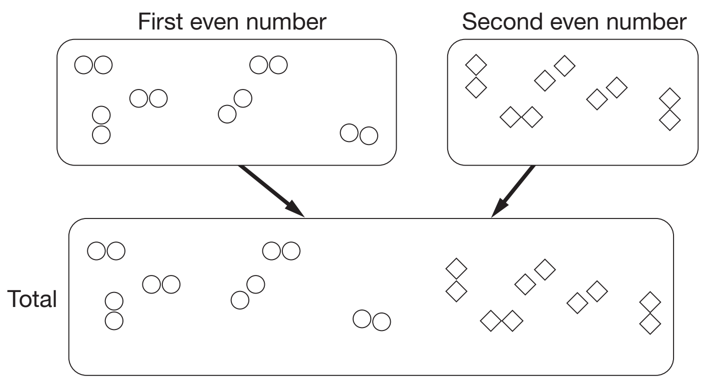
Introduction

For most adults, the use of letters to represent numbers—x's and y's—is the chief identifying feature of algebra. However, this notation in and of itself is only one aspect of algebraic thinking. The underlying reasoning of how operations work is what is critical to the understanding of algebra. This reasoning about operations is the aspect of algebra that is an important undertaking for elementary and middle school students.

In *Number and Operations, Part 1, Building a System of Tens*, participants learn to recognize how young students devise computational strategies for whole numbers and decimals. In *Number and Operations, Part 2, Making Meaning for Operations*, actions and situations modeled by the four basic operations are examined. *Number and Operations, Part 3, Reasoning Algebraically about Operations* extends the work of the earlier modules to examine the generalizations students make about the operations and the reasoning entailed in addressing the question "Does this always work?"[1] For example, consider the following vignette:

> An elementary-grade class was exploring odd and even numbers. Several children noticed that whenever they added two even numbers, the result was an even number. Based on this evidence, several of the children were ready to declare that whenever you add any two even numbers, the sum is an even number. Others acknowledged that each time they tested it, it came out that way, but you cannot tell whether it will always happen because "numbers go on forever."
>
> Myra said she knows it will happen for all even numbers. Because, she said, if you take two even numbers, you can make them into pairs with none left over. Then, when you put the numbers together, you combine the parts so that your new number has pairs with none left over.
>
> Myra illustrated what she meant with cubes and disks:



[1] *Reasoning Algebraically about Operations* was written under the assumption that seminar participants will have already worked through the ideas of *Making Meaning for Operations.*

> Felicity worked hard to understand what Myra had said. However, as she began to reiterate it, she went back to the idea that you can't know for all numbers. "There might be two even numbers somewhere that when you add them together, make an odd. You just don't know."
>
> Myra insisted she did know for all even numbers.

As students move from particular numbers and actions to patterns of results, they begin to make generalizations; that is, they begin to view numbers and operations as a system. This vignette illustrates a context in which students begin to notice generalizations and articulate what they see.

The vignette also raises several questions that will be explored in this module. What does it mean to make a generalization—a claim that something is always true? What does it mean to prove a generalization when making a claim about an infinite class of numbers and one cannot check every case? How do students engage with these questions and what constitutes "proof" at this level?

In this module, we will explore some of the generalizations that are central to our number system, generalizations that students can begin to explore in the context of their learning about numbers and computation. This module will cover three types of generalizations:

1. Properties of operations: We take for granted that when two numbers are multiplied, the answer is the same no matter how we order the factors:
   $4 \times 6 = 6 \times 4$. To add 12 to any number, say 49, we automatically assume that the 12 can be broken into 10 and 2. We then add the 10 to the 49 to get 59 and add 2 for a result of 61. Implicit in such assumptions are properties of the operations of multiplication and addition that students begin to investigate in elementary classrooms. When students explore such properties, they not only learn to make and justify generalizations, but they also come to understand more fully the computational strategies they use every day.
2. Relationships between operations: Flexible and accurate computations often require that students move between different operations in the same problem. In this module, we will study, for example, how students work through the idea that addition and subtraction are systemically related operations—any problem that can be solved by finding a missing addend can also be solved by subtracting.
3. Results of operating on particular kinds of numbers: In the vignette above, Myra and her classmates were studying what happens when you add even numbers. In this module, we will also examine operations on negative numbers, as well as the special case of 0. Many of the generalizations that are developed can easily be understood and justified if the numbers involved are the counting numbers (1, 2, 3, . . .). However, what happens when the number system is extended to include zero and negative numbers? Do all of the same generalizations still make sense? Are they still true? If so, why? For example, if we add 3 and -4, does the order of addends matter? Does $3 + -4$ yield the same result as $-4 + 3$?

When students begin to state and justify such generalizations, they tend to use diagrams, concrete objects, and words to do so—just as Myra did in her statement about adding two even numbers. As their statements become more complicated, they begin to need other ways to point at "the first number," "the bigger number," "the answer you get when you add two numbers," and so forth. This is the beginning of what later becomes conventional algebraic notation. In this module, we look at the early stages of this process and consider when it is useful and productive for students to begin to make this shift.

The cases in this casebook were written by elementary and middle school teachers recounting episodes from their classrooms. The range represents schools in urban, suburban, and rural communities. The teacher-authors, who were themselves working to understand the "big ideas" of the elementary- and middle-grade mathematics curriculum, wrote these cases as part of their own process of inquiry. They came together on a regular basis to read and discuss each other's developing work.

The K–5 teachers who participated in the program found that this work deepens students' understanding of arithmetic, the heart of their mathematics program. Middle school teachers said that the work supported students' early study of algebra. In the words of one teacher, "By explicitly stating the generalizations and then finding examples, counterexamples, and proofs, students are thinking more about the principles that underlie their work. Generalizations help students see relationships among and between numbers, and among and between operations."

Teacher collaborators report that students who tend to have difficulty in mathematics become stronger mathematical thinkers through this work. As one teacher wrote, "When I began to work on generalizations with my students, I noticed a shift in my less capable learners. Things seemed more accessible to them." When the generalizations are made explicit—through language and through visual representation used to justify them—they become accessible to more students and can become the foundation for greater computational fluency. Furthermore, the disposition to create a representation when a mathematical question arises supports students in reasoning through their confusions.

At the same time, students who generally outperform their peers in mathematics find this content challenging and stimulating. The study of numbers and operations extends beyond efficient computation to the excitement of making and proving conjectures about mathematical relationships that apply to an infinite class of numbers. A teacher explained, "Students develop a habit of mind of looking beyond the activity to search for something more, some broader mathematical context to fit the experience into."

Through the cases, you will study students' initial ideas as they develop their algebraic thinking. We will look at students' arguments as they work to "prove" such generalizations as subtracting a smaller amount produces a larger result; switching around numbers in an addition or multiplication problem gives the same answer, but switching the numbers in a subtraction or division problem does not; and a factor of a number is always a factor of that number's multiples as well.

In the seminar sessions, you will have opportunities to work through these and related mathematical ideas for yourself. You will identify patterns in the number system, articulate the generalizations you see, and work to prove them. You will explore how the consistency of the number system determines calculations with negative numbers. In addition, you will look at how students' generalizations, and your own, appear when represented in algebraic notation.