into practice

Chapter 1
The Concept of a Function

Essential Understanding 1*a*
Functions are single-valued mappings from one set—the *domain* of the function—to another—its *range*.

Essential Understanding 1*b*
Functions apply to a wide range of situations.

Essential Understanding 1*c*
The domain and range of functions do not have to be numbers.

The first big idea about functions presented in *Developing Essential Understanding of Functions for Teaching Mathematics in Grades 9–12* (Cooney, Beckmann, and Lloyd 2010) is that a function is a single-valued mapping from one set—the *domain* of the function—to another—its *range* (Carlson et al. 2002; Monk 1992; Vinner and Dreyfus 1989). Functions are a prominent feature of secondary mathematics, as reflected in the Common Core State Standards for Mathematics (National Governors Association Center for Best Practices and Council of Chief State School Officers 2010), where the function strand begins in grade 8 and continues throughout high school. But functions have multidisciplinary utility apart from any content standards. Pause in your reading and turn to the next page to respond to the questions in Reflect 1.1, which ask you to consider the usefulness of the concept of a function.

Functions are essential in every field of applied mathematics. They are useful to the statistician who deploys a set of probability functions that tells whether a certain outlying observation is significant or expected. The climatologist has a single-valued mapping of a given year to the global mean temperature of that year. Government accountants have single-valued mappings of a given year

Reflect 1.1

In what mathematical fields, careers, or real-life events is the concept of a function useful?

Under what circumstances is it useful to define a relationship between two sets as a relationship?

to the amount of revenue that the government should expect to take in under current tax policy. In both of these last cases, if the climatologist's or the accountant's model for temperature or revenue returned *more than one* value for a given year, it would not be a function—and it would not be very useful.

Computer programmers rely on functional language to such an extent that an ascendant programming paradigm is labeled "Functional Programming." A graphics programmer might begin a function with the following line of code:

```
def drawASquare(topLeftXCoordinate,topLeftYCoordinate) {}
```

This example shows the utility of *multivariable* functions, functions where the domain includes *more than one* variable—in this case, both the x- and the y-coordinate of the top left corner of the square. After some difficulty, the programmer might realize that this function does not actually map to a single square for each pair of domain values. She would realize that the square must also be defined by some measure of *size*—a side length, for instance—not just position, and she could re-form the function with this line of code:

```
def drawASquare(topLeftXCoordinate, topLeftYCoordinate, squareSideLength) {}
```

In addition to the enormous value of functions in a variety of applied fields, they can simply be intellectually stimulating. When students see a set of domain values mapping to a set of range values, they may be inspired to wonder what kinds of functions could possibly account for that transformation. In the teaching of applied and pure mathematics, framing discussions related to domain in terms of destruction can engage students as they consider, for example, "What numbers would *break* this function?"

Introducing Functions Qualitatively: Car-Wash Scenario

Classroom discussions about functions and their applications will not be fruitful unless teachers can anticipate students' misconceptions, account for them, and take steps to eliminate them. This chapter will illustrate prevalent misconceptions related to the concept of a function and ways to help students overcome them.

To begin, meet Derek and Marta, two high school students who are organizing a car wash for a service club at their school.

> Derek and Marta are co-presidents of the Community Action Club at Grover Cleveland High School. Their club wraps up its school-year activities with a daylong car wash that raises funds for the next year's activities in the community. The car wash is growing in popularity and scope, with club members providing musical entertainment for drivers while their cars are cleaned.
>
> Derek and Marta are organizing this year's car wash. Their work involves locating a site, scheduling volunteers, advertising, and buying supplies—the soap and sponges without which the car wash would be nothing more than a line of dirty cars on a hot spring day. But club members encounter the same problem every year: They do not budget the right amount of money for supplies. Last year they ran out of soap in the middle of the car wash. They had to shut the car wash down and turn away many customers while Derek ran to the store to buy any soap he could find. The year before last, they bought *too much* soap, which is not as bad as buying too little, but still irritating. *They want a better method for predicting the amount of money they should spend on supplies every single year.*
>
> At a club meeting, they bring the question to their club adviser, Mr. Ramirez, who is a mathematics teacher at the school. Derek and Marta hope that he will just take the matter off their hands and make it his responsibility. Instead, he says, "You guys should use a function. I bet that would help."
>
> Derek and Marta trade confused looks, and Marta asks, "What's a function?"

Before reading further, consider what you would do at this moment in Mr. Ramirez's place. Use the questions in Reflect 1.2 to guide your thinking.

Reflect 1.2

How would you explain the concept of a function to Marta and Derek without resorting immediately to heavy abstractions or unfamiliar vocabulary?

What metaphor might be useful to you here?

In what follows, note that Mr. Ramirez answers Marta's question carefully, closely monitoring the responses from Marta and Derek to each piece of information before proceeding:

> "A function is a mathematical tool to help us manage how two values change together," Mr. Ramirez replies to Marta. "One of your values is changing—the amount you need to spend on supplies. We need to figure out another value that is changing with it and then find a function to describe how they change."
>
> Neither Derek nor Marta has a clear idea yet of the definition of a function, so Mr. Ramirez explains: "Let's look at *relationships* between two values. A function is just a special kind of relationship." He pins four letters at intervals across the back wall: A, B, C, and D. He turns to the members of the club. "Okay, if your ride to school takes ten minutes or less, stand under A." Derek, Jamie, and Angela go to stand under A. "If your ride to school takes *longer* than ten minutes, stand under D." Denitha and Marta stand under D.
>
> "Look at this," Mr. Ramirez says as he starts drawing on the board. "Our two sets of values are *your names* and *where you stand*." He writes all the students' names and then draws arrows to the letters under which they are standing (see fig. 1.1).

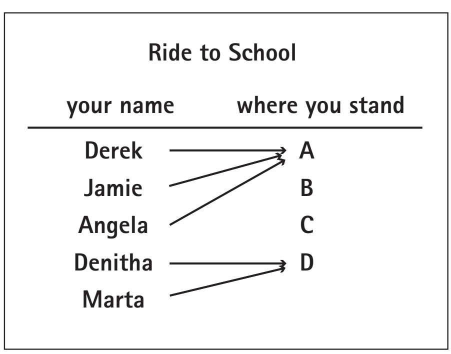

Fig. 1.1. An example of a function

Mr. Ramirez continues, "One thing you should notice here is that *none of you was confused*. Each of you had one place to go, and exactly one place to go. Let's look at another relationship. If you are wearing any shade of blue, stand under A."

Everyone but Denitha is wearing some kind of blue jeans, so all the club members start moving toward A. Mr. Ramirez then says, "If you are wearing white, stand under B. If you are wearing red, stand under C. If you are wearing black, stand under D."

Everyone freezes mid-stride. Marta says, "Mr. Ramirez, I'm wearing blue, white, and black. Where am I supposed to go?"

Mr. Ramirez says, "That is exactly my point. We have a *relationship*, but it is not a *function*." He starts drawing. "See, Marta, you're in the *domain* of the relationship: the input. But you don't know where to go in the *range* of the relationship: the output. You're confused. You have too many options. It's still a relationship, but it isn't a function. In a function, values in the domain go to exactly one value in the range." (Fig. 1.2 shows the situation created by Mr. Ramirez, with an input with more than one output.)

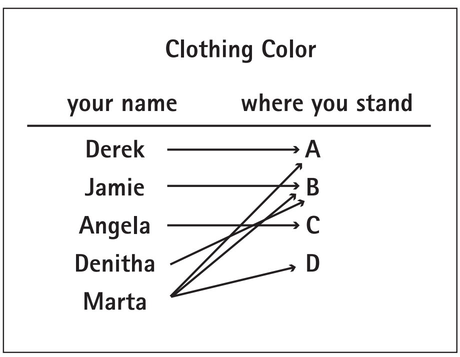

Fig. 1.2. An *input* with more than one *output*

"The rest of you have similar issues in this relationship, but even if you were each wearing exactly one color, it wouldn't matter. Marta's situation alone makes this a relationship, not a function." (Fig. 1.3 shows the relationship that Mr. Ramirez has set up—a relationship that is a not a function.)

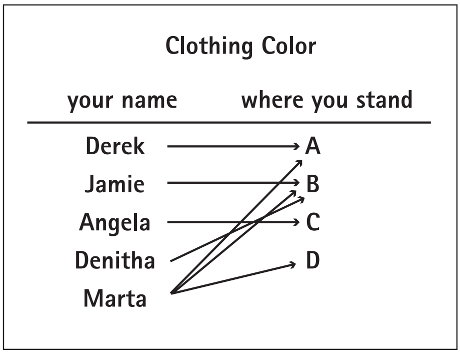

Fig. 1.3. A non-example of a function

"Let's try one more," Mr. Ramirez says. "You tell me if it's just a relationship or if it's a function. OK, here we go: If your birthday is in January, stand under A. If your birthday is in February, stand under B. If your birthday is in March, stand under C. If your birthday is in April, stand under D." (Fig. 1.4 maps the students' actions in response to Mr. Ramirez's instructions.)

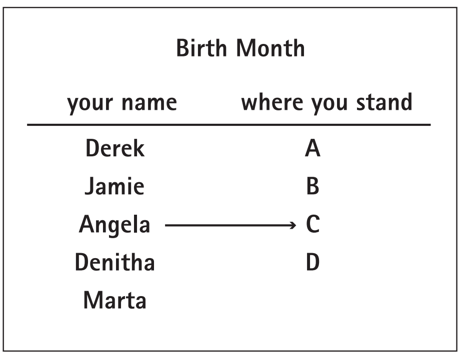

Fig. 1.4. Every *input* needs an *output*

Only Angela, whose birthday is March 15, goes to stand at the wall. Everyone else remains seated. "Function or not?" asks Mr. Ramirez.

"Well, no one is going to more than one location, like last time," says Marta. "But some of us have *no* place to go. That means it's not a function. In a function, everything in the domain goes to exactly one place in the range."

"Bingo," says Mr. Ramirez. "If you can understand functions, you will be able to figure out how many supplies to buy for your car wash."

> After pausing to let the students think about this, Mr. Ramirez poses a question: “So with our car wash, what are the different relationships we can examine, and which of them would be functions? Which function would be most relevant to our current challenge of buying supplies?”

Mr. Ramirez’s Function Wall activity illustrates the kinds of conversations and confusions that can arise when students try to comprehend the concept of a function. Notice that Mr. Ramirez first introduces the notion of a function in response to a concrete need that the students in his service club have. Derek and Marta have a problem—the over- and under-budgeting of money for supplies for the car wash—and Mr. Ramirez suggests using a function to aid in determining a solution. Not every student will have that kind of practical problem, however, so Mr. Ramirez positions his other students to experience a perturbation—a moment of cognitive conflict. The instructions that he gives his students for standing along the wall are sometimes easy to carry out (under the letter for the time for riding to school) and sometimes very difficult (under the letter for the color of clothes).

At that point, the students lack any kind of formal vocabulary to explain to themselves the difference between the easy and difficult situations, so Mr. Ramirez takes the opportunity to define important terms. In a difficult situation, he explains, “You are confused. You have too many options. It’s still a relationship, but it isn’t a function. In a function, values in the domain go to exactly one value in the range.” Use the questions in Reflect 1.3 to consider your own students’ thinking.

Reflect 1.3

What misconceptions do your students commonly have about functions?

What examples and counterexamples can you present to help them confront those misconceptions?

Common Misconceptions in Students' Thinking about Functions

In a study of 152 preservice teachers and their conceptions of functions, Ruhama Even (1993) hints at the importance of the type of qualitative introduction that Mr. Ramirez gives his students to functions. In her interviews, Even detected three trends in students’ thinking, the first of which was to consider that “functions are (or can always be represented as) equations or formulas” (p. 104). One study participant said, “A function is really an equation” (p. 105). Another said, “I think you could write all functions in terms of equations. It might be a trigonometric equation, like sin x, but in every term the y-value is going to be equal to some operation with x-value" (p. 105).

But functions can have many representations, of which equations are just one. And some functions, like the mapping of each student to a place identified by a letter along the wall, lack an equation or an algebraic representation altogether. Over the course of their secondary education, students see algebraic representations of functions more often than representations of any other type—so much so in fact that they risk conflating equations and functions completely. For that reason, it is productive to introduce functions non-algebraically, at first highlighting the single-valued mapping from domain to range in whatever form the function takes.

Three more misconceptions may arise as students develop their understanding of concepts related to functions:

1. They may think that the range must map back onto the domain with single values.
2. They may misunderstand how restrictions on the domain of a relationship affect the relationship.
3. They may rely on superficial or misleading indicators, such as the so-called vertical line test, to determine whether they are in fact dealing with a function.

It is useful to consider each of these sources of misunderstanding in turn.

Students may think that the range must map back onto the domain with single values. When the range of a function does in fact map back onto the domain with single values, the function is a special case called a "one-to-one" function. Just as a function is a special kind of relationship and therefore has characteristics and capabilities that distinguish it from a relationship (for one thing, a function allows single-valued predictions, whereas a relationship might return many values), so too does a one-to-one function have capabilities that another function that is not one-to one does not. (The inverse of a one-to-one function, for instance, is also a function, whereas the inverse of another function might not result in another single-valued function.) Later sections highlight this distinction, but for now, it is enough to say that it is important for students to understand which qualities of a relationship sufficiently describe a function and which qualities describe something more specific than a function.

The Function Wall activity is an attempt to confront this misconception. The full activity, which appears in Appendix 3, includes a task not used by Mr. Ramirez in the vignette: students who are shorter than five feet tall are to stand under A, students who are between five and six feet tall are to stand under B, students who are between six and seven feet tall are to stand under C, and students who are taller than seven feet are to stand under D. If you use this task with your students, they may laugh at this last request. They may also think that this relationship between students and their height is not a function, because no one is standing under D. (If someone *is* standing under D, you should of course create a section E for students who are taller than eight feet. You should also contact Rick Pitino, c/o the basketball program at the University of Louisville, at your earliest convenience.) Ask your students, "Do you feel confused about where you should stand? Do you feel any internal confusion at all?" If the students' response is no, say, "Then connect that clarity with the definition of a function. If all of you in the domain have exactly one place to stand, then you should call the relationship between all of you in the domain and the places to stand in the range a *function*."

Students may not understand how restrictions on the domain of a relationship affect a relationship, sometimes in effect turning a multi-valued mapping into one that is single-valued. Restricting the domain of a relationship to produce a function is an exercise of creativity in mathematics and an opportunity not to be missed. Restricting the domain of the relationship between students and the color of the clothes they are wearing to the two students in the class who are wearing *only one color* turns a relationship into a function. The capacity to do this may seem trivial at first, but it can be very important. Consider the graph of $y = \sqrt{x}$, shown in figure 1.5.

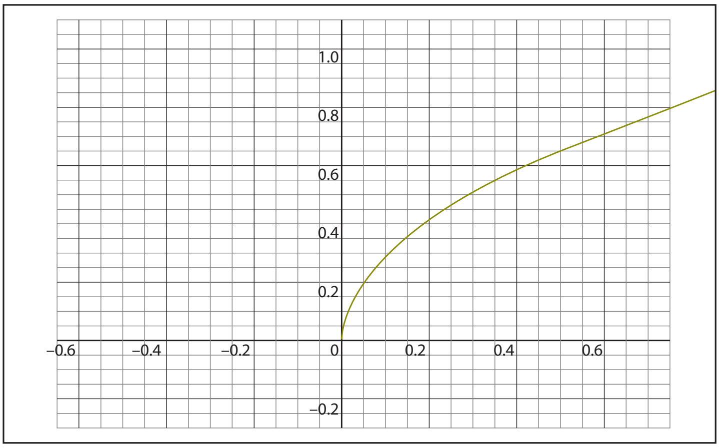

Fig. 1.5. The graph of $y = \sqrt{x}$

Does this graph show a function? A student who has a particularly rigid understanding of a function may say, "This isn't a function. Negative numbers don't have anywhere to go to in the range." In this case, you need to ensure that the student sees that *restricting* the domain creates a function: "If x is greater than or equal to zero, it *is* a function." Also take advantage of the opportunity to practice formal mathematical notation. Tell this student, "It's kind of hard to write out what you just told me. Mathematicians have a faster way to write what you said: $x \geq 0$."

Students can also *add items to the range of a relationship* to produce a function. Consider the case of the birthday month relationship in the Function Wall activity. Only January through March are in the range, and every other birth month lacks a mapping. Simply adding a fourth option—"Any other birthday"—creates a function. Anybody who has checked "Other" or "None of the above" on a survey understands that the purpose of that option is to eliminate confusion, and it does so by creating a function.

Students may rely on superficial or misleading indicators such as the so-called vertical line test. Reflect 1.4 poses questions about this method, which students frequently rely on to determine whether a relationship is a function.

Reflect 1.4

What are the limitations of the vertical line test?

Under what circumstances will it produce a false reading?

How does the vertical line test limit or enhance a student's understanding of functions?

Nearly one-third of Even's (1993) 152 prospective mathematics teachers explained functions to students by using the vertical line test. One participant described this test by saying, "[Students] can go through with a ruler or a straightedge and vertically go across the function, looking for places where there are two points" (p. 108). However, some of those same participants struggled to understand the limitations of that test. One participant looked at a circle and contradicted himself: "Like if I was going to have.... Well, uh, a circle is a function, but a circle doesn't pass the line test" (p. 110).

Rules and mnemonics like the vertical line test are only as effective as the conceptual knowledge that underpins them. In some cases, those mnemonics undermine conceptual knowledge. Consider the circle, for instance. A circle *isn't* a function with x mapping to y, because a single x-value in the domain maps to more than one value in the range. The vertical line test works here. But the same graph *does* represent a function if we shift our coordinate system from Cartesian to polar. With polar coordinates, we would create a circle by taking any angle in the domain and assigning it to the same value for the radius. Now we would have a function because every value in the domain (0°, 23°, 198°, 370°, to name just four of them) would map to only one value in the range (the radius, in every case).

If the shift to polar coordinates seems too technical, we can stay on the Cartesian plane. Consider the graph of $x = y^2$. This *is* a function, provided that we define the domain as y and the range as x. But the vertical line test makes no such distinctions: $x = y^2$ will fail the test without any consideration of domain and range. The writers of the draft progression on high school functions for the Common Core State Standards assert, "The vertical line test is problematical, since it makes it difficult to discuss questions such as 'is x a function of y' when presented with a graph of y against x (an important question for students thinking about inverse functions)" (Common Core Standards Writing Team 2012, p. 8).

Consider also the parametric scenario posed by Clement (2001), whose students struggled to answer the question shown in figure 1.6.

A caterpillar is crawling around on a piece of graph paper, as shown below. If we wished to determine the creature's location on the paper with respect to time, would this location be a function of time? Why or why not?

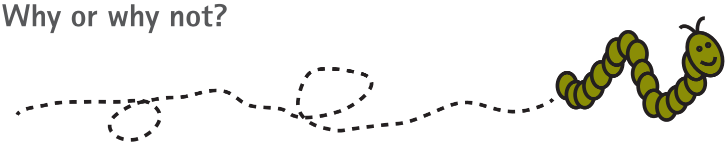

Fig. 1.6. Is a caterpillar's location a function of time? (Clement 2001, p. 745)

Three of five students that Clement interviewed applied the vertical line test directly to the caterpillar's path and decided that they were not looking at a function. They made that determination in spite of the fact that for any time in the domain the caterpillar can only have one position in the range. It cannot be in two places at once.

The vertical line test returns correct results for a limited set of functions. But the vast number of exceptions that must be taught along with the test ("Be careful with inverse functions, polar functions, parametric functions"—and so on) militate against teaching it at all. Instead, urge students to pay deliberate attention to the domain and range of a function. (The domain may not always be the horizontal axis.) Once students have located the domain, they can use their finger to map from an element of the domain to the corresponding element in the range. In some cases, this deliberate movement will rather closely resemble the vertical line test. In the case of inverse functions, though, it will look like a *horizontal* line test. And in the case of polar functions, it will appear to be a *radiating* line test. In each of these cases, students are incorporating the concepts of domain and range into a robust understanding of functions rather than integrating misleading indicators like the vertical line test into a limited understanding.

Developing a Robust Understanding of Domain and Range

A robust understanding of domain and range can be helpful in many jobs, but it is simply indispensable for anyone who designs surveys for the Web. Offer your students the screenshot shown in figure 1.7, and ask them to write a few sentences about what is wrong with it and how they would fix it. Their responses will demonstrate their understanding—or misunderstanding—of functions.

Fig. 1.7. A screenshot of a Web menu of possible survey responses

The designer of the Web menu could have profited at several points from an understanding of functions. Depending on the prompt that preceded this menu, a user might not know which option to select. If the prompt asked, "How long have you been enrolled in this university?" a student who had been enrolled for five months could truthfully select any of those options except "More than 4 years." In this case, we find that the domain (students) maps onto multiple values in the range (length of enrollment).

Furthermore, the domain almost certainly contains a student who simply has no option to select. This particular student does not map to *multiple values* in the range. She maps to no value in the range. Do you see her? She is a student who has been enrolled for 3 years and a few months. She does not have a place in this university's survey. The Web designer has misunderstood the concept of a function in a small but significant way, and the cumulative effect of that small misunderstanding might be an enormous headache months later.

Figure 1.8 shows the Function Finder activity. The activity poses several questions that may advance students' understanding of functions. As your students struggle to grasp the concepts, ask them to describe some hypothetical examples from the domain. They should choose enough examples to make a case that the relationship in question is or is not a function. You should also encourage them to restrict the domain if they think that would be useful.

Function Finder

Question: Which of these relationships are functions?

Goal: Understand the definition of a function.

Decide if each of the relationships below is a function. If it isn't a function, demonstrate how it fails the definition of a function by showing elements of each set that fail it. If it is a function, explain why, and also show several elements of each set.

Facebook user	password	student	hair color	students in our class	planet he/she lives on

state	letters in name	month	days in the month	days in the month	month

date	temperature outside	password	Facebook user	any integer	double that integer

Fig. 1.8. Function Finder activity

Some notes on each of the Function Finder relationships follow:

- **Facebook user → password.** Facebook does not have any mechanism for creating *multiple* passwords, and every Facebook user *must* have a password. Thus, this relationship is a function.

- **Student → hair color.** Whether or not this relationship is a function depends on your domain. If, in your class, nobody has colored or streaked hair, you have a function. If you expand your domain to include the entire school and even *one* student's hair has more than one color, you do not have a function. (You could *make* this a function, of course, by adding a "multicolor" option to the range.)

- **Students in our class → planet the student lives on.** This is certainly a function. Every student maps to exactly one planet. But you might take the opportunity to try to elicit and address a common misconception discussed earlier. Underneath "Earth," write the names of three other planets—perhaps Mars, Jupiter, and Venus. Say to your students, "No one maps to these planets. Can this relationship still be a function?"

- **State → letters in name.** Each state can have only one number of letters in its name (California → 10, for instance). Therefore, this relationship is a function.

- **Month → days in the month.** Some students will leap to the conclusion that this is a function, but others will take their time and recall that February will map to 28 days in roughly three out of every four years but 29 days in leap years. (Students can again exercise some creativity, this time by relabeling the domain "Months except for February" or "Months in years that are not leap years.")

- **Days in the month → month.** Take this opportunity to define the term *inverse*. This relationship inverts the previous relationship, and students should do the same with other relationships on the worksheet after they finish the first task. ("Is the inverse a function?" is a productive question.) The numbers 30 and 31 map to several different months, so this relationship is not a function.

- **Date → temperature outside.** Expect a productive disagreement about this relationship. On the one hand, the temperature may fluctuate wildly over a single day, perhaps from lower in the morning to higher in the afternoon. The relationship is not a function. But if students were to rename this domain "*mean* temperature outside" (which is often the case with historical weather data), they would have a function, since each day has only one mean temperature.

- **Password → Facebook user.** Again, students have an inverse function. In this case, the inverse is *not* a function, because Facebook does not check to make sure a password is unique.

- **Any integer → double that integer.** In considering this scenario, students are inching their way toward pure mathematical functions. Students should see that there is not any way to double 6, for example, and get any other number except 12. The relationship is a function.

Conclusion

This chapter has highlighted possible misconceptions that students may have about the concept of a function. They may hold misconceptions about the single-valued nature of the domain as well as about the multi-valued nature of the range. Students may incorrectly assume that the inverse of a function must also be a function. It is important to confront these misconceptions as early as possible since they will only cause more trouble as students begin to operate on functions, analyze them for covariation, and represent them in different ways.

We encourage you to replicate some of the conversations between Mr. Ramirez and his students. Those conversations are all adapted in one way or another from conversations that we have had with our own students. They demonstrate that it is one thing to define functions from the front of the class, but it is another thing entirely to give students an experience that calls the definition of a function into play and ties it to clarity or confusion in the students' own understanding.