2 Focusing on Linear Functions and Linear Equations

In grade 8, students learn how to analyze and represent linear functions and solve linear equations and systems of linear equations. They learn how to represent linear relationships as graphs, tables, and equations. Students learn the meaning of slope and y-intercept and how these two elements of the graph of a linear function can be identified in each of three representations of a linear relationship—algebraic, tabular, and graphical. They also gain an understanding of the meaning of linear function and that the algebraic representation of a linear function is a linear equation. The eventual goal of this Focal Point is for students to use their understanding of linear functions and linear equations to represent and solve problems.

Instructional Progression for Linear Functions and Linear Equations

The focus on linear functions and linear equations in grade 8 is supported by a progression of related mathematical ideas before and after grade 8, as shown in table 2.1. To give perspective to the grade 8 work, we first discuss some of the important ideas that students focused on before grade 8 that prepare them for learning about linear functions and linear equations in grade 8. At the end of the detailed discussion of this grade 8 Focal Point, we present examples of how students will use linear functions and linear equations in later grades. For more detailed discussions of the "before" parts of the instructional progression, please see the appropriate grade-level books, for example, *Focus in Grade 6* (NCTM 2010) and *Focus in Grade 7* (NCTM 2010).

Early Foundations for Understanding Linear Functions and Linear Equations

Before entering the grade 8 classroom, students are expected to have learned concepts and skills that they can use to understand their work in linear functions and linear equations. In grade 7, students develop efficient, accurate, and generalizable methods for operating with all rational numbers. They learn to graph and represent proportional relationships. Students also use linear equations in one variable and rational numbers to solve word problems.

Table 2.1
Grade 8: Focusing on Linear Functions and Linear Equations—Instructional Progression for Linear Functions and Linear Equations

Before Grade 8	**Grade 8**	After Grade 8
Students develop efficient, accurate, and generalizable methods for operating with rational numbers. Students recognize fractions, percents, and certain decimals as ways of representing rational numbers and convert flexibly among fractions, decimals, and percents. Students are able to explain which fractions correspond to terminating decimals. Students graph proportional relationships and recognize the graph as a line through the origin with the constant of proportionality as the slope of the line. Students express proportional relationships as $y = kx$ and distinguish them from other relationships, such as $y = kx + b$. Students use linear equations in one variable to solve word problems.	Students translate among algebraic, geometric (graphical), numerical (tabular), and verbal representations of linear functions. Students recognize the slope of a line as a constant ratio representing the change in y compared with the related change in x. Students recognize the y-intercept of a line as the point $(0, y)$ where the line crosses the y-axis. Students recognize relationships that are functions and develop an understanding of how the algebraic representations of linear functions are linear equations. Students relate systems of equations to pairs of lines that intersect, are parallel, or are the same line in the plane and understand that the solution to a system of equations is a solution to both equations. Students analyze and solve problems using linear equations and systems of linear equations.	Students understand numbers, ways of representing numbers, and relationships among numbers within different number systems (e.g., rationals, reals). Students apply operations appropriately, compute fluently, and make reasonable estimates within different number systems. Students understand relations and functions. Students represent and analyze mathematical situations and structures using algebraic symbols. Students use mathematical models to represent and understand quantitative relationships. Students analyze change in various contexts.

Rational numbers

In grade 7, students learn about the set of rational numbers along with the operations of addition and multiplication, along with their inverses of subtraction and division, as being components of the rational number system. They learn that the set of rational numbers is made up of every number that can be expressed as a/b, where a is an integer and b is an integer other than 0. The set of rational numbers includes the integers, since any integer a can be written as $a/1$. Any nonzero rational number can be expressed as a positive or negative fraction and as a positive or negative decimal that terminates or repeats; zero, although neither positive nor negative, also can be expressed as a fraction and as a terminating decimal. Students explore the properties of the operations in the rational number system, comparing and contrasting the properties of these operations that exist in the system of whole numbers and the system of rational numbers. In their study of the properties of operations, students learn about the concept of closure and how to decide whether an operation in a system is closed. Students also develop computational procedures that they use to add, subtract, multiply, and divide with rational numbers, both positive and negative.

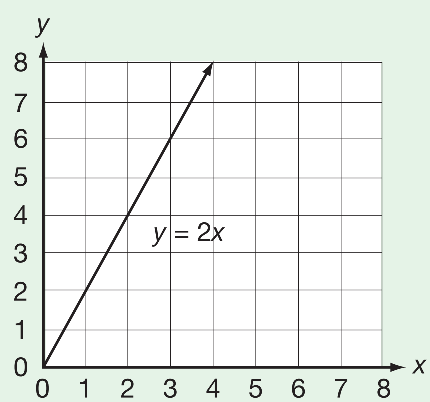
During their study of the rational number system, students develop an understanding of percent as a part-to-whole ratio where *n* percent means *n* parts out of 100 total parts. Students flexibly translate among percent, fraction, and decimal forms. For example, students who know that 76 percent means 76 out of 100 use their understanding of rational numbers and ratio to reason that

$$76\% = 76/100 = 0.76 = 19/25.$$

Proportionality

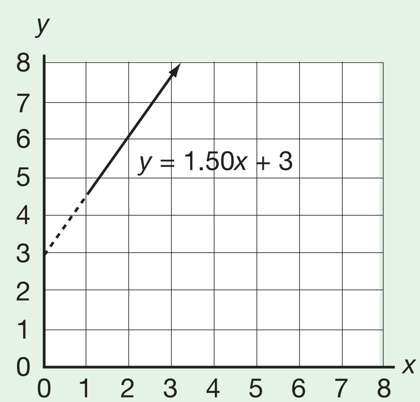
Students' work with proportionality in grade 7 also prepares them to study linear functions and linear equations in grade 8. In grade 7, students express proportional relationships algebraically as equations of the form $y = kx$, where

<table>
<tr><th>Proportional Relationship</th><th>Relationship That Is Not Proportional</th></tr>
<tr><td>
Chicken costs $2 per pound.
Let y = number of dollars.
Let x = number of pounds.
Then $y = 2x$ models this relationship.
The graph is a line through the origin.



The relationship is proportional and thus linear. Note that k, the constant of proportionality, is 2. In other words, no matter how many pounds of chicken you buy in this situation, the cost per pound is constant. One proportion for this relationship is

$$\frac{2 \text{ dollars}}{1 \text{ pound}} = \frac{4 \text{ dollars}}{2 \text{ pounds}}.$$
</td><td>
Chicken costs $4.50 for a minimum purchase of one pound and $1.50 for each additional pound.
Let y = number of dollars.
Let x = number of pounds.
Then $y = 1.50x + 3$ models this relationship, for $x \geq 1$.
The graph is a line not through the origin.



The relationship is linear but not proportional. Therefore, there is no constant of proportionality. In other words, for different amounts of chicken purchased in this situation, when the cost is spread evenly across the number of pounds purchased, the cost per pound varies.
</td></tr>
</table>

Fig. 2.1. Comparing a proportional relationship with a relationship that is not proportional

k is the constant of proportionality. Students also graph these relationships, recognizing the graph as a line through the origin whose slope is k. Students learn to distinguish proportional relationships from relationships that are not proportional, including linear functions whose equations have the form $y = mx + b$, where $b \neq 0$, as shown in figure 2.1.

In grade 8 students apply this understanding as they learn about linear functions and the linear equations that represent them. They learn that $y = mx + b$ is one form of a linear equation and that $y = kx$ represents a linear equation where $m = k$ and $b = 0$. Thus, a proportional relationship between two variables can always be modeled by a linear function, but not all linear functions represent proportional relationships.

Equations

In grade 6 students learn how to represent relationships algebraically using expressions and equations. They learn how to use substitution to evaluate expressions, and they learn various ways to solve equations. They learn that a numerical solution to an equation is a number that makes the equation true. Students also use expressions and equations to solve word problems.

Focusing on Linear Functions and Linear Equations

There are many types of relationships between quantities and many different ways to describe them in order to make predictions and to solve problems. In this Focal Point, students learn about linear relationships and how to use words, graphs, tables, and equations to represent them, as shown in figure 2.2.

Representations of linear relationships

In previous grades, students learn about equations and use equations in one variable to solve problems. For example to solve the problem

> Ed works in a restaurant. He earns \$8 per hour plus tips. One day he earned a total of \$83, including \$35 from tips. How many hours did he work that day?

students have written and solved the equation as follows:

$$\begin{aligned} 8x + 35 &= 83 \\ 8x + 35 - 35 &= 83 - 35 \\ 8x &= 48 \\ \frac{8x}{8} &= \frac{48}{8} \\ x &= 6 \end{aligned}$$

Verbal: Johanna ran 1 mile the first day and then increased the number of miles she ran each day by 2.

Algebraic:
Linear Equation

$y = 2x - 1$

or

$-2x + y = -1$

Numerical:
Table

x	y
1	1
2	3
3	5
4	7

Geometric:
Graph

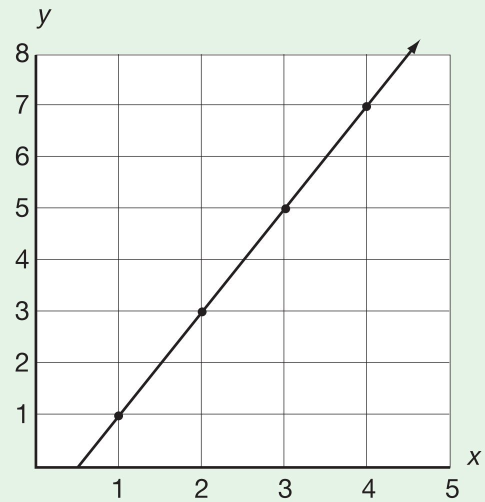


Fig. 2.2. Ways to represent a linear relationship

They have discussed that there is only one solution to this equation, $x = 6$, and that the solution, in the context of the problem, means that Ed worked six hours that day.

In grade 8, students expand their understanding of equations to include linear equations in two variables. A linear equation is an algebraic equation in which each term is either a constant or the product of a constant (called a *coefficient*) and the first power of a single variable. Some linear and nonlinear equations are shown in figure 2.3.

Linear equations in two variables are used to describe the relationship between two quantities. For example, in the equation $y = 2x$, the quantity y is two times the quantity x. In the equation $y = x + 5$, the quantity y is five more than the quantity x. Teachers can help students make the transition from linear equations in one variable to linear equations in two variables by giving them appropriate contextual situations to model, as shown in the following classroom discussion.

Teacher: Here is the problem we were working on solving by using an equation:

Linear Equations	Nonlinear Equations
$x - 3 = 7$ or $x^1 - 3 = 7$	$x^2 - 3 = 7$
$2x - y = 12$ or $2x^1 - y^1 = 12$	$2x^2 - y = 12$
$c = a + b$	$xy = 1$
$3 + x = y - 7$	$\frac{1}{x} = y + 3$

Fig. 2.3. Examples of linear and nonlinear equations

> This weekend Rick did homework for 6 hours. He did homework for twice as long as he played soccer. How many hours did Rick play soccer?

Who would like to share their equation and solution?

Maddie: I would. Rick did his homework for twice as long as he played soccer. He did his homework for 6 hours. So, I let s stand for the number of hours he played soccer and wrote the equation $6 = 2s$. Then, I divided both sides by 2 to solve the equation:

$$6 = 2s$$
$$\frac{6}{2} = \frac{2s}{2}$$
$$3 = s$$

I got $s = 3$, so Rick played soccer for three hours. This makes sense because $6 = 2(3)$ is true.

Teacher: The equation that Maddie wrote involves one variable, s. There is one solution to this equation. The solution for the equation is $3 = s$, and so the solution set is one value for s. Suppose I changed the situation to—

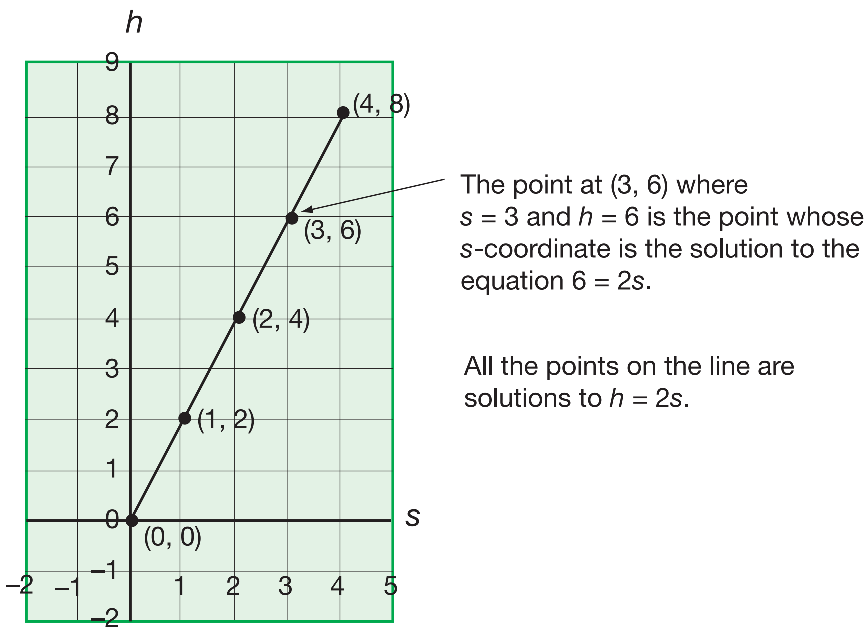
> This weekend Rick did homework and played soccer. He did homework for twice as long as he played soccer.

In this context, you don't know how long Rick did homework or played soccer, but you do know something about the relationship between the amount of time he spent doing each activity. We can use an equation to represent this relationship. Let's use h to represent the time that Rick did homework and s to represent the time that Rick played soccer. Who can tell me an equation that you can write to represent this relationship?

Shaundra: I think I can. Rick did homework twice as long as he played soccer, so the time spent on homework is 2 times the time spent on soccer, or

$h = 2s$. This is almost the same as Maddie's equation, except I used h for the homework and Maddie used 6.

Teacher: Equations such as $h = 2s$ are linear equations that involve two variables and are used to show the relationship between two quantities, in this case the relationship between the number of hours spent doing homework, h, and the number of hours playing soccer, s. A line on the coordinate plane represents the solutions to this equation. Notice that the solution to the one-variable equation, $6 = 2s$, is related to the point on the line $h = 2s$ where $h = 6$ and $s = 3$.



Teachers could then change the situation to show that if they change the relationship between the number of hours spent doing homework and the number of hours spent playing soccer, the equation that represents that relationship changes as well, as shown in figure 2.4.

For students to understand how equations can be used to represent the relationship between quantities, they also need to learn about solution sets to these equations. A solution to an equation in two variables is an ordered pair of values that makes the equation true. For example, in the equation $y = 2x$, a solution is an ordered pair in the form (x, y). There are infinitely many solutions to this equation, with a few shown in figure 2.5.

As students have learned in their previous work with coordinate graphing, the order of the numbers in an ordered pair is very important. Teachers can further emphasize this concept by having students analyze equations and their solutions in context. For example, if, in the equation $y = 2x$, x is the

Let x represent number of hours spent playing soccer; let y represent number of hours spent doing homework.	
$y = 2x$	← Rick spends twice as many hours doing homework as he does playing soccer.
$y = x + 3$	← Rick spends 3 more hours doing homework than he does playing soccer.
$x = 4$	← Rick spends 4 hours playing soccer regardless of the amount of time he spends doing homework.
$y = 5$	← Rick spends 5 hours doing homework regardless of the amount of time he spends playing soccer.

Fig. 2.4. Different equations that represent different relationships

(1, 2) is a solution to the equation $y = 2x$ because— $y = 2x$ $2 = 2(1)$ $2 = 2$	(35, 70) is a solution to the equation $y = 2x$ because— $y = 2x$ $70 = 2(35)$ $70 = 70$	(–5, –10) is a solution to the equation $y = 2x$ because— $y = 2x$ $-10 = 2(-5)$ $-10 = -10$	$\left(-\frac{2}{3}, -1\frac{1}{3}\right)$ is a solution to the equation $y = 2x$ because— $y = 2x$ $-1\frac{1}{3} = 2\left(-\frac{2}{3}\right)$ $-1\frac{1}{3} = -\frac{4}{3} = -1\frac{1}{3}$

Fig. 2.5. Using substitution to demonstrate that several ordered pairs are solutions to the equation given

number of people and *y* is the number of books, the ordered pair (2, 4) represents two people and four books; however, the ordered pair (4, 2) represents four people and two books. If the relationship is that every person has two books, the ordered pair (4, 2) would *not* be a solution for that equation.

Linear equations are so named because the graphical representation of a linear equation is a line in the coordinate plane. Students can see that this is true as they connect the solutions of two variable linear equations with what they already know about graphing in the coordinate plane. Students learn that since every ordered pair corresponds to a point in a coordinate plane, a set of ordered pairs can be graphed. To generate this set of ordered pairs, students can choose values for *x* or *y*, substitute those values in the equation, and solve for the other variable. The resulting values form an ordered pair that is a solution to the equation. Students may find it helpful to organize these values in a table, as shown in figure 2.6.