Chapter 2

Building Equations and Functions

Focus in High School Mathematics: Reasoning and Sense Making (NCTM 2009) stresses the importance of reasoning with algebraic symbols, equations, and functions. These skills are precisely the ones that cause students so much difficulty in the transition from arithmetic to algebra. Indeed, teachers' assessments of the areas that cause beginning students to struggle are overwhelmingly uniform. The following areas often present challenges:

1. Expressing generality with algebraic notation, including function notation
2. Reasoning about slope, graphing lines, and finding equations of lines
3. Building and using algebraic functions
4. Setting up the appropriate equations to solve word problems

At first glance this list looks like a collection of disparate topics. Yet, looking underneath the topics and considering the kind of reasoning that would help students master them reveals a remarkable similarity. *A key component of all of these topics is the reasoning habit of seeking and expressing regularity in repeated calculations.*

This habit manifests itself when one is performing the same calculation over and over and begins to notice the "rhythm" in the operations. Articulating this regularity leads to a generic algorithm, which is typically expressed with algebraic symbols and can be applied to any instance and transformed to reveal additional meaning, often leading to a solution of the problem at hand.

This chapter explores how this habit can be used to bring coherence to three topics in the high school curriculum:

1. Building equations to model situations
2. Finding lines of best fit
3. Calculating monthly loan payments

The habit of seeking and expressing regularity in repeated calculations runs throughout the specific components that *Focus in High School Mathematics: Reasoning and Sense Making* identifies in the reasoning habits that it describes. For example, those components include—

- identifying relevant mathematical concepts, procedures, or representations;
- seeking patterns and relationships;
- looking for hidden structure;
- making purposeful use of procedures;
- organizing the solution, including calculations.

From Calculations to Equations

Focus in High School Mathematics: Reasoning and Sense Making calls for *reasoned solving* of equations—seeing steps in the solution of an equation as logical deductions. However, before equations can be solved, they have to be constructed, by using what that publication calls the meaningful use of symbols. Teachers report that many students, even students who are quite skillful in solving linear and quadratic equations, have a very hard time building equations that model particular situations.

> Teachers of algebra typically comment, "My students can solve the equations; setting them up is the hard part."

For example, consider how hard it is for students to set up an equation that they can use to solve an algebra word problem. Reasons for their difficulties typically include the reading levels and the unfamiliar contexts of such problems. Still, there has to be more to students' difficulties than these surface features. Consider, for example, the following two problems:

Problem 1: The driving distance from Boston to Chicago is 990 miles. Rico drives from Boston to Chicago at an average speed of 50 mph and returns at an average speed of 60 mph. For how many hours is Rico on the road?

Problem 2: Rico drives from Boston to Chicago at an average speed of 50 mph and returns at an average speed of 60 mph. Rico is on the road for 36 hours. What is the driving distance from Boston to Chicago?

The problems have identical reading levels and context. But teachers report that many students who can solve problem 1 are baffled by problem 2. A significant body of research can help to explain this phenomenon (Bransford, Brown, and Cocking 1999; Breidenbach et al. 1992; Cuoco 1995; Piaget 1972; Sfard 1991; Sfard and Linchevski 1994; Slavit 1997).

> Problems 1 and 2 and others like them make no pretense of being rooted in reality. Indeed, their puzzle-like quality makes them ideal vehicles for developing the reasoning habits under consideration.

Problem 1 can be solved with isolated calculations, as shown in figure 2.1. However, problem 2 requires that the student encapsulate these isolated individual calculations into a coherent *process*—an algorithm that calculates the time on the road from the distance traveled—so that they can invert the algorithm (reasoned solving again) to come up with a distance that will produce 36 hours.

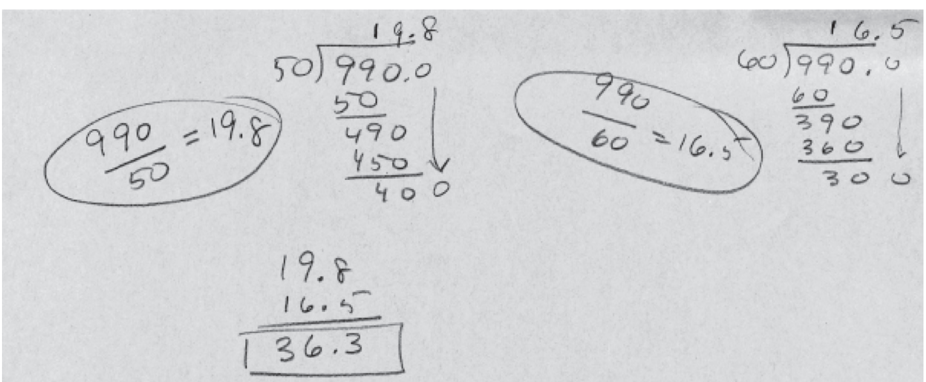

Fig. 2.1. Isolated calculations for solving problem 1

In this situation, the reasoning habit of "expressing the rhythm" in a calculation can be of great use to them. The basic idea is for them to guess at an answer to problem 2 and check their guess as if they were working on problem 1, *keeping track of their steps.* The purpose of the students' guess is not to stumble on (or to approximate) the correct answer; rather, it is to help them construct a "checking algorithm" that will work for any guess. So, students can make several guesses until they are able to express their checking algorithm in algebraic symbols. The following example shows how a student might approach this problem; figure 2.1 shows the student's calculations.

> Students who solve this problem with the aid of a calculator typically hit the "=" key very often.

Student: I began by guessing that the distance is 1000 miles. I then divided 1000 by 50 and 1000 by 60. Then I added the answers together to see if I got 36. I didn't, so I made another guess—950 miles. Let's see: 950 divided by 50 plus 950 divided by 60. Is that 36? No, but a general method is evolving that might allow me to check *any* guess. My guess-checker is

$$\frac{\text{guess}}{50} + \frac{\text{guess}}{60} \stackrel{?}{=} 36.$$

So my *equation* is

$$\frac{\text{guess}}{50} + \frac{\text{guess}}{60} = 36,$$

or, letting x stand for the unknown correct guess,

$$\frac{x}{50} + \frac{x}{60} = 36.$$

In the classroom

In the following vignette, two teachers sort out the difference between the solution method described for problem 2 and traditional guess-and-check. Mr. Thomas Gradgrind and Ms. Maria Agnesi are talking about their algebra classes. Tom is sharing his concerns about a lesson involving the relationships among distance, rate, and time.

> "Guess-and-check" has long been a popular method for finding or approximating solutions to all kinds of problems. What we present here isn't quite the same—the guesses are just *scaffolds* to help students build equations. The real goal is to build a generic "guess checker"—the equation that can be solved to produce the exact solution.

Tom: Maria, I just don't know what to do. Right now in my class we are working on distance-rate problems. We had already talked about the relationship among distance, rate, and time. I then gave students a problem like this: "The driving distance from Boston to Chicago is 990 miles. Rico drives from Boston to Chicago at an average speed of 50 mph and returns at an average speed of 60 mph. How many hours was Rico on the road?" Almost every student was able to come up with the correct solution.

Maria: How do you know students understood what they were doing?

Tom: When I walked around to see what students were doing, I saw that they were dividing the one-way distance by each respective speed and then adding both times to get the total hours. I asked students to explain why they were dividing, and they were able to talk about $d = rt$.

Maria: So what exactly is your concern?

Tom: After asking students to determine the total time for the problem, I switched the problem a bit. I gave students the same speeds as before but told them this time that Rico was making a round trip from Fort Lauderdale, Florida, to Reston, Virginia. I asked them to figure out the one-way distance between the two cities if the total driving time was 38.5 hours. They didn't even know how to begin the problem. So I ended up just telling them how to set up the equation to solve the problem.

Maria: What understanding do you think your students have about the problem?

Tom: None. I gave them a formula of sorts that can help them solve these types of problems. What else was I supposed to do?

Maria: This is a great opportunity to help students develop as problem solvers while at the same time giving them a chance to make meaning out of algebraic symbols. Let me show you what I mean. Given two different rates, one each for the trip out and back, your students were able to determine the total trip time. Well, have them use this method to help solve the second problem.

Tom: I'm not sure that I follow. Students didn't set up an equation initially, and they clearly couldn't set up an equation for the second problem.

Maria: Suggest to students that they "guess" a distance and use it to check if they are correct by calculating if they get the same total driving time.

Tom: But how does guessing help them? I don't want them to keep guessing and checking. It's not efficient, and they may never get the right answer.

Maria: The "guessing" is just the means for them to develop an algorithm. Have students keep track of the steps they are using to check their guess. Here, let's try one. Begin with a guess of 500 miles and conjecture what students will do.

Tom: They will divide 500 by 50 and then divide 500 by 60 and add them together to get the total time—just as they did for the initial problem.

Maria: Suggest they try another number for the distance between the cities, like maybe 800 miles. What will they do?

Tom: The same thing as before. They will divide 800 by 50 and then divide 800 by 60 and add them together. Oh, I see what you're getting at. After a couple of times, students can begin to see a pattern. I can coach them to come up with a type of verbal description, like

$$\frac{\text{Miles between cities}}{50 \text{ mph}} + \frac{\text{Miles between cities}}{60 \text{ mph}} = 38.5 \text{ hours}.$$

Maria: Now students can simply replace "miles between cities" with x and they have an equation where the variables and equation make sense to them. They have also developed a method that will come in very handy in the future for setting up equations.

This habit of trying numerical examples until the structure of an algorithm becomes clear captures a very common process that is a useful tool throughout algebra: we carry out several concrete examples of a process that we don't quite "have in our heads" to find regularity and build a generic algorithm that describes every instance of the calculation. As another example, let's look at how this same reasoning can be used to find equations of lines and other curves.

Equations of lines and other curves

Roger Howe (forthcoming) makes a careful analysis of word problems, showing how arithmetic and algebraic approaches can be developed and used in tandem.

Suppose that a student who is new to algebra and comes to it with no formulas is asked to find the equation of the vertical line l that passes through the point with coordinates (5, 4). Students can draw the line, and, just as in the word problem example, they can guess points and check to see if they are on l. For example, trying some points, like (5, 1), (3, 4), (2, 2), and (5, 17), leads to a generic guess-checker: *To see if a point is on l, you check that its x-coordinate is 5*. This leads to the guess-checker $x \stackrel{?}{=} 5$ and the equation $x = 5$.

To be completely rigorous, students should check that a point is on l *if and only if* its x-coordinate is 5. The equation $x = 5$ is often referred to as a *point tester* for l.

This method works well for vertical and horizontal lines, and even for special lines like the one that bisects the first and third quadrant. But what about lines for which there is no simple guess-checker? The idea is to find a geometric characterization of such a line and then to develop a guess-checker based on that characterization. One such characterization uses *slope*.

In first-year algebra, students study slope, and one fact about slope that often comes up is that three points on the coordinate plane but not all on the same vertical line are collinear if and only if the slope between any two of them is the same. Figure 2.2 shows three points that satisfy this condition and three points that do not.

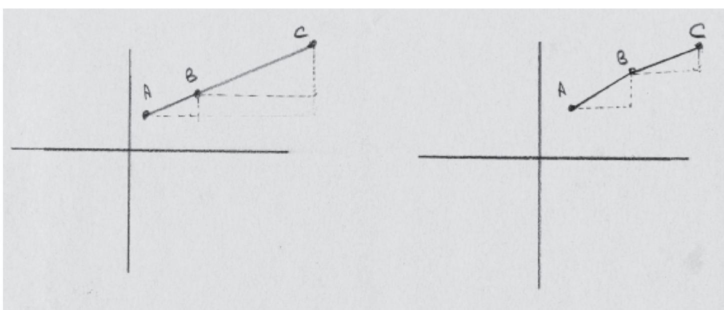

Fig. 2.2. On the left, points *A, B,* and *C* are collinear. On the right, they are not.

In the figure, if we let $m(A, B)$ denote the slope between A and B (calculated as change in y divided by change in x), then the collinearity condition can be stated like this:

Three points A, B, and C that do not all lie on the same vertical line are collinear if and only if $m(A, B) = m(B, C)$.

To prove this characterization of collinearity, one needs some facts about similar triangles. In figure 2.2, the two triangles on the left are similar; the two triangles on the right are not.

This criterion for collinearity can be used to find the equation of a line between two points. Suppose, for example, that students are asked to find an equation for $\overleftrightarrow{AB}$ if A is the point (5, 1) and B is the point (–3, 6). Imagine again that they have no knowledge of $y = mx + b$ or related formalisms. They can reason as follows: The slope between A and B is $-\frac{5}{8}$, and we can guess at some points C and check to see whether or not C is collinear with A and B by checking slopes:

C	$m(B, C)$	C on $\overleftrightarrow{AB}$?
$(1, 3)$	$-\frac{3}{4}$	no
$(7, 0)$	$-\frac{3}{5}$	no
$(13, -4)$	$-\frac{5}{8}$	yes
$(-6, 7)$	$-\frac{1}{3}$	no

How might the students check a generic guess, say $C(x, y)$? They could calculate the slope between $C(x, y)$ and $B(-3, 6)$, and see if the slope is $-\frac{5}{8}$. The guess-checker is $m(B, C) \stackrel{?}{=} -\frac{5}{8}$, or

$$\frac{y-6}{x+3} \stackrel{?}{=} -\frac{5}{8}.$$

So an equation of $\overleftrightarrow{AB}$ is

$$\frac{y-6}{x+3} = -\frac{5}{8}.$$

From here, the students can simplify the equation to get it into a more standard form.

This outline glosses over some important details that would need classroom discussion. For example, the special case when $x = -3$ needs attention, and students should check this result against the result obtained when one checks the slope from C to A instead of from C to B.

It is certainly true that algebra students need to become fluent in understanding the correspondence between linear equations and their graphs. In many applications, they will need to be able to read the slope and y-intercept of a line from its equation, and given these features, they will need to be able to draw a line.

So, why not jump directly to the development of these skills without the guess-checking activities? A number of reasons support starting with an approach like the one outlined here:

> Students frequently think that $y = 3x + 4$ is a "code" that means "put a point at (0, 4), then go over 1 and up 3, put a point there, and then draw a line between these two points."

1. Several research studies (Greenes et. al. 2007; Goldenberg 1988, 1991) show that students who can fluently graph equations like $y = 3x + 4$ often can't use the equation to see if a given point is on the graph. Building equations from the repeated testing of numerical examples reinforces the "Cartesian connection" that a point is on the graph of an equation if and only if its coordinates satisfy the equation.
2. This same reasoning habit can be applied to other equations and their graphs. For example, to find an equation for the circle with center $C(3, 7)$ and radius 5, students who are used to thinking this way might ask, "How can I check to see if a given point P is on the circle?" They might then follow up this question by asking, "Is the distance from P to C equal to 5?" Students equipped with the Pythagorean theorem would be able to write down the equation from this characterization long before learning about the formula $(x - h)^2 + (y - k)^2 = r^2$.
3. The very act of articulating a guess-checking algorithm in a way that can be formulated with algebraic symbols is a skill that will serve students well throughout mathematics and related fields.

Automaticity in graphing is very important. However, jumping directly to the automatic applications of methods like using "$y = mx + b$" can disconnect students' skill in graphing equations from the underlying meaning that connects equations and their graphs.

> Articulating a guess-checking algorithm as described will serve students well in dealing with algebra word problems and area formulas. Eventually, students should be able to go from a problem directly to an equation or function that models the problem's situation, but jumping directly to "problems by type" or rules like "let $x = \ldots$" or $A = \frac{1}{2}(b_1 + b_2)h$ can disconnect the symbols from their meaning for students.

Fitting Lines to Data

Imagine a class in which students have developed automaticity with the connection between lines and their equations. One application of this set of skills is to provide some insight into the sometimes-mysterious calculator button that calculates the line of best fit for a set of data. After students have had appropriate informal experiences with data trends, many high school curricula give a definition of a best-fit line in a manner such as the following:

For a set of points $\{(x_i, y_i)\}_{i=1}^{n}$, the line of best fit minimizes the sum of the squares of the deviations in y-values. In other words, it is a line with equation $y = ax + b$ so that the sum

$$\sum_{i=1}^{n}\left(y_i - (ax_i + b)\right)^2$$

is as small as possible.

> Notice that a and b are the variables here. Think of the set of all possible lines dancing through the data, each one with its own "badness," or lack of fit (its particular sum of squares of deviations in y-heights from the data points). The use of dynamic geometry software can make this image precise (Cuoco and Goldenberg 1996).

The actual derivation of the a and b that minimize this sum is usually left for linear algebra or calculus. However, a little knowledge of quadratic functions (and how to minimize them), along with the habit of abstracting from calculations, can take students quite a bit further.