2 Focusing on Proportionality

In grade 7, students develop an understanding of proportionality and apply that understanding to solve problems involving proportional relationships, including similarity. The goal is for students to be able to write a proportion to represent a problem involving proportionality and use the properties of proportionality to solve the problem.

Instructional Progression for Proportionality

The focus on proportionality in grade 7 is supported by a progression of related mathematical ideas before and after grade 7, as shown in table 2.1. To give perspective to the grade 7 work, we first discuss some of the important ideas that students focused on before grade 7 that prepare them for learning about proportionality in grade 7. At the end of the detailed discussion of this grade 7 Focal Point, we present examples of how students will use their proportionality understandings and skills in later grades. For more detailed discussions of the "before" and "after" parts of the instructional progression, please see the appropriate grade-level books, for example, *Focus in Grade 3*, *Focus in Grade 4*, *Focus in Grade 5*, *Focus in Grade 6*, and *Focus in Grade 8*.

Table 2.1 represents an instructional progression for the conceptual understanding of proportionality before grade 7, during grade 7, and after grade 7.

Early Foundations of Proportionality

Students develop skills and understandings prior to grade 7 that they can use as they work with the concepts of proportionality taught in grade 7. Students have previously applied multiplicative thinking, developed an understanding of ratios including equivalent ratios, and solved ratio and rate problems using a variety of strategies involving multiplication and division. All these skills work together to facilitate students' understanding of the concepts in this Focal Point.

In grade 6, students strengthen and apply their understanding of multiplicative reasoning. That is, they understand that *2 times* has a different meaning than *2 more than*, *3 times* has a different meaning than *3 more than*, and so on. As students analyze and compare the representations shown in tables and in graphs with the same scales, as shown in figure 2.1, they begin to see the differences between an additive relationship and a multiplicative relationship.

Table 2.1
Grade 7: Focusing on Proportionality—Instructional Progression for Proportionality

Before Grade 7	**Grade 7**	After Grade 7
Students develop multiplicative thinking. Students develop an understanding of equivalent ratios. Students solve rate and ratio problems using a variety of strategies involving multiplication and division.	Students develop an understanding of proportional relationships. Students graph proportional relationships and recognize the graph as a line through the origin with the constant of proportionality as the slope of the line. Students express proportional relationships as $y = kx$ and distinguish them from other relationships, such as $y = kx + b$, where $b \neq 0$. Students use understanding of percent as a ratio to solve a variety of problems including problems involving discounts, interest, taxes, tips, and percent of increase or decrease. Students develop an understanding of similarity as a geometric relationship involving proportionality in which scale factors can be used to solve problems, for example, finding lengths in similar figures, distances on maps, and so on. Students use proportionality to understand π and its use in determining the circumference and area of a circle.* Students use their knowledge of proportionality to solve a wide range of problems involving ratios and rates.	Students understand the relationships among the angle measures, side lengths, perimeters, areas, and volumes of similar objects and use these relationships to solve problems. Students understand the relationship of a line's slope to the similar triangles formed by using two points on the line as two vertices of a right triangle and the intersection of the "rise" and "run" segments related to those two points as the third vertex.**

*Appears in the Grade 7 Connections to the Focal Points (NCTM 2006).
**Appears in the Grade 8 Connections to the Focal Points (NCTM 2006).

In grade 6, students also explore multiplication as scaling. When a quantity is multiplied by a number greater than 1, it is scaled up. When a number is multiplied by a positive number less than 1, it is scaled down. When a number is multiplied by 1, the identity element for multiplication, the number remains the same. The number by which you multiply is called the *scaling factor* (or *scale factor*).

In grade 6, the focus on ratio and rate plays an important role in students' building the foundation necessary to understand proportionality. Students learn that a ratio is a multiplicative comparison of two numbers, and they build from what they know about fractions to understand ratios. For example, students use ratios to represent part-to-whole relationships, part-to-part relationships, and whole-to-part relationships, as is illustrated in figure 2.2.

Situation: Barb was 2 years old when her brother Jim was born.	***Situation:*** You use 2 cups of grape juice for every cup of apple juice when you make fruit punch.

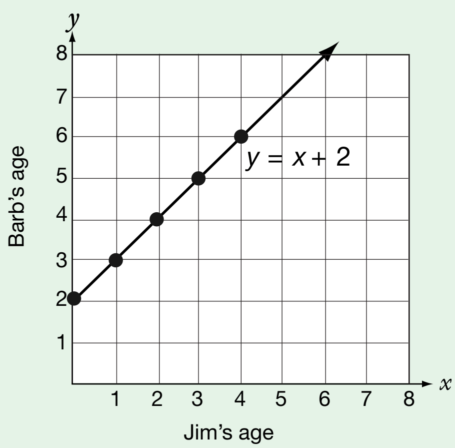
Table representation:

Jim's age (x)	0	1	2	3	4
Barb's age (y)	2	3	4	5	6

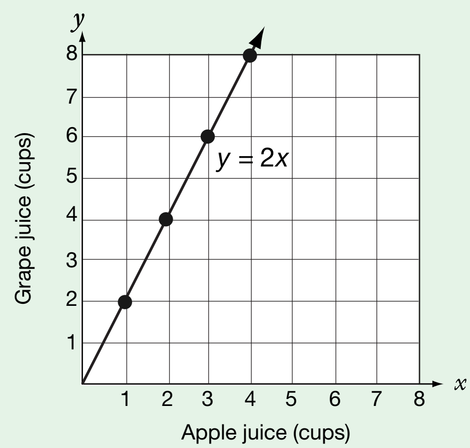
Table representation:

Cups of apple juice (x)	1	2	3	4
Cups of grape juice (y)	2	4	6	8

Graphical representation:



Graphical representation:



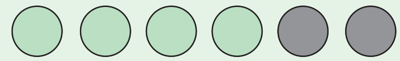
Algebraic representation: $y = x + 2$	***Algebraic representation:*** $y = 2x$
An additive relationship	**A multiplicative relationship**

Fig. 2.1. Example showing an additive and a multiplicative relationship

Students also use their understanding of equivalent fractions to determine whether two ratios are equivalent, that is, whether the resulting fractions are equal. They learn that equivalent ratios represent the same multiplicative comparison. They use various methods to model equivalent ratios, as shown in figure 2.3.

In grade 6, students use a type of ratio, called a *rate,* to describe a comparison of two measurements that often involve different units, as shown in figure 2.4.

Students also write a rate as a unit rate, that is, as an equivalent ratio with a denominator of 1 unit. Students learn that every rate situation can be written in two ways with two different unit rates. For example, in the situation in which you can buy 6 pounds of apples for $3, the relationship can be described as the rate



Relationship	Comparison	Ratio
Part-to-whole	Green to all	$\frac{4}{6}$
	Gray to all	$\frac{2}{6}$
Part-to-part	Gray to green	$\frac{2}{4}$
	Green to gray	$\frac{4}{2}$
Whole-to-part	All to green	$\frac{6}{4}$
	All to gray	$\frac{6}{2}$

Fig. 2.2. Relationships that can be described using ratios

$$\frac{6 \text{ pounds}}{3 \text{ dollars}}$$

and as the unit rate in terms of pounds per dollar: 2 pounds per dollar or

$$\frac{2 \text{ pounds}}{1 \text{ dollar}}.$$

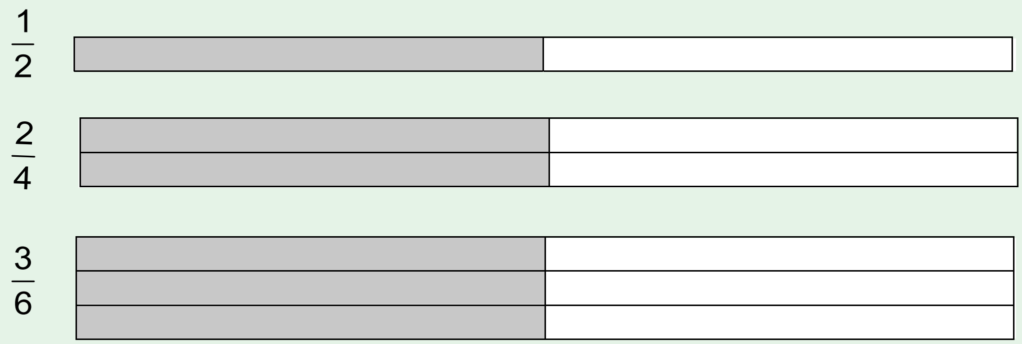
The same situation can be described by the rate

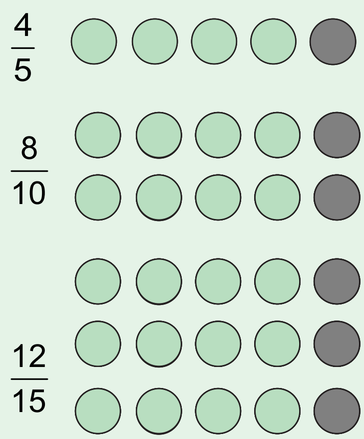
$$\frac{3 \text{ dollars}}{6 \text{ pounds}}$$

and as the unit rate in terms of dollars per pound:

$$\frac{1}{2} \text{ dollar per pound or } \frac{\frac{1}{2} \text{ dollar}}{1 \text{ pound}}, \ \frac{0.50 \text{ dollar}}{1 \text{ pound}}, \text{ or } \frac{\$0.50}{1 \text{ pound}}.$$

Equivalent Ratios





	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	8	10	12	14	16	18	20	22	24
3	3	6	9	12	15	18	21	24	27	30	33	36
4	4	8	12	16	20	24	28	32	36	40	44	48
5	5	10	15	20	25	30	35	40	45	50	55	60
6	6	12	18	24	30	36	42	48	54	60	66	72
7	7	14	21	28	35	42	49	56	63	70	77	84
8	8	16	24	32	40	48	56	64	72	80	88	96
9	9	18	27	36	45	54	63	72	81	90	99	108
10	10	20	30	40	50	60	70	80	90	100	110	120
11	11	22	33	44	55	66	77	88	99	110	121	132
12	12	24	36	48	60	72	84	96	108	120	132	144

$\frac{3}{10} = \frac{6}{20} = \frac{9}{30}$, and so on.

Fig. 2.3. Representations of equivalent ratios

$$\text{Rates:} \quad \frac{80 \text{ words}}{2 \text{ minutes}} \quad \frac{48 \text{ inches}}{4 \text{ feet}} \quad \frac{100 \text{ students}}{4 \text{ buses}} \quad \frac{20 \text{ miles}}{3 \text{ hours}}$$

Fig. 2.4. Examples of rates

Students then use their understanding of ratio and rate to solve a variety of problems, such as the ones shown in figure 2.5.

***Problem 1:* Class Party Problem**

You are planning a menu for a class party. Each student will receive one drink at the party. Your teacher tells you that, for a class this size, you can expect 3 out of 5 students to prefer cola, whereas 2 out of 5 will prefer lemonade. If, in fact, this proves true for your class of 30 students, how many students will want cola? How many will want lemonade?

Solution:

$$\frac{3 \text{ colas}}{5 \text{ students}} = \frac{3 \text{ colas}}{5 \text{ students}} \times \frac{6}{6} = \frac{3 \text{ colas} \times 6}{5 \text{ students} \times 6} = \frac{18 \text{ colas}}{30 \text{ students}}$$

$$\frac{2 \text{ lemonades}}{5 \text{ students}} = \frac{2 \text{ lemonades}}{5 \text{ students}} \times \frac{6}{6} = \frac{2 \text{ lemonades} \times 6}{5 \text{ students} \times 6} = \frac{12 \text{ lemonades}}{30 \text{ students}}$$

***Problem 2:* Least - Unit Price Problem**

Callie's Cats sells cat food for \$0.89 a can. Pet Mart sells 5 cans of cat food for \$4.00. At Jay's Pets, cat food is 8 cans for \$6.50. The cans are all the same size. At which store is cat food the least expensive? (Remember, fractions of cents are rounded up.)

Solution:

$$\text{Callie's Cats: } \frac{\$0.89}{1 \text{ can}}$$

$$\text{Pet Mart: } \frac{\$4.00}{5 \text{ cans}} = \frac{\$0.80}{1 \text{ can}}$$

$$\text{Jay's Pets: } \frac{\$6.50}{8 \text{ cans}} = \frac{\$0.82}{1 \text{ can}}$$

Pet Mart has the least expensive cat food when compared by unit price.

Fig. 2.5. Sample problems involving ratio and rate

Focusing on Understanding and Applying Proportionality

As students begin their exploration of proportionality in grade 7, it is important that they understand the language associated with proportions and proportional reasoning. Students also need to be able to distinguish proportional relationships from relationships that are not proportional. The goal is for students to gain the ability to use proportions to solve problems. When proportionality becomes a focus in the curriculum in grade 7, students begin to see proportional relationships in problems that they previously solved.

Reflect As You Read

Considering the importance of understanding ratios and rates as essential background knowledge, reflect on your school's curriculum and how much your students know about these topics prior to the grade 7 focus on proportionality.

Using language to develop understanding of proportionality

As students begin to develop an understanding of proportionality, they need to understand and properly use related mathematical terms, such as *ratio*, *rate*, *proportion*, *proportionality*, *proportional relationship*, *unit rate*, *constant of proportionality*, and *scale factor*.

A *proportion* is an equation stating that two ratios are equivalent. For example,

$$\frac{3}{4}=\frac{6}{8}$$

can be interpreted as a proportion stating that the ratios 3/4 (3 to 4) and 6/8 (6 to 8) are equivalent. This proportion can be thought of as "3 is to 4 as 6 is to 8." Because a proportion is a statement that two ratios are equivalent, students can build on their understanding of ratios and equivalent ratios to develop an understanding of a proportion. The term *proportionality* refers to the property that one quantity is a constant times another. A *proportional relationship* exists when a relationship between two quantities can be described by a set of equivalent ratios. *Proportional reasoning* is used along with known quantities to find unknown quantities.

A teacher might introduce the characteristics of a proportional relationship to students using the following situation:

> Suppose you walked 3 miles in 1 hour. Now suppose that you repeated this for a second hour and a third hour and a fourth hour.

The instructional goal is for students, after working with proportionality over time in various contexts, to be able to summarize, in their own words, the following ideas about the given proportional relationship:

- With the additional assumption that all the walking takes place at the same speed, the distance traveled is proportional to the time elapsed, and for elapsed time period "x", the distance traveled is "$3x$," so the proportional relationship can be represented with the equation $y = 3x$.
- Substituting the elapsed time values 1, 2, 3, and 4 into the equation for x gives the related distance-traveled values 3, 6, 9, and 12.
- A table that includes the ordered pairs (1, 3), (2, 6), (3, 9), (4, 12) is a partial representation of the proportional relationship.
- The points located by these ordered pairs lie on a line that goes through the origin that is described by the equation $y = 3x$.
- The ratios $y{:}x$ for every point on the line are equivalent.

To help students reach this learning goal, teachers can provide opportunities for students to engage in the following types of experiences and discussions.

In the given situation, the proportional relationship described is between distance traveled and time elapsed. Students should notice that one may choose any two distance-time pairs and use them to write a proportion. For example, students may choose 3 miles in 1 hour and 6 miles in 2 hours and can write the proportion

$$\frac{3\text{ miles}}{1\text{ hour}} = \frac{6\text{ miles}}{2\text{ hours}}.$$

Students also can write

$$\frac{3}{1} = \frac{6}{2}$$

if the measurement units of miles and hours are understood.

The ratios 3 miles/1 hour and 6 miles/2 hours are representations of a *rate;* each is a ratio of two quantities that, in this situation, are measured in different units. A *unit rate* is a rate in which the numerical part of the denominator is 1, so 3 miles/1 hour is a unit rate. The numerical part of the numerator of a unit rate can be called the *constant of proportionality* for that proportional relationship. In the proportional relationship represented by

$$\frac{3\text{ miles}}{1\text{ hour}} = \frac{6\text{ miles}}{2\text{ hours}},$$

the constant of proportionality is 3 when comparing distance traveled in miles to elapsed time in hours; that is, the distance traveled in miles is 3 times the number of hours of time elapsed. The constant of proportionality is also called the *scale factor*.

By observing the patterns in a set of equivalent ratios that describe a proportional relationship, students may conjecture that the relationship between the numerators (y) and denominators (x) of the equivalent ratios can be described by the equation

$$y = kx \text{ or } \frac{y}{x} = k,$$

(which are equivalent except when $x = 0$) where k represents the constant of proportionality (or scale factor) of the relationship. The proportional relationship that exists when $k = 1$ is represented by the equation $y = x$. In the distance-time relationship represented by the equivalent ratios

$$\frac{3 \text{ miles}}{1 \text{ hour}} = \frac{6 \text{ miles}}{2 \text{ hours}} = \frac{9 \text{ miles}}{3 \text{ hours}} = \frac{12 \text{ miles}}{4 \text{ hours}},$$

the proportional relationship can be represented by the equations

$$y = 3x$$

and

$$\frac{y}{x} = 3,$$

so the constant of proportionality, or scale factor, is 3.

For any ratio that describes a relationship, another ratio can be formed using the reciprocal to describe an equivalent relationship. For example, the relationship described by the proportion

$$\frac{3 \text{ miles}}{1 \text{ hour}} = \frac{6 \text{ miles}}{2 \text{ hours}}$$

can also be described by a proportion using the reciprocals of the ratios:

$$\frac{1 \text{ hour}}{3 \text{ miles}} = \frac{2 \text{ hours}}{6 \text{ miles}}.$$

The unit rate for this second proportion can be obtained by dividing the numerator and denominator by 3:

$$\frac{1 \text{ hour} \div 3}{3 \text{ miles} \div 3} = \frac{\frac{1}{3} \text{ hour}}{1 \text{ mile}},$$

or

$$\frac{1}{3} \text{ hour per mile.}$$

The constant of proportionality, k, is 1/3 when comparing hours of time elapsed to miles traveled; that is, the number of hours elapsed is 1/3 times the number of miles traveled. Figure 2.6 shows in table form and graphically the two representations of the proportional relationship in the given hiking situation. As students explore proportional relationships, they learn that the points in the graph of any given proportional relationship are collinear and lie on a line that passes through the origin.

Comparing proportional relationships with relationships that are not proportional

As students focus on proportionality, they begin to develop the ability to distinguish between relationships that are proportional and relationships that are not proportional. A proportional relationship is described by a constant of proportionality. Figure 2.7 shows two linear relationships; one is a proportional relationship and one is not a proportional relationship.

The equation $y = 2x$ that models the relationship in the left-hand column of figure 2.7 is in the form $y = kx$ and models a proportional relationship in which k, the constant of proportionality, is equal to 2. The equation $y = 2x + 2.50$ that models the relationship in the right-hand column of figure 2.7 is in the form $y = mx + b$. An equation in this form models a linear relationship but not a proportional relationship—unless $b = 0$. If $b = 0$, the equation $y = mx + b$ is equivalent to $y = mx$, which models a proportional relationship whose constant of proportionality is represented by m rather than k. A proportional relationship is a special case of a linear relationship. The graph of a linear relationship is a line; if that line is not vertical and passes through the origin, then the linear relationship is also a proportional relationship. Examples of other relationships that involve multiplication but are not proportional are shown in figure 2.8.

Students can distinguish proportional relationships from relationships that are not proportional by analyzing the graph of the relationship. The ratio of any x-value to its corresponding y-value should be equivalent to the ratio of any other x-value to its corresponding y-value. In a proportional relationship, every y-value is a product of the constant of proportionality, k, times the corresponding x-value. But in a relationship that is not proportional, no constant of proportionality is involved. Thus, we recognize that a relationship is not proportional when we learn that the ratio y/x can be different for different points (x, y), as illustrated in figure 2.9.

Reflect As You Read

How would your students think about this problem:

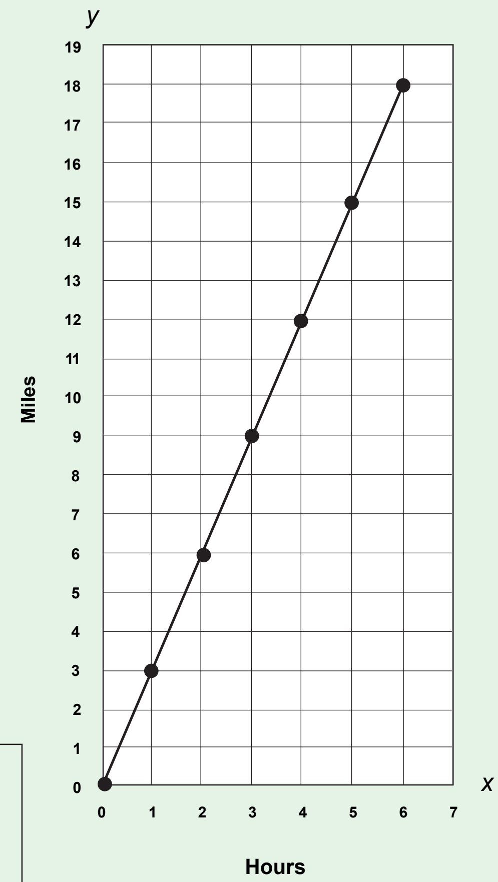
You buy 2 packages of trading cards for \$6. How much would it cost to buy 10 packages?

What representations are they comfortable using? Can they reason through any procedures for solving this problem?

hours, x	1	2	3	4	5	6
miles, y	3	6	9	12	15	18

$y = 3x$

Hours x	Miles y
1	3
2	6
3	9
4	12
5	15
6	18



miles, x	1	2	3	4	5	6
hours, y	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3}{3}=1$	$\frac{4}{3}=1\frac{1}{3}$	$\frac{5}{3}=1\frac{2}{3}$	$\frac{6}{3}=2$

$y=\frac{1}{3}x$

Miles x	Hours y
1	$\frac{1}{3}$
2	$\frac{2}{3}$
3	$\frac{3}{3}=1$
4	$\frac{4}{3}=1\frac{1}{3}$
5	$\frac{5}{3}=1\frac{2}{3}$
6	$\frac{6}{3}=2$

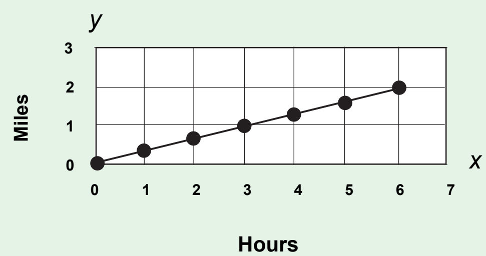


Fig. 2.6. Tables and graphs for the proportional relationship that can be represented by $y = 3x$ and $y = (1/3)x$, where the constants of proportionality are $k = 3$ and $k = 1/3$

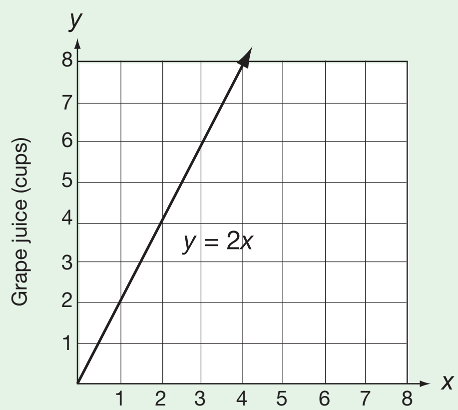
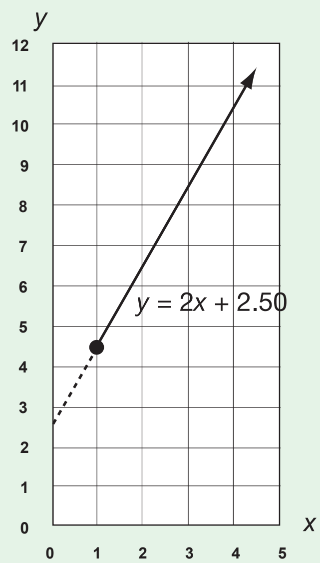
Proportional Relationship	Relationship That Is Not Proportional
Chicken costs \$2 per pound. Let y = number of dollars. Let x = number of pounds. Then $y = 2x$ models this relationship. The graph is a line through the origin.	Chicken costs \$4.50 for the first pound, with a miminum purchase of one pound, and \$2 for each additional pound. Let y = number of dollars. Let x = number of pounds. Then $y = 2x + 2.50$ for $x \geq 1$ models this relationship. The graph is a line not through the origin.
	
You can write many proportions for this relationship; one proportion is $\frac{2 \text{ dollars}}{1 \text{ pound}} = \frac{8 \text{ dollars}}{4 \text{ pounds}}$. The unit rate (unit cost) is $\frac{2 \text{ dollars}}{1 \text{ pound}}$, or 2 dollars per pound, and 2 is the constant of proportionality. In other words, no matter how many pounds of chicken you buy in this situation, the cost per pound is constant.	This is not a proportional relationship. and therefore it involves no constant of proportionality. In other words, when you buy different numbers of pounds of chicken in this situation, the cost per pound varies.

Fig. 2.7. Comparing a proportional relationship and a relationship that is not proportional

- $y = 3x^2 + 1$ (a quadratic relationship)
- $y = 2^x$ (an exponential relationship)
- $y = 5 \cdot \frac{1}{x}$ (an inverse varying relationship)

Fig. 2.8. Examples of relationships that involve multiplication but are not proportional

Proportional Relationship

$y = 2x$

x	1	2	3	4
y	2	4	6	8
Ratio: $\frac{y}{x}$	$\frac{2}{1}$	$\frac{4}{2}$	$\frac{6}{3}$	$\frac{8}{4}$
Ratio in $r/1$ form	$\frac{2}{1}$	$\frac{2}{1}$	$\frac{2}{1}$	$\frac{2}{1}$

The constant of proportionality is 2.

Relationship That Is Not Proportional

$y = 2x + 2.50$

x	1	2	3	4
y	4.50	6.50	8.50	10.50
Ratio: $\frac{y}{x}$	$\frac{4.5}{1}$	$\frac{6.5}{2}$	$\frac{8.5}{3}$	$\frac{10.5}{4}$
Ratio in $r/1$ form	$\frac{4.5}{1}$	$\frac{3.25}{1}$	$\frac{2.8\overline{3}}{1}$	$\frac{2.625}{1}$

No constant of proportionality is involved.

Fig. 2.9. Example illustrating equivalent ratios and a constant of proportionality in a proportional relationship but not in a relationship that is not proportional

Developing proportional reasoning

Students have used proportional reasoning to solve problems prior to their formal study of proportionality. Thus, they should be encouraged to use representations related to their previous understandings as they learn about the properties of proportionality, as illustrated in the following classroom discussion of this problem:

> If the cost of trading cards is two packs for \$6, how much will it cost to buy 10 packages?

Teacher: As I was walking around, I saw that you used different methods to find the answer. Larry, can you explain how you solved the problem?

Larry: Sure. I knew that if I could figure out the cost of 1 package, then I could use multiplication to find the cost of 10 packages. You can buy 2 packages for \$6, so you can divide both 2 and 6 by 2 to find that you can buy 1 package for \$3. And, since you can buy 1 package for \$3, the cost of 10 packages is 10 × \$3, or \$30.

The teacher should realize that Larry's representation includes an important and useful concept—unit rate. Unit rate plays an important role in proportionality because it can be used to connect students' intuitive understandings of proportionality to an understanding of constant of proportionality, or scale factor.

Teacher: Interesting approach, Larry. Maura, I noticed that you used a table as part of your solution process. Can you show your table and explain how you used it to solve the problem?