2 Number and Operations

Overview

The topic of number and operations has three major components: the number core, the relations core, and the operations core.

- The *number core* focuses on four components:
  - Seeing cardinality (seeing how many there are)
  - Knowing the number-word list (one, two, three, four, etc.)
  - One-to-one counting correspondences when counting
  - Written number symbols (1, 2, 3, etc.)
- The *relations core* concerns finding the relationship between two groups of objects or two numbers: Is one of these more than, or less than, or equal to the other?
- The *operations core* involves adding or subtracting two groups of objects or numbers to make a third.

Each of these is discussed more fully in table 2.1.

Table 2.1

Progression of Ideas about Number and Operations (and Algebra from Kindergarten On)

Kindergarten	Grade 1	Grade 2
The Number Core	**The Number Core**	**The Number Core**
Integrate all core components for teen numbers (10 to 19) to see a ten and some ones in teen numbers (e.g., 18 = 10 + 8) and relate 10 ones to 1 ten. Extend the core components: say the tens list 10, 20, …, 100 and count to 100 by ones, count up to twenty-five things in a row with effort, read and write 1 to 19.	See, say, count, draw, and write tens units and ones units from 1 to 100, seeing and counting the groups of ten both as decades (ten, twenty, thirty, …) and as tens (1 ten, 2 tens, 3 tens, …).	See, say, count, draw, and write hundreds units, tens units, and ones units from 1 to 1,000; use place-value terms in explaining multidigit addition and subtraction.
The Relations (More Than/Less Than) Core	**The Relations (More Than/Less Than) Core**	**The Relations (More Than/Less Than) Core**
Show comparing situations with objects or in a drawing, and match or count to find out *which is more* and *which is less* for two numbers ≤ 10. Use = and ≠ symbols for groups of things, numerals, and pictures of fingers.	Solve additive comparison word problems that ask "How many more (less) is one group than another?" for two numbers ≤ 18 by counting or matching with objects or drawings or by knowing numerical relationships (such word problems describe relations between two numbers more precisely: the difference is now involved). Use the words *more/fewer-less* and > and < to compare numbers to 10 and use the concepts of tens and ones developed in the number core and multiunit objects or math drawings to compare numbers to 100.	Use the words *more/fewer-less* and > and < to compare numbers to 1,000 using knowledge of hundreds, tens, and ones.

(*Continued on next page*)

Table 2.1 (Continued)

Progression of Ideas about Number and Operations (and Algebra from Kindergarten On)

The Operations (Addition/Subtraction) Core	The Operations (Addition/Subtraction) Core	The Operations (Addition/Subtraction) Core
Use objects or fingers or pictures or math drawings to solve change-plus, change-minus, and put-together/take-apart situation problems and also such word, oral number, and written symbolic problems with totals ≤ 10. Learn to decompose 3, 4, 5 into partners; work on decomposing 6 and 7 (e.g., 6 = 5 + 1, 6 = 4 + 2, 6 = 3 + 3, 6 = 2 + 4, 6 = 1 + 5); and see equations with one number on the left and the partners (addends) on the right.	Use objects or fingers or math drawings and equations to solve change-plus, change-minus, and put-together/take-apart situation problems with all unknowns and also such word, oral number, and written symbolic problems with addends from 1 to 9. Pose as well as solve such problems. After working with additive comparison situations and word problems (see the relations core above), mix all types of word problems. Learn to decompose numbers from 3 to 10 into partners (e.g., 10 = 9 + 1, 10 = 8 + 2, 10 = 7 + 3, 10 = 6 + 4, 10 = 5 + 5) and use these relationships to relate addition and subtraction in problem situations, to add and subtract quickly for totals ≤ 6, and to build the prerequisite knowledge for addition and subtraction strategies. Count on for addition problems with totals ≤ 18. Think of subtraction as finding an unknown addend (e.g., think of and rewrite 14 – 8 = ? as 8 + ? = 14). Count on fluently and accurately to find an unknown addend. Work with derived-fact strategies such as make-a-ten and doubles +1 or –1. Give unknown totals or unknown addends (including for subtraction) quickly for totals ≤ 6. Use the concepts of tens and ones developed in the number core and use multiunit objects or math drawings to add and subtract tens and ones (e.g., 60 + 3) and tens and tens (e.g., 40 + 20) and to add two-digit numbers and ones (e.g., 58 + 6) and 2 two-digit numbers starting with problems requiring regrouping (e.g., 38 + 26) (do not do such subtraction problems with or without regrouping). Relate mathematics drawings to written number (symbolic) work.	Develop fluency in solving all types of addition and subtraction word problems (change-plus, change-minus, put-together /take-apart, and comparison situations) with all unknowns. Solve two-step problems and problems with extra information or not enough information, using all types of word problems with all unknowns for addends 1 to 9. Pose as well as solve such problems. Use partners of single-digit numbers to understand and use partners of decades and of hundreds (e.g., 100 = 90 + 10, 100 = 80 + 20, 100 = 70 + 30, 100 = 60 + 40, 100 = 50 + 50). Develop fluency with addition facts and related subtraction facts: Use level 2 (counting on) or level 3 (derived-fact) solution procedures or rapid known facts to solve additions and subtractions and to solve single-digit additions and subtractions within multidigit problems. Develop fluency with addition and subtraction to 1,000: Use the concepts of hundreds, tens, and ones developed in the number core and multiunit objects or math drawings to add and subtract two-digit numbers to 100 requiring regrouping, then problems to 200 with and without regrouping, and then pairs of all sizes of numbers to 1,000 including problems given horizontally. Relate steps with objects or drawings to steps with written numbers, then later do written number work with understanding without objects or drawings. Solve change-plus, change-minus, put-together/take-apart, and comparison situations with all unknowns using numeric math drawings or situation equations and numeric methods for totals to 200 and later for totals to 1,000. Pose as well as solve such problems. Relate word problems and numerical work continually.

The Number Core

The number core developed in preschool and kindergarten

The number core for kindergarten is outlined in the table 2.1 and discussed in detail in the book *Focus in Kindergarten* (NCTM 2010). A brief summary is included in this book to foster understanding of what children with adequate learning experiences have been able to learn. Children without such experiences will need extra time and support at school and at home at the beginning of grade 1 to build this knowledge. Suggestions for this support are included here in the kindergarten summaries. The number core content for prekindergarten is overviewed in *Focus in Kindergarten* and discussed in more detail in *Focus in Prekindergarten* (NCTM 2010).

Initially the four number core aspects (cardinality, number-word list, counting correspondences, and written number symbols) are separate. Then children make vital connections. *First,* they connect saying the number-word list with one-to-one correspondence to begin counting objects. Initially, counting is just an activity and does not have cardinal meaning, because young children do not understand that the last word is special, that it tells them the total amount (its cardinality). If a child at this level is asked the question "How many are there?" after she or he has counted, she or he may count again (repeatedly) or give a number word different from the last counted word. A crucial *second step* is connecting counting and cardinality so that the count tells how many there are. This step in the learning path coordinates the first three aspects of the number core. The *third step* connects counting and cardinality in the opposite direction: 4s/pre-Ks come to be able to count out a specified number of objects (e.g., six). Doing so requires that counting be so automatic for them that they have mental capacity to remember the word *six* while they are counting. Therefore, children in grade 1 who cannot yet do this will need lots of practice counting groups of objects or pictures and telling how many there are (see later kindergarten section). Counting can then become fluent enough that they will have mental space to remember the word to which they are counting. A few first graders may also need support to learn the first two steps (see the sidebar "Teaching the Cardinality Principle").

Prekindergarten and kindergarten children also master the relationships among count words, written numbers, and quantities for numbers from 1 to 10. They count or see (perceptually subitize) quantities, relating the quantity to a number word that tells how many there are (the cardinality). They also relate written number symbols to number words and to seen or counted quantities. They begin to see two small quantities and see/know their total; this is called *conceptual subitizing* (e.g., see two and two and say *four*). Five-groups (see figure 2.1) are especially important as a basis for later work with groups of tens, and they relate to fingers and to coins (nickels). Other groups (twos and fours) can also be helpful visualizations.

Teaching the Cardinality Principle

The adult puts out three to seven objects in a row in front of the child each time.

"When you count, the *last* word you say tells you how many things there are. Watch me. One, two, three, four, five, six" (adult counts blocks pointing to each block). "*Six.* There are *six* blocks" (adult gestures in a narrow ellipse over the set of six blocks).

"Watch again." Repeat for five small toy pigs. "When you count, the *last* word you say tells you how many things there are. Watch me. One, two, three, four, five" (adult counts toy pigs, pointing to each pig). "*Five.* There are *five* pigs" (adult gestures in a narrow ellipse over the set of five pigs).

"Watch again." Repeat for seven pennies. "When you count, the *last* word you say tells you how many things there are. Watch me. One, two, three, four, five, six, seven" (adult counts pennies pointing to each penny). "*Seven.* There are *seven* pennies" (adult gestures in a narrow ellipse over the set of seven pennies).

"So the *last* word you say in counting tells you how many things you have.

Now you try it. How many trucks are here?"

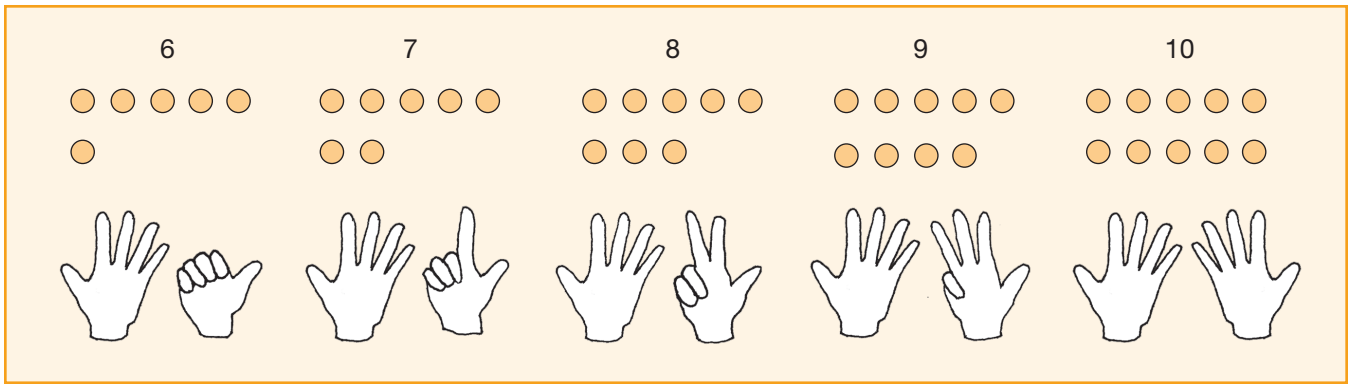

Figure 2.1. Seeing 5-groups that show 6 as 5 + 1, 7 as 5 + 2, 8 as 5 + 3, 9 as 5 + 4, and 10 as 5 + 5

Kindergarten children can consolidate and extend earlier number core experiences in the four aspects so that they—

- see numbers 6 to 9 with a 5-group (dot arrays, fingers) within teen numbers 16 to 19;
- say the number-word list to 100 and say the tens list 10, 20, 30, …, 90, 100;
- count up to twenty-five things in a row with effort and count out *n* things up to twenty;
- read and write numerals 1 to 19.

The major new advance in kindergarten is the crucial step described in two National Research Council reports (Kilpatrick, Swafford, and Findell 2001; Cross, Woods, and Schweingruber 2009): Kindergarten children integrate all the core components of number to see that teen numbers are made up of a ten and some ones. Before discussing this major step, we need to review the difficulties English number words make for understanding teen numbers and numbers from 20 to 100.

Difficulties in English number words for 11 to 19

The first ten number words are arbitrary in most languages. But then most languages begin to have patterns that make the number words easier to learn. English has a partial pattern for words from eleven to nineteen, but this pattern is marred by irregularities. These problematic issues are summarized in table 2.2 and listed below. For simplicity we will call these numbers *teen* numbers even though not all number words for these numbers have the *-teen* suffix.

- The partial pattern is to say one of the first nine counting words and add *-teen* to it (the *-teen* means *ten*), as in *fourteen, sixteen, seventeen, eighteen, nineteen*.
- However, the first two "teen" words (*eleven, twelve*) do not have the *-teen* ending.
- The third and fifth words have modified the number below ten so that it may not be recognizable: *thirteen* instead of *threeteen* and *fifteen* instead of *fiveteen*.

Children frequently see and use the *-teen* structure before they learn the exceptions, as in this diary example for a child aged three years two months: *"Eight, nine, ten, eleventeen, twelveteen, thirteen."* Parents and teachers can discuss these irregularities.

- "What would be better number words than *eleven* and *twelve?* (*oneteen* and *twoteen*)
- "What would be clearer to say than *fifteen*?" (*fiveteen*)
- "Yes, it is harder to have some of the number words we have, but we need to use them or other people will not understand us."

Many children skip over fifteen even when they can say the end of the teens ("sixteen, seventeen, eighteen, nineteen") correctly. This may be because it is in the middle and because it is irregular. So parents and teachers may need to focus children's attention and work particularly on the word *fifteen*.

Table 2.2
Difficulties in Learning English Words from 1 to 100

English number words for the first four places name the values for the third and fourth places from the right (*hundred* and *thousand*). But there are many irregularities for the second place, where we never say *ten* but only say *-teen* or *-ty.* So 2222 is *two thousand two hundred twenty two*. In a totally regular named-value system, such as Chinese, Japanese, and related systems, the number would be *two thousand two hundred two ten two one.*

Difficulties in English number words for 10 to 19. These teen words follow the reversals of German languages in which the ones are said before the tens.

a. Written numerals show the ones second, and teen words say them first (if at all): 18 but eighteen.
b. Written numerals do not show that the 1 is really one ten or ten ones. It looks like 1, not 10.
c. The teen words never say *ten,* and only some say *teen*.
d. *Ten* is not *one ten*, *eleven* is not *oneteen*, and *twelve* is not *twoteen*.
   We say *thirteen*, not *threeteen*, and *fifteen*, not *fiveteen*.
   Only *fourteen, sixteen, seventeen, eighteen, nineteen* have any pattern that can relate easily to some ones and a ten, and this pattern is late in the teens.
   This is in contrast to East Asian number words that are said as *ten one, ten two, ten three, …, ten nine, two ten* (20).
e. The English words for 20 to 100 say the ones second, opposite to the pattern for the teens.
f. There is auditory confusion between teen and decade words: fourteen and forty, sixteen and sixty, seventeen and seventy, and so on.

Difficulties in English number words for 20 to 99. These teen words do not follow the reversals of German languages but do have irregularities.

g. We say *-ty* instead of *ten*: the suffix *-ty* is added to the digit word (one, two, three, etc.) to say 20, 30, …, 90, but *-ty* does not say *ten* clearly.
h. Four of the earliest decade words are irregular and do not even show this *digit-ty* structure clearly: 10 is said as *ten* not *one ten,* 20 as *twen-ty* (not *two tens* or *two-ty*), 30 as *thir-ty* (not *three tens* or *three-ty*), and 50 as *fif-ty* (not *five tens* or *five-ty*); only 40 (*four-ty*), 60 (*six-ty*), 70 (*seven-ty*), 80 (*eight-ty*), 90 (*nine-ty*) show this structure clearly.

This *ones-before-tens* structure of the teen words is opposite to the *tens-before-ones* structure in the written teen number symbols. We say "four" first in "fourteen" but write 4 second in 14 (1 ten 4 ones). This reversal, and the irregularities listed in the foregoing, make the pattern-finding activity of relating written numerals to teen number words particularly complex for children speaking English. They need help and support to learn to say the teen numbers correctly. These are discussed in the kindergarten number core.

A final difficulty in understanding the meaning of the teens words is that English words do not explicitly say the *ten* that is in the teen number (*teen* does not mean *ten* even to many adults). This anomaly is in contrast with number words in East Asia that are said "ten, ten one, ten two, ten three," and so on, for 10, 11, 12, 13, and so on. Therefore, English-speaking children need particular help seeing the ten inside teen quantities. This is a goal for kindergarten and is discussed more fully after the next section.

Difficulties in English number words for 20 to 99

The *ones-before-tens* structure for English teen numbers comes from German. The same structure is used in German for all numbers from 11 to 99. But English changes for the words from twenty to ninety-nine to the *tens-before-ones* structure. This tens-before-ones structure is the same order in which numerals are written: "twenty" is said first in "twenty-seven," and 2 is written first in 27 (2 tens, 7 ones). So it is easier for children to relate the patterns in the written numerals to English number words from twenty to ninety-nine, a goal for kindergarten. Full understanding of the quantities of tens and ones in these numbers is a goal for grade 1.

The transition in English words from nine to ten is not clear: *ten* just sounds like another word with no special significance. Therefore, at first children often do not stop at twenty-nine but continue to count "twenty-nine, twenty-ten, twenty-eleven, twenty-twelve, twenty-thirteen." This error can be a mixture of not yet understanding that the pattern ends at nine and difficulty stopping the usual counting at nine so as to shift to another decade.

Irregularities in English decade number words complicate saying the number-word list correctly to one hundred (see the bottom of table 2.2). *Forty, sixty, seventy, eighty,* and *ninety* have a regular pattern: The ones word followed by *-ty* (which means ten). But most of the early decade words are irregular:

- *Twenty,* not *twoty* or *two-ten*
- *Thirty,* not *threety* or *three-ten*
- *Fifty,* not *fivety* or *five ten*

As with the teen words, the *ten* is not said explicitly but is said as a different suffix, *-ty*. Therefore, children need to work explicitly with groups of tens and ones to understand these meanings for the decade words from twenty to one hundred as groups of ten.