read through them now, but do not think that you must absorb them fully at this point. The chapter will discuss each one in turn in detail.

Big Idea 1. Addition, subtraction, multiplication, and division operate under the same properties in algebra as they do in arithmetic.

> **Essential Understanding 1*a*.** The fundamental properties of number and operations govern how operations behave and relate to one another.
>
> **Essential Understanding 1*b*.** The fundamental properties are essential to computation.
>
> **Essential Understanding 1*c*.** The fundamental properties are used more explicitly in some computation strategies than in others.
>
> **Essential Understanding 1*d*.** Simplifying algebraic expressions entails decomposing quantities in insightful ways.
>
> **Essential Understanding 1*e*.** Generalizations in arithmetic can be derived from the fundamental properties.

Big Idea 2. A mathematical statement that uses an equals sign to show that two quantities are equivalent is called an *equation*.

> **Essential Understanding 2*a*.** The equals sign is a symbol that represents a relationship of equivalence.
>
> **Essential Understanding 2*b*.** Equations can be reasoned about in their entirety rather than as a series of computations to execute.

Essential Understanding 2*c*. Equations can be used to represent problem situations.

Big Idea 3. Variables are versatile tools that are used to describe mathematical ideas in succinct ways.

Essential Understanding 3*a*. The meaning of *variable* can be interpreted in many ways.

Essential Understanding 3*b*. A variable represents the measure or amount of an object, not the object itself.

Essential Understanding 3*c*. The same variable used more than once in the same equation must represent identical values in all instances, but different variables may represent the same value.

Essential Understanding 3*d*. The same variable may play one or more roles within a given application, problem, or situation.

Essential Understanding 3*e*. A variable may represent either a discrete or a continuous quantity.

Big Idea 4. Quantitative reasoning extends relationships between and among quantities to describe and generalize relationships among these quantities.

Essential Understanding 4*a*. Two quantities can relate to each other in one of three ways: (1) they can be equal, (2) one quantity can be larger than the other, or (3) one quantity can be smaller than the other.

Essential Understanding 4*b*. Known relationships between two quantities can be used as a basis for describing relationships with other quantities.

Big Idea 5. Functional thinking includes generalizing relationships between covarying quantities, expressing those relationships in words, symbols, tables, or graphs, and reasoning with these various representations to analyze function behavior.

Essential Understanding 5*a*. A function is a special mathematical relationship between two sets, where each element from one set, called the *domain*, is related uniquely to an element of the second set, called the *co-domain*.

Essential Understanding 5*b*. Functions can be viewed as tools for expressing covariation between two quantities.

Essential Understanding 5*c*. In a functional relationship between two covarying quantities, a variable is said to be either *independent* or *dependent* and will represent either a discrete or a continuous quantity.

Essential Understanding 5*d*. In working with functions, several important types of patterns or relationships might be observed among quantities that vary in relation to each other: recursive patterns, covariational relationships, and correspondence rules.

Essential Understanding 5*e*. Functions can be represented in a variety of forms, including words, symbols, tables, and graphs.

Essential Understanding 5*f*. Different types of functions behave in fundamentally different ways, and analyzing change, or variation, in function behavior is one way to capture this difference.