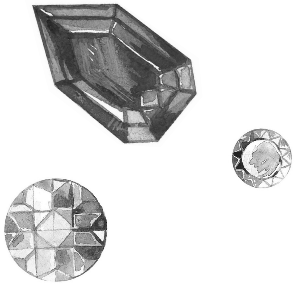
1



Demystifying Magic Squares

OVERVIEW

Magic squares have fascinated mathematicians for thousands of years, dating back to more than 2000 years BCE. References to magic squares are found in the historical records of Chinese, Arab, Indian, Japanese, African, and European mathematics. The 16th-century German Renaissance artist Albrecht Dürer created a famous engraving in which a magic square is prominently displayed on the wall, and even Benjamin Franklin is remembered for the remarkable magic squares that he developed.

Derived from "Demystifying Magic Squares" in *Navigating through Reasoning and Proof in Grades 9–12* (NCTM 2008b, pp. 22–26).

In the classroom, magic squares, if they are introduced at all, are most often presented as number puzzles, recreational curiosities, or challenge problems outside the mainstream curriculum. But magic squares present us with opportunities to combine number sense, algebraic reasoning, and problem solving in a context of exploring and making sense of interesting, often fascinating, mathematics. This activity opens the gateway to such mathematical reasoning by delving into several cases of magic squares, a first foray into a virtually boundless area of mathematical thinking and inquiry.

GOALS

- Use algebraic representations and properties of numbers to identify patterns and verify properties of magic squares.
- Apply reasoning by processes of elimination, examining cases, and other strategies to justify the structure of magic squares.
- Communicate discoveries and justifications in algebraic terms.
- Use logical reasoning to rule out possibilities.

MATERIALS NEEDED

For each student, copies of the following activity sheets:

- Demystifying Magic Squares
- 3×3 Magic Square Grid
- 3×3 Magic Square Numbers
- 3×3 Magic Squares Recording Sheet
- 5×5 Magic Square Grid
- 5×5 Magic Square Numbers
- 5×5 Magic Squares Recording Sheet
- Albrecht Dürer's Magic Square
- Benjamin Franklin's Magic Square
- More than Meets the Eye

Scissors

(Optional) Calculator or access to a spreadsheet program (for calculating sums)

Engage

Introduce this activity by having the students examine the magic square presented in problem 1 on the activity sheet Demystifying Magic Squares (see figure 1.1). Discuss with them the defining properties of magic squares and the vocabulary used in describing them. Important points of discussion include the following:

FIG. 1.1

A magic square of order 4

1	8	10	15
12	13	3	6
7	2	16	9
14	11	5	4

- A magic square of size $n \times n$ consists of n^2 distinct numbers arranged such that the sum of the numbers in every row, every column, and both diagonals is the same number, called the **magic sum**. Verify that the square in problem 1 on the activity sheet is a magic square, and find its magic sum. [*34*]
- Magic squares are classified according to their size, and a square of size $n \times n$ is called an **order-n** square. The example given above is an order-4 magic square.
- Most magic squares use the numbers 1, 2, 3, . . ., n^2 as the entries, although that is not a requirement. Squares that do use the consecutive numbers 1 through n^2 are generally known as "normal" magic squares. Note that in the order-4 example, the entries are 1, 2, 3, . . ., 16.

The example used in the introduction here is not the only order-4 magic square. In fact, it has been shown that 880 distinct normal order-4 magic squares exist, excluding reflections and rotations, and as the size of the squares increases, their numbers quickly become mind boggling. So we will begin our investigation by considering the smallest possible magic square, that of order 3. Problem 2 on the activity sheet gives the placement of the numbers 1 and 2, as shown in figure 1.2, and asks the students to complete the magic square. To facilitate their search, give students copies of the activity sheets 3 × 3 Magic Square Grid and 3 × 3 Magic Square Numbers. They can cut apart the nine number squares and manipulate them on the grid as they investigate this and subsequent problems. The 3 × 3 Magic Squares Recording Sheet will help them keep a record of the squares they find as the activity progresses.

FIG. 1.2

Complete the magic square

	1	
		2

Unless students have had prior experience with magic squares, they are likely to approach this task using a trial-and-error strategy, so allow sufficient time for them to explore the possibilities and encourage them to keep track of any strategies they may have used. When they have completed their squares, ask them to compare their solutions and to describe their methods. Did they all find the same magic square? [*Probably yes.*] What is the magic sum? [*15*].

Before going on, engage students in a discussion of the questions posed in item 2 of the activity sheet, beginning with these queries:

- **Why must the magic sum be 15?** [*The numbers 1 through 9 appear in the square, and their total is* $1 + 2 + \ldots + 9 = 45$. *If the three rows (or three columns) all have the same magic sum, then the total of 45 must be distributed equally across rows (or columns), so that each row (column) totals 1/3 of 45, or 15.*]
- **Explain why 3 cannot occupy the center square. What positions are possible for the 3?** [*The 3 cannot be in the same row, column, or diagonal with either 1 or 2, because both* $(3 + 1 + x = 15)$ *and* $(3 + 2 + x = 15)$ *require that x be greater than 9. The only place that 3 can go is in the first column, second row, as shown in figure 1.3.*]
- **Explain why 4 cannot occupy the center square. What positions are possible for the 4?** [*The 4 cannot be in the same row, column, or diagonal with 1 because that would require the third addend to be 10, which is not available. There are two possible locations for the 4: cells (a) and (b) in figure 1.4. Try each case to determine that 4 must be located at (a).*]
- **Why can none of the numbers 6, 7, 8, or 9 occupy the center square?** [*Try each of the numbers in the center square and identify a contradiction or impossibility. (See figure 1.5.) For example, 6 in the center square would require another 6 to complete the second row; 7 in the center square would require another 7 to complete the second column. Each case leads to a roadblock. Call on students to demonstrate and show reasons why each of the numbers 6, 7, 8, 9 must be eliminated.*]

FIG. 1.3

Place the number 3 in the magic square

	1	
3		
		2

FIG. 1.4

Place the number 4 in the magic square

	1	
3		(b)
(a)		2

FIG. 1.5

Can 6, 7, 8, or 9 occupy the center square?

	1	
3		
4		2

- **By process of elimination, what number do you conclude must occupy the center square?** [*The center number must be 5. See the final solution in figure 1.6. Note that 5 is both the median and the mean of the set of numbers used. We will return to that observation later.*]

FIG. 1.6

The order-3 magic square

8	1	6
3	5	7
4	9	2

We have now found one order-3 magic square. Could there be others? The third item on the activity sheet challenges students to try to make magic squares, using the numbers 1 through 9 with magic sum 15, for the six starting positions specified in figure 1.7.

FIG. 1.7

Try to make magic squares with magic sum 15

(a)

	1	

(b)

1		

(c)

	1	

(d)

1		

(e)

	1	
2		

(f)

		1

Students can use their 3×3 magic square grid and numbers to explore possibilities for the six cases. They should discover that they cannot make squares (a) and (b) magic. There is only one solution for square (e), with its two fixed starting numbers, whereas squares (c), (d), and (f) each have two solutions. All the possible magic squares, together with the original square from before, are shown in figure 1.8.

FIG. 1.8

Arrangements of the order-3 magic square

Original

8	1	6
3	5	7
4	9	2

$R_{90°}$

4	3	8
9	5	1
2	7	6

$R_{180°}$

2	9	4
7	5	3
6	1	8

$R_{270°}$

6	7	2
1	5	9
8	3	4

F_V

6	1	8
7	5	3
2	9	4

F_{D+}

2	7	6
9	5	1
4	3	8

F_H

4	9	2
3	5	7
8	1	6

F_{D-}

8	3	4
1	5	9
6	7	2

Call on individuals to describe the patterns, similarities, or consistent features that they notice among the successful squares. Likely responses include the following:

- In every case, the 5 is in the center cell.
- The 1 is never in the center or in a corner cell.
- The numbers in the corners are always even.
- All the 3×3 magic squares are reflections or rotations of one another.

Let's examine those observations more closely by looking at all the ways that three distinct numbers from the set {1, 2, . . . 9} can total 15:

$1 + 5 + 9 = 15 \quad 1 + 6 + 8 = 15$

$2 + 4 + 9 = 15 \quad 2 + 5 + 8 = 15 \quad 2 + 6 + 7 = 15$

$3 + 4 + 8 = 15 \quad 3 + 5 + 7 = 15$

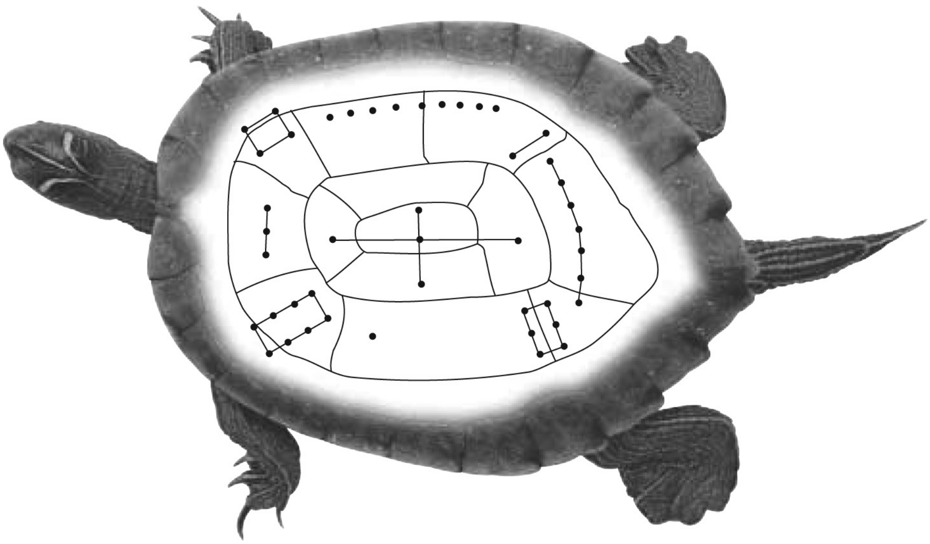
$4 + 5 + 6 = 15$

We can tally the number of times each digit appears among the addends:

Number	1	2	3	4	5	6	7	8	9
Frequency	ǀǀ	ǀǀǀ	ǀǀ	ǀǀǀ	ǀǀǀǀ	ǀǀǀ	ǀǀ	ǀǀǀ	ǀǀ

These results explain why the even numbers, each of which occurs in three combinations, must occupy the corner cells where each belongs to one row, one column, and one diagonal. Likewise, the odd numbers 1, 3, 7, and 9, which each occur in two of the combinations, must occupy the noncorner cells on the sides of the square, where they each belong to one row and one column. And we see another reason why 5, with its four tallies, must occupy the center position where it belongs to one row, one column, and two diagonals.

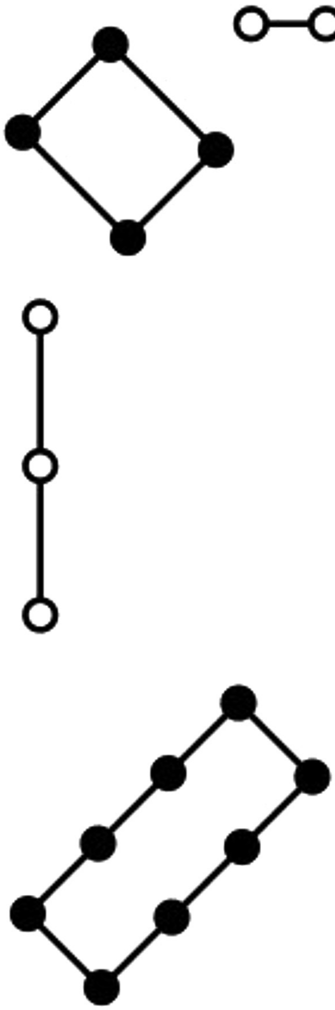
An ancient Chinese legend told the story of the Emperor Yu who, around 2200 BCE, was walking along the Lo River, a branch of the Yellow River, when he saw a tortoise with a unique diagram on its shell.



The diagram, which he called lo shu, *is a representation of the order-3 magic square.*

LO-SHU DOTS

The *lo shu* diagram



For more on the history of magic squares, see Anderson (2001), available at more4U.

Finally, let's consider the observation that the magic squares in figure 1.8 are all reflections or rotations on one another. If we consider the magic square from figure 1.6 to be the "original" square, then the other seven can be classified as follows:

$R_{90°}$—rotate the original square 90 degrees clockwise.

$R_{180°}$—rotate the original square 180 degrees clockwise.

$R_{270°}$—rotate the original square 270 degrees clockwise.

F_H—reflect (flip) over the horizontal axis.

F_V—reflect (flip) over the vertical axis.

F_{D+}—reflect (flip) over the positive-sloping diagonal.

F_{D-}—reflect (flip) over the negative-sloping diagonal.

Because all the arrangements shown in figure 1.8 are reflections or rotations of one another, the squares are not different. In other words, only one normal order-3 magic square has the numbers 1 through 9 and the magic sum of 15.

But this is just the beginning of our investigation.

Explore

Let's see how much more we can discover about 3×3 magic squares by posing some questions to investigate. To motivate this inquiry, first ask students to solve the two problems posed in item 4a on the activity sheet:

1. Form a magic square using the entries 5, 6, 7, 8, . . . 13 and find its magic sum.
2. Form a magic square using the entries 2, 4, 6, 8, . . . 18 and find its magic sum.

Solutions for the two squares are given in figure 1.9. But each of these examples leads to a broader generalization:

- What happens if you add or subtract the same constant in every cell of a magic square?
- What happens if you multiply or divide by the same nonzero constant in every cell of a magic square?

FIG. 1.9

Form two magic squares

12	5	10
7	9	11
8	13	6

Magic sum 27

16	2	12
6	10	14
8	18	4

Magic sum 30

Suggest that the students use variables to represent the entries in a 3×3 magic square as shown in figure 1.10. They might work in pairs to develop and discuss their responses before presenting their reasoning to the class. Sample explanations include these:

FIG. 1.10

Representation using variables

a	*b*	*c*
d	*e*	*f*
g	*h*	*i*

- Since $a + b + c = S$ in the original magic square, then adding a constant, n, to each cell yields $(a + n) + (b + n) + (c + n) = (a + b + c) + 3n = S + 3n$. Likewise, each row, column, and diagonal will total $S + 3n$, the new magic sum, and the resulting square is also magic. [*In question 1 above,* $n = 4$ *and* $S = 15 + 3 \times 4 = 15 + 12 = 27$.]
- Since $a + d + g = S$ in the original magic square, then multiplying each cell by $n \neq 0$ yields $(a \times n) + (d \times n) + (g \times n) = (a + d + g) \times n = S \times n$. Likewise, each row, column, and diagonal will total $S \times n$, the new magic sum, and again the resulting square remains magic. [*In question 2 above,* $n = 2$ *and* $S = 15 \times 2 = 30$.]

In all the 3×3 magic squares that we've seen so far, there appear to be several relationships involving the number in the center cell. Ask the students to describe and prove all the relationships that they can discover. Here are some properties that they should observe:

> ***"The role of the student is to be actively involved in making sense of mathematics tasks by using varied strategies and representations, justifying solutions, making connections to prior knowledge or familiar contexts and experiences, and considering the reasoning of others."*** (NCTM 2014, p.11)

- **The number in the center cell is one-third of the magic sum.** [*We know that the magic sum, S, is 1/3 of the sum of the nine numbers $a + b + \ldots + i$. The entry e, in the center cell, belongs to four of the row/column/diagonal combinations, which yield $(d + e + f) + (b + e + h) + (a + e + i) + (g + e + c) = 4S$. Regrouping terms gives $4S = (a + b + c + d + e + f + g + h + i) + 3e = 3S + 3e$. Hence $S = 3e$ or $e = S/3$.*]
- **The number in the center cell is the mean of the nine numbers making up the magic square.** [*Let the total of the nine numbers be T. We know that S is $T/3$, and we now also know that e is $S/3$. Thus $e = 1/3\ (T/3) = T/9$, which is the mean of the nine entries.*]
- **On any row, column, or diagonal, the pair of numbers symmetric to the center cell sum to twice the center value, and the center number is the average of the pair.** In other words, $(a + i) = (b + h) = (d + f) = (g + c) = 2e$. Since, for example, $a + e + i = S$ and $e = S/3$, then $a + i = S - S/3 = 2S/3 = 2e$, and $e = (a + i)/2$. [*In the 3×3 square, all the symmetric pairs total 10.*]

Drawing on these observations, we can pose additional questions, such as these from the activity sheet:

1. Can the set of numbers {1, 3, 5, 7, 8, 10, 12, 14, 16} be arranged to form a 3×3 magic square? [*No. Using what was proved above, the magic sum would have to be 76/3, which is not an integer and clearly not the sum of any three integers from the list. Also, the entry in the center cell would have to be 76/9, which is also not an integer and not one of the numbers in the set.*]
2. Andy claims that if he is given any "start-up" whole number, a, for the center of a 3×3 square and any whole-number values for x and y, he can make the square "magic." His procedure begins as shown in figure 1.11. Complete Andy's magic square, expressing the missing entries in terms of a, x, and y (see figure 1.12). Can you use Andy's method to create a magic square if $a = 8$, $x = 3$, and $y = 2$? (See figure 1.13.)

FIG. 1.11

Andy's start-up square

$a + x$		$a + y$
	a	
$a - y$		$a - x$

3. Does Andy's method of creating 3×3 magic squares always work? Prove or disprove your answer. [*Be careful! If $x = y$, or if either $x = 0$ or $y = 0$, Andy's method will not work because*

FIG. 1.12

Andy's completed square

$a + x$	$a - x - y$	$a + y$
$a - x + y$	a	$a + x - y$
$a - y$	$a + x + y$	$a - x$

FIG. 1.13

Magic square using Andy's method

11	3	10
7	8	9
6	13	5

the entries in the nine cells will not all be distinct. It is important to specify those restrictions: $x \neq y$, $x \neq 0$, $y \neq 0$.]

We have already seen that many mathematical patterns are embedded in the order-3 magic square. But what will happen with larger squares? Before we explore that question, we should note that odd-order squares (such as the 3×3 square) will have different properties from even-order squares (such as the 4×4 square in figure 1.1) because of the fact that the odd-order squares contain a single center cell whereas even-order squares do not. Therefore, let's expand our inquiry by considering normal order-5 magic squares containing the integers {1, 2, . . . 25} that we can relate to the 3×3 square that we have been investigating. For this activity, give students copies of the 5×5 Magic Square Grid, the 5×5 Magic Square Numbers, and the 5×5 Magic Squares Recording Sheet. They can cut out the numbers and manipulate them on the grid as they search for magic squares, which they can then record on the sheet.

It has been reported that the number of distinct normal order-5 magic squares, excluding reflections and rotations, exceeds 275 million. **(Gardner 1988, p. 216)**

Before students begin searching for magic squares, ask them to predict what the magic sum will be. On the basis of what they know from their work with order-3, they should realize that the magic sum, S, must equal $(1 + 2 + 3 + \ldots + 25) / 5$ or $325/5 = 65$.

In an arithmetic sequence a, $(a + d)$, $(a + 2d)$, . . . $[a + (n - 1)d]$, the total of the first n terms is given by $T = \frac{n(a_1 + a_n)}{2}$. In a normal order-n square, the total of the n^2 terms is $T = 1 + 2 + \ldots + (n^2) = \frac{n^2(n^2+1)}{2}$ and $S = \frac{T}{n} = \frac{n(n^2+1)}{2}$. For order-5, $T = \frac{25 \times 26}{2} = 325$ and $S = \frac{325}{5} = 65$.

The students can use their grids and numbers to attempt to construct a magic square, but it should quickly become obvious that, because of the increased size of the square and the greater number of possible 5-addend combinations, trial-and-error is not an

Activity_Gems_9-12.indd 19 12/08/21 6:21 PM

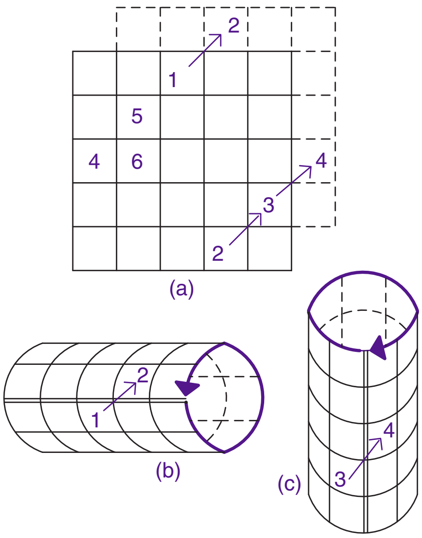
efficient strategy. Fortunately, a number of methods for constructing odd-order squares are known, and we will introduce and compare two of the best-known methods in this activity. (Interested students can find great delight in researching other strategies as follow-up activities later.)

One of the oldest and most popular methods of constructing odd-order squares is the "diagonal method" described below. Introduce this to the students and have them use their magic square grid and numbers to follow you through the first few steps; then let them complete the square on their own. The strategy is as follows:

- Place the number 1 in the center cell of the top row. Successive numbers will follow in order by moving one cell diagonally up and to the right.
- We see immediately that the diagonal move will take the 2 off the top of the grid (figure 1.14a). When that happens, imagine the square being rolled into a horizontal cylinder (figure 1.14b) where the bottom row wraps around to meet the top row, and the 2 is placed as shown in figure 1.14.
- Continue the diagonal moves to locate the numbers 3 and 4. Once again, the move takes the 4 off the grid on the right side. This time, imagine the square rolled into a vertical cylinder (figure 1.14c) where the left side wraps around to meet the right side, and the 4 is located as shown in figure 1.14.

FIG. 1.14

The diagonal method for constructing an odd-order magic square



- The next diagonal move locates the 5 (see figure 1.14a), but the subsequent diagonal move takes us to a cell that is already occupied by the 1. When that happens, drop down one row and place the 6 under the 5, and then continue the pattern as before.

At this point, stop filling the square and let students place the remaining numbers. When they finish, they should check to be sure that all the rows, columns, and diagonals have the magic sum of 65. If any do not, backtrack to locate where an error occurred.

We can now examine the 5×5 square and compare its properties to those we observed in the 3×3 square. For example:

- Look again at the 3×3 magic square in figure 1.6. Show that, in fact, it conforms to the diagonal-move strategy.

- The number (5) in the center cell of the 3×3 square is one-third of the magic sum (15); in the 5×5 square, the number in the center cell (13) is one-fifth of the magic sum (65).
- In both squares, the number in the center cell is the mean of the n^2 numbers making up the magic square. [*5 = 45/9; 13 = 325/25.*] The center number is also the median of the n^2 numbers.
- Every pair of numbers symmetric to the center cell total twice the center value. The center number is $(n^2 + 1)/2$ and the symmetric pair sums to $(n^2 + 1)$. For the order-3 square, $5 = (9 + 1)/2$ and the pairs total $9 + 1 = 10$. In the order-5 square, $13 = (25 + 1)/2$ and the pairs sum to $25 + 1 = 26$.

An* associative magic square *is one in which every pair of numbers symmetrically opposite the center cell sum to $n^2 + 1$.

Magic squares that exhibit the property described above are called **associative** (or **associated**) magic squares. Figure 1.15 illustrates three examples of symmetric pairs that total 26. Students should verify that all the symmetric pairs in this square share that property.

FIG. 1.15

An associative order-5 magic square

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

If every number in a magic square is subtracted from $(n^2 + 1)$, we obtain another magic square called the **complement** of the first square. Figure 1.16 shows the complement of the square in figure 1.15. Verify that it is also an associative magic square. What else do you notice about the square and its complement? [*The complement is a 180-degree rotation of the first square.*]

FIG. 1.16

Complement of the magic square in figure 1.15

9	2	25	16	11
3	21	19	12	10
22	20	13	6	4
16	14	7	5	23
15	8	1	24	17

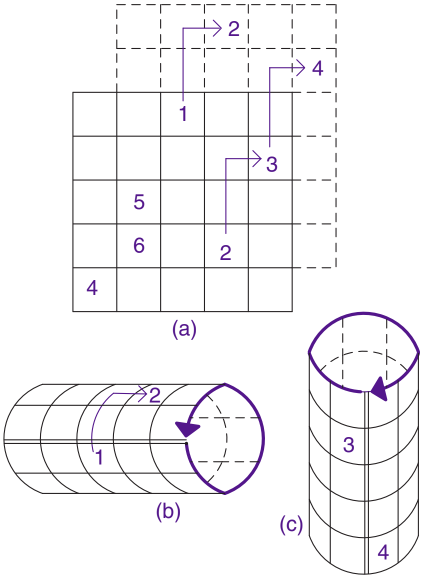
We have seen that the diagonal-move strategy is effective for constructing an associative magic square, but let's introduce one more popular method for building odd-order magic squares and examine its outcomes. This method is known as the knight's move strategy. As before, you can demonstrate this to students and have them use their magic square grid and numbers to follow you through the first few steps; then let them complete the square on their own. The strategy is as follows:

- Place the number 1 in the center cell of the top row. The method gets its name because successive numbers will be placed by following what, in chess, is the pattern in which the knight moves on the chessboard: in an L-shaped pattern of two cells in one direction (horizontally or vertically) and one cell at right angles to that. In building this square, we will follow the pattern of moving two squares "up" and one square to the right.

- The first knight's move takes the 2 off the top of the grid (figure 1.17a), so we again imagine the horizontal cylinder (figure 1.17b) to place the 2 as shown.
- Continue the knight's move to place the numbers 3 and 4. This time, the move takes the 4 off the grid at the top-right corner (figure 1.17a), and we need to imagine both a horizontal and a vertical cylinder to locate the placement for the 4.
- The next knight's move locates the 5 (figure 1.17a), but the subsequent move takes us to the cell that is already occupied by the 1. When that happens, we again drop down one row and place the 6 under the 5, and then continue the pattern as before.

FIG. 1.17

Filling a magic square using the knight's move



Pause here to let the students place the remaining numbers on the grid. When they finish, they should verify that they have a magic square with magic sum 65. Then check this new magic square for the following properties:

- Verify that the square constructed using the knight's move is different from the square constructed by the diagonal method—that is, the squares are not reflections or rotations of each other.
- Verify that this knight's move square is associative. (Figure 1.18 presents three examples of symmetric pairs; students should check the rest.)

FIG. 1.18

This knight's move square is associative

10	18	1	14	22
11	24	7	20	3
17	5	13	21	9
13	6	19	2	15
4	12	25	8	16

The knight's move square also possesses a new property not present in the order-5 squares that we have examined so far: It is a **pandiagonal magic square** (also called a **panmagic** or **diabolical** or **Nasik** square). In a pandiagonal magic square, all the "broken diagonals" also sum to the magic number. Figure 1.19 illustrates four examples of broken diagonals. Note that when the magic square is rolled into a cylinder, as above, the broken diagonals spiral around the cylinder. If we make

FIG. 1.19

Four examples of broken diagonals in a panmagic square

10	18	**1**	14	22
11	**24**	7	20	**3**
17	5	13	**21**	9
23	6	**19**	2	**15**
4	**12**	25	**8**	16

10	**18**	1	**14**	22
11	24	**7**	20	**3**
17	5	13	**21**	9
28	**6**	19	2	**15**
4	12	**25**	8	16

four copies of the square, as shown in figure 1.20, we see how the broken diagonals “straighten out” across the pattern.

FIG. 1.20

A “magic carpet”

10	18	1	14	22	10	18	1	14	22
11	24	7	20	3	11	24	7	20	3
17	5	13	21	9	17	5	13	21	9
23	6	19	2	15	23	6	19	2	15
4	12	25	8	16	4	12	25	8	16
10	18	1	14	22	10	18	1	14	22
11	24	7	20	3	11	24	7	20	3
17	5	13	21	9	17	5	13	21	9
23	6	19	2	15	23	6	19	2	15
4	12	25	8	16	4	12	25	8	16

A ***pandiagonal magic square*** *is one in which all the broken diagonals sum to the magic constant.*

- There are eight broken diagonals in this order-5 square. Identify them and verify that all eight sum to 65.
- Show that the order-5 square from figure 1.15 is *not* a pandiagonal square.
- Show that the order-4 square from figure 1.1 *is* a pandiagonal square.

These examples allow us to draw the following conclusions:

- A magic square can be associative but not panmagic (figure 1.15, for example).
- A magic square can be panmagic but not associative (figure 1.1, for example).
- A magic square can be both associative and panmagic (figure 1.18, for example). In fact, order-5 magic squares are the smallest that can possess both qualities.

But let’s return to the repeated pattern illustrated in figure 1.20. Tiling of the plane forms a sort of “magic carpet” from which we can observe one more interesting property of the 5×5 squares constructed using the knight’s move: You can outline *any* 5×5 square on the carpet, and it will be a panmagic square. What that also means is that any of the numbers from 1 through 25 can occupy the center cell of the magic square—although only the square with 13 in the center will be both associative and panmagic. Note that we have just found 24 new order-5 magic squares! The students should check some (or all 25) of the magic squares that can be outlined on the magic carpet to verify these assertions.

Extend

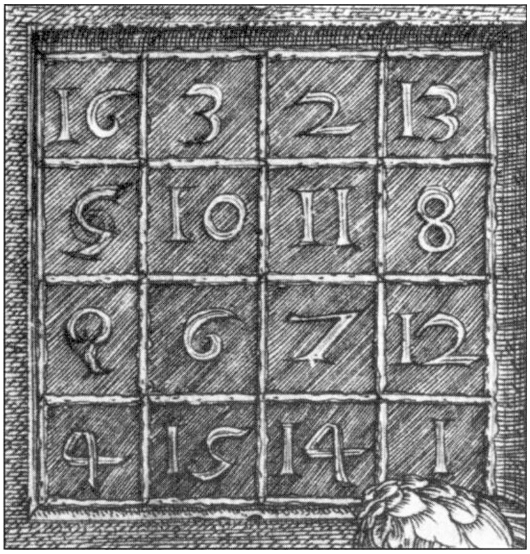
It has been said of pretzels and potato chips that you cannot eat just one. The same is true of magic squares once you have had a taste of their intriguing structure and beauty. Although your classroom time to spend on demystifying magic squares may be limited, many students can find great enjoyment and satisfaction in extending their inquiry into these mathematically appetizing patterns. Here are some suggestions for individual or small-group projects.

1. Construct larger odd-order magic squares, such as 7×7 or 9×9 or larger, using both the diagonal and the knight’s move methods. Examine those squares to see if the properties we identified for the 5×5 squares extend to the larger squares.
2. Other methods exist for constructing odd-order squares in addition to the two introduced in this activity. Research some of those strategies and examine the properties of the resultant squares.

3. Even-order squares are constructed using different strategies than those for odd-order cases. Research some even-order strategies and examine the properties of the squares they produce.
4. The German artist Albrecht Dürer's engraving *Melencholia* from the year 1514 includes a depiction of a magic square hanging on the wall. (Dürer was known to have signed his works with his initials and the date. Note that here the two center cells of the bottom row contain the numbers 1514, the year it was created, and the corner cells in the bottom row, 4 and 1, represent the fourth and first letters of the alphabet, *D* and *A*, his initials.) That 4×4 square, shown in figure 1.21, has been described as "super-magic" because of its many patterns. Not only do the rows, columns, and diagonals sum to 34, but there are numerous other patterns of four squares with the same magic sum. Figure 1.22 provides a glimpse of some of those internal patterns. There are more. A copy of the Dürer square is available in the appendix (p. 267) and at this book's more4U page for students who choose to delve into its hidden beauty.

FIG. 1.21

Albrecht Dürer's magic square

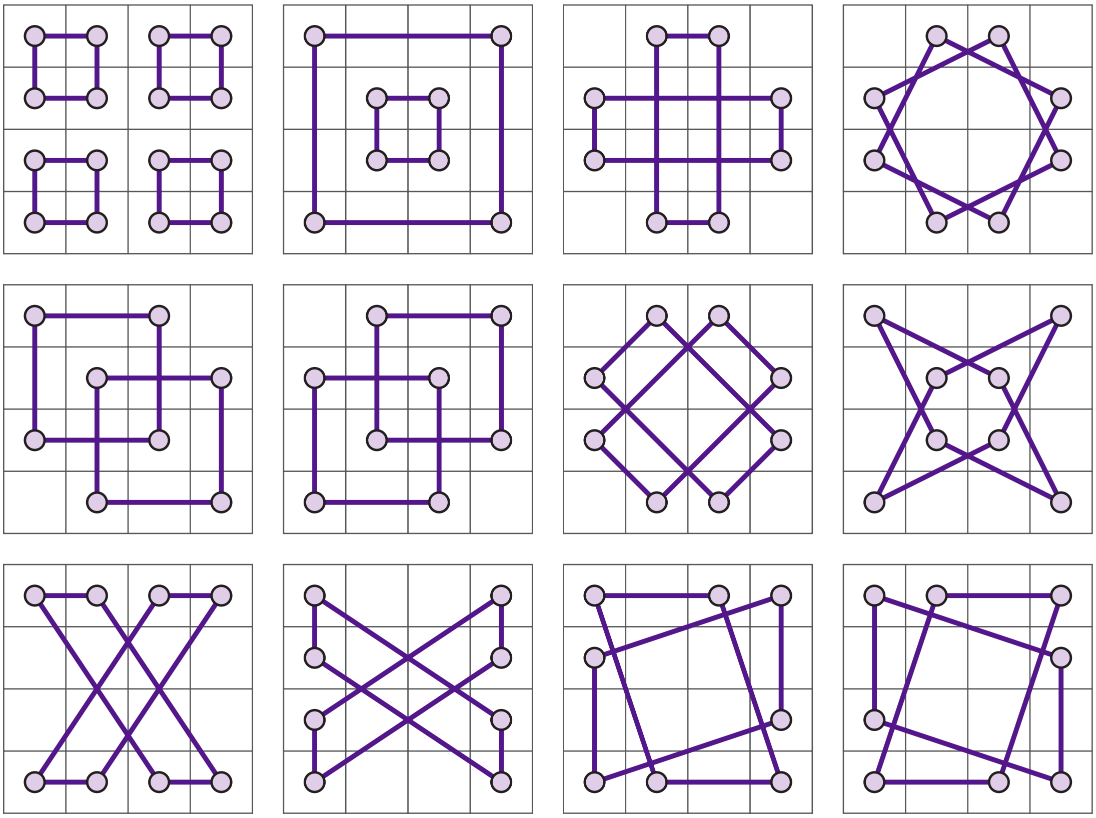


5. Benjamin Franklin did more than fly kites and study lightning in his free time. He also created some amazing magic squares, one of which appears in figure 1.23. This one, which he created as a young man, is also known as Franklin's "little" magic square. Franklin would later write in his autobiography:

> **I was at length tired with sitting there to hear debates, in which, as clerk, I could take no part, and which were often so unentertaining that I was induced to amuse myself with making magic squares, or circles, or anything to avoid weariness. (Pasles 2008, p. 74)**

FIG. 1.22

A sample of patterns in Dürer's magic square



Students can amuse themselves by examining the square in figure 1.23 (also in the appendix, p. 268, and at more4U) for its numerous patterns of eight squares that total 260, the magic sum, as well as patterns of four squares that total 130, half the magic sum, all contained within the given square. They are also encouraged to research Franklin's 16 × 16 magic square and describe all the patterns they can discover. Franklin himself once wrote in a letter to a friend that his 16 × 16 square was "the most magically magical of any magic square ever made by any magician" (Pasles 2008, p. 134).

FIG. 1.23

Benjamin Franklin's "little" magic square

52	61	4	13	20	29	36	45
14	3	62	51	46	35	30	19
53	60	5	12	21	28	37	44
11	6	59	54	43	38	27	22
55	58	7	10	23	26	39	42
9	8	57	56	41	40	25	24
50	63	2	15	18	31	34	47
16	1	64	49	48	33	32	17

6. There are also magic squares that contain within themselves subregions that are magic squares in their own right. Three examples are offered here (appendix pp. 269–271 and more4U). Students are encouraged to research other squares of

FIG. 1.24

More Than Meets the Eye—some hints

2	11	12	13	77	78	79	81	16
6	18	27	26	61	62	65	28	76
7	59	30	35	51	53	36	23	75
8	58	32	38	45	40	50	24	74
73	57	49	43	41	39	33	25	9
72	22	48	42	37	44	34	60	10
68	19	46	47	31	29	52	63	14
67	54	55	56	21	20	17	64	15
66	71	70	69	5	4	3	1	80

(a)

71	1	51	32	50	2	80	3	79
21	41	61	56	26	13	69	25	57
31	81	11	20	62	65	17	63	19
34	40	60	43	28	64	18	55	27
48	42	22	54	39	75	7	10	72
33	53	15	68	16	44	58	77	5
49	29	67	14	66	24	38	59	23
76	4	70	73	8	37	36	30	35
6	78	12	9	74	45	46	47	52

(b)

71	64	69	8	1	6	53	46	51
66	68	70	3	5	7	48	50	52
67	72	65	4	9	2	49	54	47
26	19	24	44	37	42	62	55	60
21	23	25	39	41	43	57	59	61
22	27	20	40	45	38	58	63	56
35	28	33	80	73	78	17	10	15
30	32	34	75	77	79	12	14	16
31	36	29	76	81	74	13	18	11

(c)

this type, or to attempt to create their own. Figure 1.24 gives hints as to the hidden features of these three sample squares with more magic than meets the eye.

Figure 1.24a is an example of a bordered magic square. What can you discover about its properties and magic numbers?

Figure 1.24b is an example of a cornered magic square. What can you discover about its properties and magic numbers?

Figure 1.24c is an example of a composite magic square, in this case a virtual "magic square of magic squares." Do you recognize any familiar patterns?

"Learners should have experiences that enable them to engage with challenging tasks that involve active meaning making and support meaningful learning." (NCTM 2014, p. 9)

SUMMARY

Magic squares offer an engaging context in which students can search for patterns; build number sense; reason algebraically; and describe, represent, and justify mathematical relationships that go well beyond what is apparent at first glance. One of the most attractive features is that engagement with magic squares is virtually unending, inviting students to continue to explore and enjoy well beyond the parameters of any classroom lesson or activity. We hope that many will accept the challenge and enjoy the experience.