3 Focusing on Ratio and Rate

In grade 6, students develop an understanding of ratio and rate. The focus is on understanding how ratio and rate are connected to multiplication and division. The eventual goal of this Focal Point is for students to be able to use ratio and rate to describe relationships and use equivalent ratios to solve a variety of problems.

Instructional Progression for Ratio and Rate

The focus on ratio and rate in grade 6 is supported by a progression of related mathematical ideas before and after grade 6, as shown in table 3.1. To give perspective to the grade 6 work, we first discuss some of the important ideas that students focused on before grade 6 that prepare them for understanding ratio and rate in grade 6. At the end of the detailed discussion of this grade 6 Focal Point, we present examples of how students will use the ratio and rate understandings and skills in later grades. For more detailed discussions of the "before" and "after" parts of the instructional progression, please see the appropriate grade-level books from NCTM, for example, *Focus in Grade 3, Focus in Grade 4, Focus in Grade 5, Focus in Grade 7,* and *Focus in Grade 8.*

Table 3.1 on the next page represents an instructional progression for the conceptual understanding of multiplication, division, fractions, ratio and rate, and proportions before grade 6, during grade 6, and after grade 6.

Early Foundations in Ratio and Rate

Before grade 6, students are expected to develop an understanding of several concepts that form the foundation for understanding ratio and rate and the connection of ratio and rate to multiplication and division. These concepts include an understanding of, and fluency with, whole-number multiplication and division, recognizing and generating factors and multiples, and understanding fractions and fraction equivalence.

In grade 4 students are expected to develop an understanding of, and fluency with, multiplication of whole numbers. They work with multiplication as scaling; for example, 3×4 can be interpreted as a length of 4 "stretched" to be 3 times as long. In grade 6 they extend this understanding to multiplication with fractions. For example, they interpret

$$3\times\frac{2}{3}$$

Table 3.1

Grade 6: Focusing on Ratio and Rate—Instructional Progression for Ratio and Rate

Before Grade 6	Grade 6	After Grade 6
Students develop an understanding of, and fluency with, multiplication and division of whole numbers. Students recognize and generate factors and multiples. Students develop an understanding of fractions and fraction equivalence.	Students apply multiplicative reasoning to explain the meanings of ratios and rates (considering rate as a special kind of ratio).* Students recognize and use different ratios to describe different aspects of a given situation (e.g., comparing two parts of a set or comparing a part of a set to the whole set). Students model equivalent ratios and rates in a variety of ways and connect their knowledge of equivalent fractions to equivalent ratios.* Students solve ratio and rate problems using a variety of strategies reflecting their understanding of equivalent fractions and multiplication and division (e.g., "If 5 items cost $3.75 and all items are the same price, then I can find the cost of 12 items by first dividing $3.75 by 5 to find out how much one item costs and then multiplying the cost of a single item by 12").	Students develop an understanding of proportional relationships. Students graph proportional relationships and recognize the graph as a line through the origin with the constant of proportionality as the slope of the line. Students express proportional relationships as $y = kx$ and distinguish them from other relationships, such as $y = kx + b$. Students use understanding of percent as a ratio to solve problems involving discounts, interest, taxes, tips, and percent of increase or decrease. Students develop an understanding of similarity as a geometric relationship in which relationships of lengths within an object are preserved, and use scale factors to solve problems (e.g., in similar figures, maps, enlargement,…). Students use proportionality to understand π and its use in determining the circumference and area of a circle (introduce formulas).** Students use their knowledge of proportionality to solve a wide range of problems involving ratios and rates. Students understand the slope of a line as a ratio.

*Appears in the Grade 6 Connections to the Focal Points (NCTM 2006).
** Appears in the Grade 7 Connections to the Focal Points (NCTM 2006).

as scaling up, where a quantity with a "size" of 2/3 is multiplied by a factor of 3, and interpret

$$\frac{2}{3}\times 3$$

as scaling down, where a quantity with a "size" of 3 is multiplied by a factor of 2/3.

In grade 4, students use place value, basic facts, and patterns to find products involving powers of 10 (10, 100, 1,000, and so on) and multiples of those powers (such as 10, 20, 30, 100, 200, 300, 1,000, 2,000, 3,000). In grade 5, students build on patterns for multiplying by powers of 10 to apply patterns for dividing multiples of powers of 10, for example, 6 ÷ 3 = 2, 60 ÷ 3 = 20, and 600 ÷ 3 = 200.

In grade 5 students learn about representing a quotient with a fraction. For example, 5 ÷ 2 is equal to 5/2. In other words, division problems can be written as fractions, and the fraction also represents the quotient; it is the new kind of number that is needed to answer the division problem, since the answer is not a whole number.

Before grade 6, students develop understandings about fraction and fraction equivalence. In grade 3, students use fractions to represent part of a whole, part of a set, a point on a number line, or a distance on a number line. They compare and order fractions by using models, benchmark fractions, or common numerators or denominators. They also identify equivalent fractions by using models, including the number line. In grade 4 students use such techniques as modeling to justify methods for generating equivalent fractions, such as multiplying or dividing the numerator and denominator by the same nonzero number. Students' exploration of equivalent fractions in grade 4 builds the foundation for understanding fraction and decimal equivalence.

In grade 5, students continue to develop and use their understanding of equivalent fractions as they add and subtract fractions. In grade 5 students also recognize and generate factors and multiples. They learn, for example, that 10, 20, 30, 40, 50, and so on, are multiples of 10 and that 1, 2, 5, and 10 are factors of 10. They use their understanding of factors and multiples to explore prime and composite numbers, common factors, and common multiples.

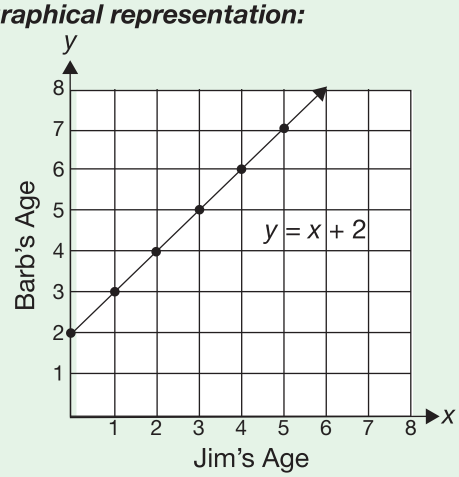
The development of the previously mentioned concepts prior to grade 6 builds the foundation that students need to understand ratio and rate, connect ratio and rate to multiplication and division, and use ratio and rate to solve problems.

Focusing on Ratio and Rate

To develop students' conceptual understanding of ratio and rate, it is beneficial to start with helping students move from additive to multiplicative reasoning. It is also important to connect ratios to concepts that students already understand, for example, fractions. By linking these two concepts, multiplicative reasoning and fractions, a solid platform for the understanding of ratio and rate can be constructed.

Moving from additive to multiplicative reasoning

Before connecting multiplication and division to the concepts of ratio and rate, students first need to build a connection from additive reasoning to multiplicative reasoning. That is, they need to understand that *2 times* has a different meaning than *2 more than*, *3 times* has a different meaning than *3 more than*, and so on. Students will benefit from constructing various representations of additive relationships and multiplicative relationships as shown in figure 3.1. As students analyze and compare the representations shown on graphs with the same scales, they begin to see the differences between an additive relationship and a multiplicative relationship.

<table>
<tr>
<td>Situation:
Barb was 2 years old when her brother Jim was born.</td>
<td>Situation:
You use 2 cups of grape juice for every cup of apple juice when you make fruit punch.</td>
</tr>
<tr>
<td>Table representation:
<table>
<tr><td>Jim’s age (x)</td><td>0</td><td>1</td><td>2</td><td>3</td><td>4</td></tr>
<tr><td>Barb’s age (y)</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td></tr>
</table>
</td>
<td>Table representation:
<table>
<tr><td>Cups of apple juice (x)</td><td>1</td><td>2</td><td>3</td><td>4</td></tr>
<tr><td>Cups of grape juice (y)</td><td>2</td><td>4</td><td>6</td><td>8</td></tr>
</table>
</td>
</tr>
<tr>
<td>Graphical representation:

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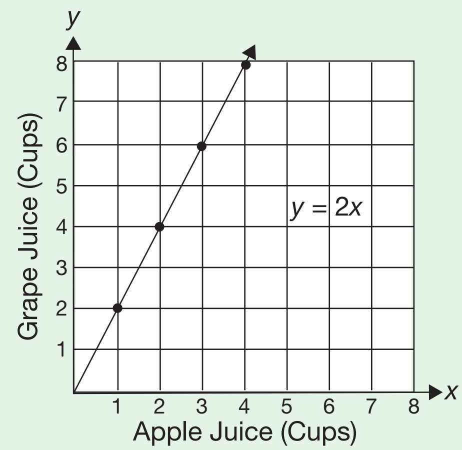
<td>Graphical representation:

</td>
</tr>
<tr>
<td>Algebraic representation:
$y = x + 2$</td>
<td>Algebraic representation:
$y = 2x$</td>
</tr>
<tr>
<td>An additive relationship</td>
<td>A multiplicative relationship</td>
</tr>
</table>

Fig. 3.1. Example showing an additive and a multiplicative relationship

A useful activity to help students build a connection from multiplication to ratio and rate is to explore multiplication as scaling. When a quantity is multiplied by a number greater than 1, it is scaled up. When a number is multiplied by a positive number less than 1, it is scaled down. (When a number is multiplied by 1, the identity element for multiplication, the number remains the same.) The number by which one multiplies is called the *scaling factor* (or *scale factor*). Opportunities to examine such problems as those shown in figure 3.2 can help students understand the concept of scaling.

Students also need to build a connection from division to ratio and rate. This can be accomplished when they learn that a ratio is a comparison of two

Problem: A cake recipe calls for $\frac{2}{3}$ cup of sugar. How much sugar is needed for 4 cakes? Do you scale up or scale down to solve the problem? What is the scaling factor? ***Solution:*** $4 \times \frac{2}{3} = \frac{8}{3} = 2\frac{2}{3}$ cups of sugar are needed for 4 cakes. You are scaling up because you are multiplying by a number greater than 1. The scaling factor is 4.	***Problem:*** A recipe for a gallon of punch calls for $\frac{2}{3}$ cup of lime juice. How much lime juice is needed for $\frac{1}{4}$ gallon of punch? Do you scale up or scale down to solve the problem? What is the scaling factor? ***Solution:*** $\frac{1}{4} \times \frac{2}{3} = \frac{2}{12} = \frac{1}{6}$ cup of lime juice is needed for $\frac{1}{4}$ gallon of punch. You are scaling down because you are multiplying by a number less than 1. The scaling factor is $\frac{1}{4}$.
Scaling up	**Scaling down**

Fig. 3.2. Examples showing scaling up and scaling down with multiplication

numbers by division; for example, the ratio 4:5 can be written 4/5 and is the quotient of 4 ÷ 5. Students will use this relationship when they write ratios as decimals.

Using fractions to build an understanding of ratios

A ratio is a multiplicative comparison of two numbers. As students begin to use what they know about fractions to understand ratios, they will learn, for example, if there are 4 children and 2 adults in a family, the ratio that compares the number of children to adults is 4 to 2. Students will also learn that they can make other comparisons. They can compare the number of children to the total number of people in the family with the ratio 4 to 6, they can compare the number of adults to the total number of people in the family with the ratio 2 to 6, and so on. In general, students will learn that they can write part-to-part ratios, part-to-whole ratios, and whole-to-part ratios. Students will also apply the idea of ratio when working with rates. Although there is no universally accepted definition of *rate,* when discussing the ratio between two measurements, especially measurements involving different units, we often use the word *rate* instead of ratio. Rates are usually expressed using the word *per,* for example, miles per gallon. In later grades, students will also encounter an *extended ratio*, the form of *a*:*b*:*c*, as a way of comparing three or more quantities. For example, the extended ratio 1:2:3 describes the ratio of sand to cement to gravel for a concrete mix. The ratio of the measures of the angles of a triangle might be compared with the extended ratio 1:1:2. Although extended ratios are still ratios, they cannot be written in fraction form.

As students begin their investigation of ratio, it is beneficial to start with a context that gives students a reason to learn about ratio, for example:

> *Problem:* You are planning a menu for a class party. Each student will receive one drink at the party. Your teacher tells you that, for a class this size, you can expect 3 out of 5 students to prefer cola, whereas 2 out of 5 will prefer lemonade. If, in fact, this guideline proves true for your class of 30 students, how many students will want cola? How many will want lemonade?

As the following classroom discussion shows, before students learn about ratios, they may use their prior knowledge of multiplication, division, or fractions to solve this problem.

Reflect As You Read

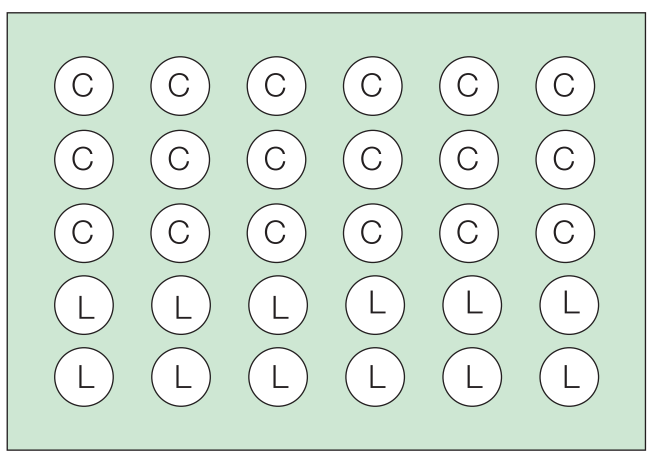
Before continuing, think about the following questions:

How would your students think about this problem?

Can you and your students represent a solution in more than one way? Are some solutions more powerful than others?

Teacher: How did you find the number of colas and the number of lemonades you should order?

Barb: I drew a picture to represent the problem. I drew a column of 5 circles to represent 5 students. I labeled 3 with C for cola and 2 with L for lemonade. Then I drew more of these columns of 5 circles until I had 30 circles.



Teacher: How does your model show the answer?

Barb: My circles show the number of Cs (colas) and Ls (lemonades) for 30 students. There are 18 Cs and 12 Ls, so I would order 18 colas and 12 lemonades.

Teacher: That makes sense. I can see that if you have 18 colas for 30 students and 12 lemonades for 30 students, then you have 3 colas for every 5 students and 2 lemonades for every 5 students. Did anyone else use a drawing to solve the problem?

Stephen: I started to, but I realized that I didn't need to draw all the circles.

Teacher: Tell us why, Stephen.

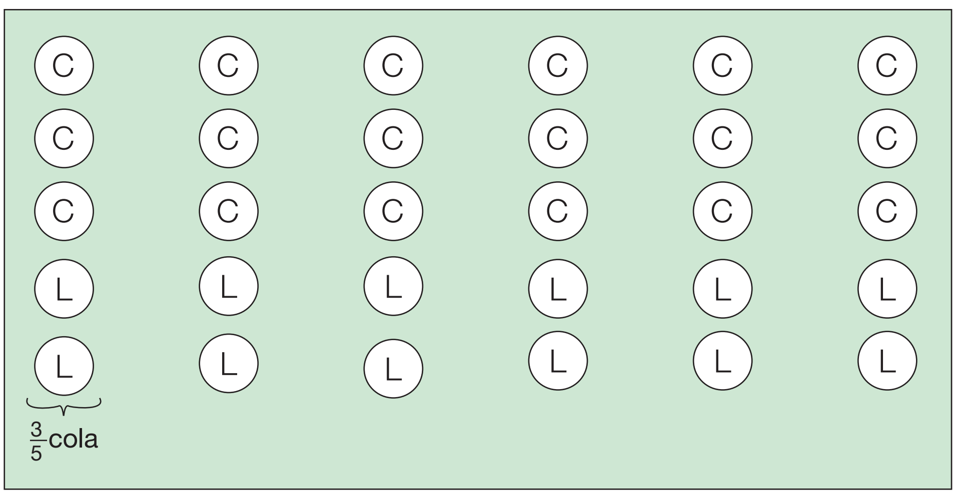
Stephen: I knew I needed 30 circles in all—that's 6 groups because there are 5 in each group. There are 3 Cs in each group, so there are 6 groups of 3 Cs, or 6 × 3 =18 Cs in all. There are 2 Ls in each group, so there are 6 groups of 2 Ls, or 6 × 2 = 12 Ls in all. I used multiplication.

Teacher: I understand. You used division and then multiplication to make sure that there were 3 colas for every 5 students and 2 lemonades for every 5 students. You ended up with 18 colas for 30 students and 12 lemonades for 30 students.

Yonnie: I used division and multiplication, but I didn't draw a model. I know that 30 ÷ 5 = 6, so there are six 5s in 30. That means that I would need six 3s for the number of colas and six 2s for the number of lemonades. Then 6 × 3 = 18, so I need 18 colas, and 6 × 2 = 12, so I need 12 lemonades. I added 18 + 12 = 30, so I know my answer is right.

Teacher: So you also ended up with 18 colas for 30 students and 12 lemonades for 30 students, which is also 3 colas for every 5 students and 2 lemonades for every 5 students.

Trish: I drew the same picture as Barb, but I used fractions to solve the problem.



The fraction of 5 students that want cola is 3/5. The fraction of 5 students that want lemonade is 2/5. Then I used equivalent fractions to find the fraction of 30 students that want cola and lemonade:

$$\frac{3}{5} = \frac{?}{30}$$
$$\frac{3 \times 6}{5 \times 6} = \frac{18}{30}$$

So I need to order 18 colas.

$$\frac{2}{5} = \frac{?}{30}$$
$$\frac{2 \times 6}{5 \times 6} = \frac{12}{30}$$

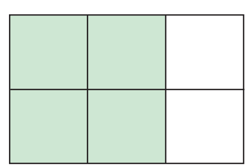
So I need to order 12 lemonades. And 18 + 12 = 30, so my answer is right.

Teacher: Again, there are 18 colas for 30 students (or 3 colas for every 5 students) and 12 lemonades for 30 students (or 2 lemonades for every 5 students). So what we've done today is use the idea called *ratio*. When we describe this situation by saying things like *3 for every 5* and *18 for every 30*, we are using ratio language. We were able to use what we know about fractions and multiplication and division in this ratio situation because those ideas are also related to the idea of comparing some number to another number.

As students are presented with such contexts as the Class Party Problem, teachers can connect students' prior knowledge to representations that support their use of ratios and equivalent ratios to solve the problems. Through experiencing such problems, students will have a context through which to learn and understand ratios.

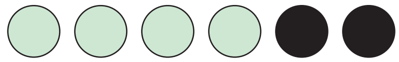
Students in grade 6 understand that a fraction is a way of comparing a part to a whole. They can use this knowledge to develop an understanding of ratios. The link between students' prior understanding of using a fraction to model a part of a whole and the representation of using a ratio to model other relationships, such as comparing a part to a part, can be made through classroom discussions such as the one that follows.

Teacher: Can you make a model to show 4/6 and explain what it means?
Amanda: I used a grid model.



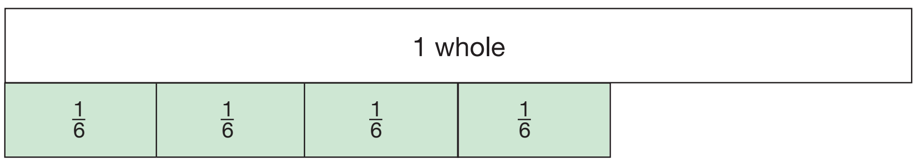
I divided a rectangle into 6 equal parts and shaded 4 of the parts, so 4/6 of the rectangle is shaded.

Brent: I used counters.



My model has 4 green counters and 6 counters in all, so 4/6 of the counters are green.

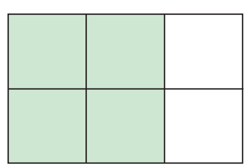
Camila: I used a fraction strip model.



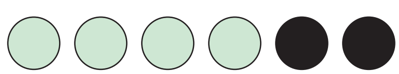
I put 4 sixth strips next to each other and under a one whole strip to show 4/6.

Teacher: All of your models show 4/6. Each model shows 4 equal parts compared to a whole that is the same size as 6 of those equal parts. So each model shows a fraction comparing a part to a whole. A ratio also is a comparison of two quantities, so each of your examples also shows a ratio that compares a part to a whole.

Look at Amanda's model. Her grid model shows that the comparison, or ratio, of shaded squares to all squares is 4 to 6, 4 shaded parts for every 6 parts.



Brent's set model shows that the comparison, or ratio, of green counters to all counters is 4 to 6, 4 green counters for every 6 counters.



To use Camila's model to show a comparison between the number of parts taken (4) and the number of parts in the whole, we need to modify it a little. Camila, what is missing in your diagram that we need to show in the comparison?

Camila: My model only shows the 4 sixths; it doesn't show the other 2 sixths.

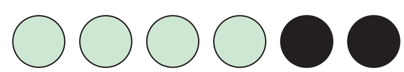
Teacher: That's right. To change the fraction strip model to show a comparison of parts taken to the whole, we need to show all 6 of the sixths in the whole. [Note that the teacher shifts the language here from 4 and 2, which is additive thinking, to 4 and 6, which is the multiplicative relationship that is the focus.] We can circle or shade 4 of the parts to show that 4 of the 6 parts are taken, so you also have 4 shaded pieces for every 6 pieces.

1 whole					
$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

So a ratio can represent a comparison of a part to a whole and can be represented by a fraction.

Discussions such as this one help students link their prior knowledge of a fraction as part *of* a whole to a ratio as a comparison of part *to* a whole. Teachers can build off of this prior knowledge to discuss how ratios can also be used to represent part-to-part and whole-to-part comparisons, as is evident in the continuation of this classroom discussion.

Teacher: A ratio can represent a comparison of a part to the whole, but it can also represent a comparison of a part to another part or of the whole to a part. Let's look at Brent's set model.



We used the ratio 4/6 (4 to 6) to compare the number of green counters to the number of counters in all. But we can make many other comparisons with these counters. Can anyone think of other comparisons we can make?

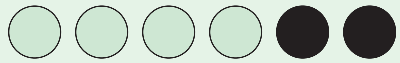
Joshua: We can compare the number of black counters to the number of counters in all. That would be 2/6 (2 to 6).

Teacher: Okay. How could we describe a comparison between the different colors?

Jeff: We can compare green to black. But would you write it 4/2?

Teacher: Yes, we can use the fraction 4/2 to describe the part-to-part ratio, and we think *4 greens for every 2 blacks.* The fraction in this situation does not describe the comparison of a part to a whole; it describes a comparison of a part to a part.

Then the teacher would guide students to write in fraction form the different ratios for each comparison. The comparisons and ratios are summarized in figure 4.3.



Comparison	Ratio
Green to all	$\frac{4}{6}$
Black to all	$\frac{2}{6}$
Black to green	$\frac{2}{4}$
Green to black	$\frac{4}{2}$
All to green	$\frac{6}{4}$
All to black	$\frac{6}{2}$

Fig. 3.3. Ratios that describe four green counters and two black counters

Students should also be exposed to sets that can be separated into more than two parts, for example, the set in figure 3.4.

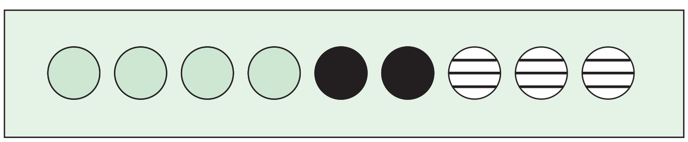


Fig. 3.4. Set that can be separated into 3 different parts

Students learn that even more ratios can be written to compare the elements in this set. For example, students can compare green circles to black circles (4/2), black circles to striped circles (2/3), striped circles to green circles (3/4), all circles to green circles (9/4), black circles to all circles (2/9), and so on.

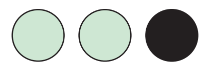
In these contexts, students can learn how to read and write the different formal representations of a ratio. For example, in the counter model comparing 4 green counters with 6 counters in all, the ratio can be written as 4 to 6, 4/6, or 4:6. As students develop their understanding of the ratios used to describe different situations, they can then begin to create situations to represent given ratios. For example, teachers could give students a ratio such as 2:3 and ask them to use counters to create a situation that can be described by the ratio 2:3. As is clear from the following classroom discussion, students will quickly begin to develop an appreciation of the fact that one ratio (e.g., 2:3) can be represented in a variety of ways.

Teacher: Use black and green counters to model the ratio 2 to 3.

Corey: I used 5 counters. My model shows 2 green and 3 black counters: 2 green to 3 black.

(green to black)

Rae: I modeled the same ratio, and I only used 3 counters. Look, I did 2 green counters and 1 black counter; my model shows 2 green to 3 counters in all.



(green to all)

Teacher: Both of your models are correct ways to show the ratio 2:3. Corey's model shows it as a part-to-part ratio. Rae's model shows it as a part-to-whole ratio. Does anyone else have a different model?

Through classroom discussion, students would begin to realize that all the models shown in figure 3.5 are correct ways of modeling the ratio 2:3.

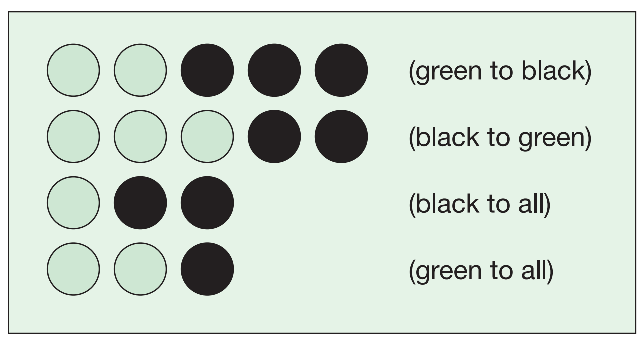


Fig. 3.5. Different ways to represent the ratio 2:3

Experiences such as this one, including ones in which the set contains more than two different elements, will help students realize that the order of the numbers in a ratio is extremely important and that the ratio one wants to use depends heavily on the context.

Reflect As You Read

After reading this section about ratios and their meaning, think about these questions:

How does your curriculum material develop these ideas?

Do students have enough time and various experiences to fully understand the ideas of ratio before moving on to related concepts?