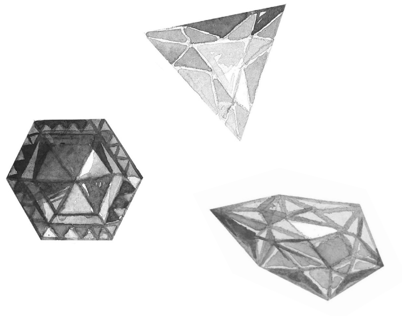


Introduction

> **Imagine a classroom, a school, or a school district where all students have access to high-quality, engaging mathematics instruction. There are ambitious expectations for all, with accommodation for those who need it. Knowledgeable teachers have adequate resources to support their work and are continually growing as professionals. The curriculum is mathematically rich, offering students opportunities to learn important mathematical concepts and procedures with understanding. Technology is an essential component of the environment. Students confidently engage in complex mathematical tasks chosen carefully by teachers. They draw on knowledge from a wide variety of mathematical topics, sometimes approaching the same problem from different mathematical perspectives or representing the mathematics in different ways until they find methods that enable them to make progress. Teachers help students make, refine, and explore conjectures on the basis of evidence and use a variety of reasoning and proof techniques to confirm or disprove those conjectures. Students are flexible and resourceful problem solvers. Alone or in groups and with access to technology, they work productively and reflectively, with the skilled guidance of their teachers. Orally and in writing, students communicate their ideas and results effectively. They value mathematics and engage actively in learning it. (NCTM 2000, p. 3)**

FORTY years ago, NCTM initiated widespread dialogue about the teaching and learning of mathematics with the publication of *An Agenda for Action* (1980) and its leadoff recommendation that problem solving should be the focus of school mathematics. Over the ensuing years, the dialogue has been reiterated, refined, and expanded through such vehicles as the *Curriculum and Evaluation Standards for School Mathematics* (1989), *Professional Standards for Teaching Mathematics* (1991), *Assessment Standards for School Mathematics* (1995), *Principles and Standards for School Mathematics* (2000), and other documents both from NCTM and in collaboration with other educational partners. Among the relevant documents that we consider here, the following are major sources:

Principles and Standards for School Mathematics (NCTM 2000) [PSSM]

Common Core State Standards for Mathematics (2010) [CCSSM]

Principles to Actions: Ensuring Mathematical Success for All (NCTM 2014) [PtA]

Closely aligned and complementary to the above reports are two additional NCTM documents, *Curriculum Focal Points for Pre-K–Grade 8 Mathematics: A Quest for Coherence* (2006) and *Focus in High School Mathematics: Reasoning and Sense Making* (2009), along with *Adding It Up: Helping Children Learn Mathematics* (2001), a report from the National Research Council.

Each of these documents considers the process of teaching and learning mathematics, along with curricular content, and offers a vision of high-quality mathematics programs and the educational experiences, practices, and outcomes that are deemed essential components of such programs. Although the language varies a bit, common themes resonate. Consider, for example, the PSSM Process Standards (see figure Intro.1), the CCSSM Standards for Mathematical Practice (see figure Intro.2), and the P-to-A Mathematics Teaching Practices (see figure Intro.3). Comparing these three sets of standards or practices, we observe strong convergence on the following:

- an emphasis on reasoning and sense making to develop conceptual understanding and facilitate problem solving;
- flexibility in investigating and representing mathematical problems and developing mathematical arguments and proofs;
- connections among mathematical concepts, representations, and processes, and using those connections to extend understanding and solutions;
- communication of mathematical ideas and reasoning and evaluation of the communication of others; and
- development of procedural fluency anchored in conceptual understanding and flexible thinking.

FIG. INTRO.1

NCTM Process Standards (*Principles and Standards for School Mathematics,* NCTM 2000)

NCTM Process Standards (PSSM)

Problem Solving: *Instructional programs from prekindergarten through grade 12 should enable all students to—*

- Build new mathematical knowledge through problem solving
- Solve problems that arise in mathematics and in other contexts
- Apply and adapt a variety of appropriate strategies to solve problems
- Monitor and reflect on the process of mathematical problem solving

Reasoning and Proof: *Instructional programs from prekindergarten through grade 12 should enable all students to—*

- Recognize reasoning and proof as fundamental aspects of mathematics
- Make and investigate mathematical conjectures
- Develop and evaluate mathematical arguments and proofs
- Select and use various types of reasoning and methods of proof

Communication: *Instructional programs from prekindergarten through grade 12 should enable all students to—*

- Organize and consolidate their mathematical thinking through communication
- Communicate their mathematical thinking coherently and clearly to peers, teachers, and others
- Analyze and evaluate the mathematical thinking and strategies of others
- Use the language of mathematics to express mathematical ideas precisely

Connections: *Instructional programs from prekindergarten through grade 12 should enable all students to—*

- Recognize and use connections among mathematical ideas
- Understand how mathematical ideas interconnect and build on one another to produce a coherent whole
- Recognize and apply mathematics in contexts outside of mathematics

Representation: *Instructional programs from prekindergarten through grade 12 should enable all students to—*

- Create and use representations to organize, record, and communicate mathematical ideas
- Select, apply, and translate among mathematical representations to solve problems
- Use representations to model and interpret physical, social, and mathematical phenomena

FIG. INTRO.2

CCSSM Standards for Mathematical Practice (NGA Center and CCSSO 2010)

CCSSM Mathematical Practices
1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

FIG. INTRO.3

Mathematics Teaching Practices (*Principles to Actions,* NCTM 2014)

	Principles to Actions **Mathematics Teaching Practices**
1	**Establish mathematics goals to focus learning.** Effective teaching of mathematics establishes clear goals for the mathematics that students are learning, situates goals within learning progressions, and uses the goals to guide instructional decisions.
2	**Implement tasks that promote reasoning and problem solving.** Effective teaching of mathematics engages students in solving and discussing tasks that promote mathematical reasoning and problem solving and allow multiple entry points and varied solution strategies.
3	**Use and connect mathematical representations.** Effective teaching of mathematics engages students in making connections among mathematical representations to deepen understanding of mathematics concepts and procedures and as tools for problem solving.
4	**Facilitate meaningful mathematical discourse.** Effective teaching of mathematics facilitates discourse among students to build shared understanding of mathematical ideas by analyzing and comparing student approaches and arguments.

Principles to Actions Mathematics Teaching Practices *(continued)*	
5	**Pose purposeful questions.** Effective teaching of mathematics uses purposeful questions to assess and advance students' reasoning and sense making about important mathematical ideas and relationships.
6	**Build procedural fluency from conceptual understanding.** Effective teaching of mathematics builds fluency with procedures on a foundation of conceptual understanding so that students, over time, become skillful in using procedures flexibly as they solve contextual and mathematical problems.
7	**Support productive struggle in learning mathematics.** Effective teaching of mathematics consistently provides students, individually and collectively, with opportunities and supports to engage in productive struggle as they grapple with mathematical ideas and relationships.
8	**Elicit and use evidence of student thinking.** Effective teaching of mathematics uses evidence of student thinking to assess progress toward mathematical understanding and to adjust instruction continually in ways that support and extend learning.

We can compare the enterprise of mathematics education to a woven fabric in which two sets of threads, the *warp* and the *weft* (or *woof*), are interlaced to produce a finished cloth. In weaving, the warp threads are stretched lengthwise on the loom and form the longitudinal elements of the fabric, while the weft threads are woven transversely, over and under the warp, to yield the finished cloth. In mathematics education, the warp can be likened to the curriculum strands, such as the elements elaborated in the PSSM content standards of number and operations, algebra, geometry, measurement, and data analysis and probability, which span the PK–12 curriculum and beyond, whereas the weft is embodied in the practices of reasoning and sense making essential for mathematical proficiency at all levels. Those process threads have also been summarized succinctly as five interwoven and interdependent strands of mathematical proficiency (National Research Council 2001, p. 115):

- *conceptual understanding*—comprehension of mathematical concepts, operations, and relations;
- *procedural fluency*—skill in carrying out procedures flexibly, accurately, efficiently, and appropriately;
- *strategic competence*—ability to formulate, represent, and solve mathematical problems;
- *adaptive reasoning*—capacity for logical thought, reflection, explanation, and justification; and
- *productive disposition*—habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy.

Both components of the woven fabric are essential to the strength, usefulness, integrity, as well as the beauty, of the finished cloth, just as the components of both content and process are essential for mathematical proficiency and understanding. PSSM stressed that the content and process standards are inextricably linked, and NCTM complemented the publication of PSSM by commissioning the series of 35 *Navigations* books and accompanying CDs as guides to help educators set a course for successful implementation of the PSSM vision by providing examples of high-quality tasks that promote reasoning and sense making across the fabric of mathematics education PK–12.

Yet despite our belief in the essential connectedness of mathematical understanding and competencies, discussions (and assessments) of mathematics education, both among educators and the general public alike, run the risk of focusing disproportionately on the warp of the educational fabric while overlooking the vital role of the weft. With that in mind, our goal with this series of books has been to select activities from throughout the Navigations series (in each of the four grade bands and across mathematics topics) that embody the mathematics teaching practices described in P-to-A and illustrate how each selected activity can provide opportunities for students to develop the mathematical understanding and proficiency that we value as goals for all students.

We will be paying particular attention to those teaching practices throughout this book. Each activity will establish clear goals that drive the flow of the lesson. The tasks are chosen to promote reasoning and problem solving. Multiple representations of concepts are not only used but are connected in classroom discussion. Throughout each activity, teachers are guided and encouraged to facilitate mathematical discourse, posing purposeful questions to guide discussions. For these young learners, the emphasis is on developing conceptual understanding, with procedural fluency following as each student is ready.

Perhaps the most difficult practice for many teachers is supporting productive struggle. All teachers have a strong desire to help students. Often, at the first sign of struggle, the teacher steps in and offers assistance to lift students over the trouble spot, despite knowing at some deep level that rich learning can occur for students who figure out a way to climb over the trouble spots on their own or with just a tiny boost. The challenge for teachers is to offer just enough encouragement and small hints so that students can continue finding solutions without removing the thinking and satisfaction of finally solving the problem on their own.

Finally, effective teachers monitor students' thinking throughout the lesson, adjust as needed, and reflect soon afterward to note suggestions for improving the lesson for the future. Reflecting on the activities sampled in this book, and trying them with your own students, can provide opportunities for gaining deeper insight into your students' thinking and understanding. Doing this together with colleagues and examining the teaching practices that you employed and their effects with your own students can contribute even more to your professional growth.

Developing Reasoning and Sense Making in Grades 9–12

"A high school mathematics program based on reasoning and sense making will prepare students for citizenship, for the workplace, and for further study." (NCTM 2009, p. 1)

> Reasoning can be thought of as the process of drawing conclusions on the basis of evidence or stated assumptions. . . . Reasoning in mathematics is often understood to encompass formal reasoning, or proof, in which conclusions are logically deduced from assumptions and definitions. However, mathematical reasoning can take many forms, ranging from informal explanation and justification to formal deductions, as well as inductive observations. Reasoning often begins with explorations, conjectures at various levels, false starts, and partial explanations before a result is reached. . . .
>
> "We define sense making as developing understanding of a situation, context, or concept by connecting it with existing knowledge. (NCTM 2009, p. 4)

Throughout the recent history of North America, the needs of students and society have evolved; school and higher-education mathematics programs have adapted accordingly. The first institutions of higher education in North America were the Royal and Pontifical University of Mexico (1551), Harvard (1638), and Laval (1663), all of which originally served students preparing for ministry. Finding information about school programs is difficult, as well as higher-education programs from the early days, so we will use Harvard University as a proxy for this period because college entrance requirements directly affect offerings and requirements at the secondary level, and we do have some information about that for Harvard.

Initially, entrance requirements for Harvard specified having studied Greek, Latin, and Hebrew. Harvard did not hire a professor of mathematics until 1726, and the entrance requirement for mathematics at that time was proficiency in arithmetic. By 1820, algebra was required for entrance, and by 1844, so was geometry. By 1908, almost all secondary schools offered one year of algebra and one of geometry, although high school entrance examinations and other considerations limited high school attendance to only about 5 percent of the cohort age group (Willoughby 1967, pp. 4, 7).

Since 1908, societal needs for mathematics proficiency have increased dramatically. High school mathematics programs have grown, but they always lag somewhat behind society's needs for well-prepared high school graduates, especially among some groups of students. Many graduates are unprepared for today's workplaces, to be well-informed citizens of our world, and for further study in a postsecondary institution (NRC 1989, p. 11).

"The best opportunities for jobs and advancement will go to those prepared to cope confidently with quantitative, scientific, and technological issues. Mathematics power provides the key to these opportunities" (NRC 1989, p. 12).

"In this changing world, those who understand and can do mathematics will have significant opportunities and options for shaping their futures. A society in which only a few have the mathematical knowledge needed to fill crucial economic, political, and scientific roles is not consistent with the values of a just democratic system or its economic needs" (NCTM 2000, p. 5).

Rather than attempting to prepare all students for calculus, a goal neither achievable nor necessary, all students, including those legitimately looking toward calculus as future engineers, scientists, mathematicians, or computer scientists, need intensive opportunities to study a broad range of topics during high school, including not only geometry and measurement, but modeling, data analysis, chance, and discrete mathematics. All of these are replete with opportunities for real-life examples, applications, and connections. Many such examples will be found in the chapters of this volume, with the added advantage of appealing to and engaging a broad spectrum of students.

In this collection of *Navigations* Gems, drawn from the nine volumes of the Navigations series, reasoning and sense making are ever present to ensure that students both understand and are able to use what they learn. Here is a list of teacher moves to develop strong reasoning and sense-making skills in students:

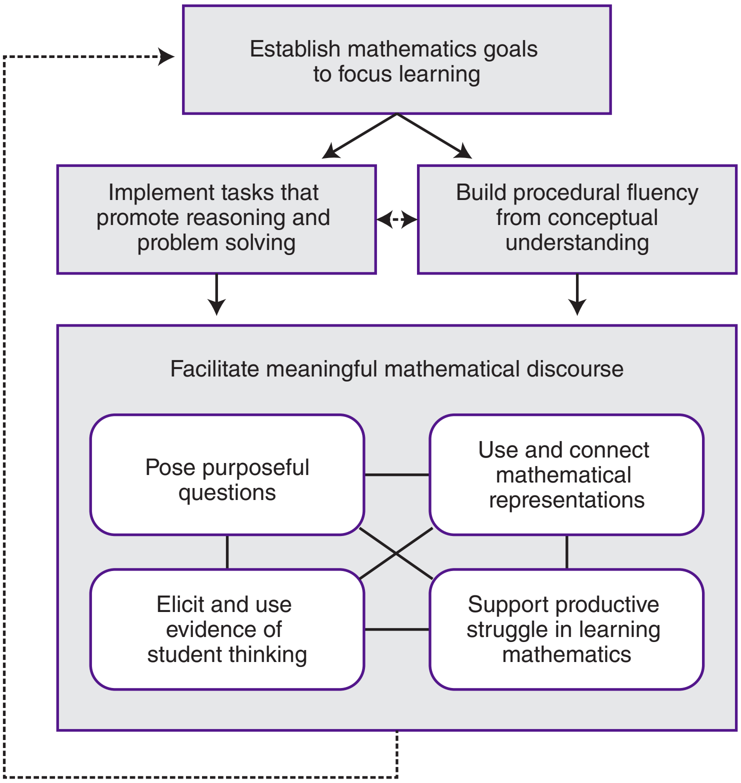
- Provide tasks that require students to figure things out for themselves.
- Ask students to restate the problem in their own words, including any assumptions they have made.
- Give students time to analyze a problem intuitively, explore the problem further by using models, and then proceed to a more formal approach.
- Resist the urge to tell students how to solve a problem when they become frustrated; find other ways to support students as they think and work.
- Ask students questions that will prompt their thinking—for example, "Why does this work?" or "How do you know?"
- Provide adequate wait time after a question for students to formulate their own reasoning.
- Encourage students to ask probing questions of themselves and one another.
- Expect students to communicate their reasoning to their classmates and the teacher, orally and in writing, through using proper mathematical vocabulary.
- Highlight exemplary explanations and have students reflect on what makes them effective.
- Establish a classroom climate in which students feel comfortable sharing their mathematical arguments and critiquing the arguments of others in a productive manner.

(NCTM 2009, p. 11)

These teacher moves are consistent with the outcomes described in *Principles to Actions* Mathematics Teaching Practices (see figure Intro.3) and are practices that, as illustrated in figure Intro.4, are interconnected and mutually supportive.

FIG. INTRO.4

Mathematics Teaching Framework (from Berry 2019)



The original NCTM Navigations series offered a rich collection of activities to support teachers and students, addressing content and practice, goals and outcomes, through implementation of the strategies reiterated in the framework shown in figure Intro.4. The Navigations Gems assembled in the current volume provide examples of how we envision high school teachers and students actively exploring, thinking, connecting, justifying, and advancing in their mathematical understanding and confidence.