NAVIGATIONS SERIES

GRADES 9–12

NAVIGATING *through* ALGEBRA

Chapter 3

Expanding the Notion of Function Representation

A recursive function *is defined from an initial condition or conditions in such a way that later terms are defined in terms of earlier ones. For example, $f(1) = 4; f(n) = f(n - 1) + 3$, where n is a natural number, is a recursive function that produces the sequence 4, 7, 10, 13, ….*

An explicit function *can be written in the form $y = f(x)$. For example, $y = x^2 + 1$ is an explicit function of x.*

Processes can often be modeled discretely by sequences, as seen in chapter 2. Sequences are functions whose inputs are natural numbers. Sequences are useful in modeling situations in which the process can be described recursively. Such recursive descriptions, when coupled with the study of difference sequences (taking successive differences of a sequence of numbers) and web plots (graphs, seen later in this chapter, that follow the path of a function's orbit), can be very effective in studying the properties of the process and in generating explicit functions as models of the process. *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics [NCTM] 2000) calls for students to explore a variety of means of representing processes.

A search for a formula to model a situation uses the basic idea that arithmetic sequences have a common difference occurring when the first difference between successive terms is found. The pattern 3, 6, 9, …, studied earlier in this book, is a good example that students could consider. This pattern has a common difference of 3 between successive terms. As pointed out earlier, it leads to the concept of recursive functions. The "Add 3" pattern is a recursive pattern by its easy description: the "next" term is found by "adding 3" to the current term. Recursively, the formula could be described as

$$\text{NEXT} = \text{NOW} + 3$$

or as

$$\left\{\begin{aligned} y_1 &= 3 \\ y_{n+1} &= y_n + 3 \end{aligned}\right\}.$$

An explicit formula for the sequence is $f(n) = 3n$, where n is a natural number. As noted above, for this sequence, the first difference sequence contains only the number 3. Any such sequence that has a constant first difference sequence is linear and is of the form $f(n) = an + b$, where a and b are real numbers and n is a natural number.

The function $f(n) = an + b$, where a and b are real numbers and n is a natural number, produces a constant-difference sequence (shown in the spreadsheet in fig. 3.1), each of whose terms is a, when n takes on successive natural numbers.

For quadratic functions, two differences are required to reach a constant sequence difference. The notion of finite differences can be generalized for any polynomial function. An example is seen in figure 3.2. For geometric sequences, the notion of common differences does not work, as seen in figure 3.3. In the spreadsheet in figure 3.3, there is no apparent common difference, but looking diagonally down starting with column B, one sees the sequence 3, 9, 27, 81. If this sequence continues, it contains the powers of 3. The diagonal sequence directly underneath is 12, 36, 108, 324. These numbers are 4 times the powers of 3. The spreadsheet offers insight into what the sequence might be but gives no clear way to describe the sequence with a formula. A closer examination of column B shows that the terms might be described in a recursive way. For example, to define this geometric sequence, we could use the recursive formula

$$a_1 = 3,$$
$$a_n = 4a_{n-1}.$$

The recursive formula defines the sequence 3, 12, 48, ….

Fig. **3.1.**
A constant-difference sequence as shown on a spreadsheet

	A	B	C
1	n	f(n)	Difference
2	1	a + b	
3	2	2a + b	(2a+b)-(a+b)=a
4	3	3a + b	(3a+b)-(2a+b)=a
5	4	4a + b	(4a+b)-(3a+b)=a

Fig. **3.2.**
Successive differences seen on a spreadsheet

	A	B	C
1	n	Term	Difference in Successive Terms
2	1	3	
3	2	6	3
4	3	9	3
5	4	12	3
6	5	15	3
7	6	18	3

(a) Difference for a linear function

	A	B	C	D
1	n	Term	First Difference in Successive Terms	Second Difference in Successive Terms
2	1	3		
3	2	6	3	
4	3	10	4	1
5	4	15	5	1
6	5	21	6	1
7	6	28	7	1

(b) Difference for a quadratic sequence

The Fibonacci sequence is typically shown as 1, 1, 2, 3, 5, 8, 13, ….

Difference equations and a computer algebra system (CAS) or spreadsheet can be used to develop formulas for polynomial patterns or some other patterns, such as the Fibonacci sequence. Using a sequence

Fig. **3.3.**

A spreadsheet showing differences for a geometric sequence

	A	B	C	D	E
1	n	Term	First Difference in Successive Terms	Second Difference in Successive Terms	Third Difference i
2	1	3			
3	2	12	9		
4	3	48	36	27	
5	4	192	144	108	81
6	5	768	576	432	324
7	6	3072	2304	1728	1296
8	7	12288	9216	6912	5184
9	8	49152	36864	27648	20736
10	9	196608	147456	110592	82944

of differences to obtain a formula to represent a set of data is seen again in chapter 5. Good models for exploring difference equations and sequences of differences are found in *Mission Mathematics, Grades 9–12* (House 1997, pp. 21–26).

Recursive formulas and difference sequences are examples of *iterative change*. Iterative change has many appealing possibilities for algebraic investigation. The following examples of recursive processes help to illustrate the benefits of this approach.

The first example in the activity Would You Work for Me? was adapted from an operations-research problem from decades ago.

A good example of the mix of mapping diagrams and graphs with a sine curve can be found in Brieske (1980, p. 275).

Finite Differences, a spreadsheet applet on the CD-ROM, allows users to explore difference sequences.

Would You Work for Me?

Goals

- Use recursive or iterative forms to represent relationships
- Approximate and interpret rates of change from numerical data
- Draw reasonable conclusions about a situation being modeled

p. 63

Materials and Equipment

- A copy of the activity pages for each student
- Spreadsheet software or a graphing calculator, if needed

Activity

Would you work for me or for my sister with the following salary schemes? We will pay you \$1 for your first day's work and \$0.50 for the second day's work. Each day after the second, your salary will be computed as follows:

$$\text{Tomorrow's salary} = \left(2\tfrac{1}{2}\right)(\text{Today's salary}) - \text{Yesterday's salary}$$

Furthermore, we don't like to bother with pennies but disagree on how they should be dealt with in computing your salary. As the glass-is-half-full optimist, I always round up to the nearest dime. My sister, however, as the glass-is-half-empty person, always rounds down to the nearest dime. Would you work for either of us? Why or why not?

Discussion

This problem can be solved with technology; technology handles recursive relationships well, and graphing each salary scheme is interesting. The equations are simple and based on a geometric relationship.

The second problem has been called The Devil and Daniel Webster. It was suggested by Boyd Henry of the College of Idaho.