Chapter 2

Algebra and Technology: Using Technology to Make Sense of Symbols and Graphs and to Reason about General Cases

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In algebra, technology affords unique opportunities for analyzing problems, implementing strategies, seeking and using connections, and reflecting on solutions. Technology can be used to rapidly produce accurate examples and linked representations, creating venues in which students can better understand by-hand work and learn new methods to solve fundamental problems. In this chapter, we chose examples using graphs and symbols to highlight two big ideas:

1. Different methods for solving equations take advantage of different representations and generalize to different classes of equations with varying degrees of accuracy.
2. Function families can be characterized by their symbolic forms, with different forms giving different information, some of which generalizes across families.

Coordinating and Generalizing Methods for Solving Equations

The generalized by-hand procedure for solving linear equations in one unknown is a staple of secondary school mathematics. Solving these equations—though often limited to work with equations of the form $ax + b = cx + d$—is standard achievement-test fare. Graphing tools and symbolic manipulators help students learn both how to make sense of the by-hand symbolic process through rapidly produced linked symbolic and graphical representations and how to use technology-based methods to transcend the limitations of the by-hand procedure.

Making sense of the by-hand process requires understanding of properties of real number operations and properties of equality as basic tools. Solving equations by hand requires properties of equality (e.g., adding 7 to both sides of the equation produces an equivalent equation) and properties of operations and number (e.g., "combining like terms" draws on properties of real number addition and multiplication). Example 2.1 illustrates how technology can be used to reconcile symbolic and graphical approaches to solving linear equations.

Example 2.1. Seeing Equivalent Equations

Task

An equation such as $6x + 3 = 12 + 3x$ is a statement about equivalence. One way to interpret the equation is as a claim about a value of the input value, x, for which $f1(x) = 6x + 3$ and $f2(x) = 12 + 3x$ have the same output value. That value of x is the solution of the equation.

a. Graph $f1(x)$ and $f2(x)$ in the same graphing window. These graphs constitute the first *graph pair*. How do these graphs illustrate the value of x for which $6x + 3 = 12 + 3x$?

b. Think about solving $6x + 3 = 12 + 3x$ by hand. What would be a first step? Perform that step with the equation. Then, produce a second graph pair by graphing $f3(x)$ and $f4(x)$, the functions that correspond to the new equation.

c. Continue producing and graphing new graph pairs that result from subsequent steps in the solution process.

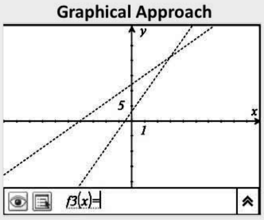
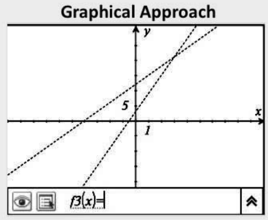
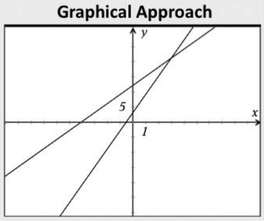
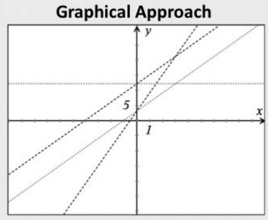
d. Look at the symbolic work, and label each step with either a property of equality (e.g., addition property of equality) or a property of number and operation (e.g., distributive property of multiplication over addition).

e. Solve the same equation using an alternate approach, such as using a different first step. Write the symbolic steps and create the corresponding graph pairs.

f. How do the graph pairs change when a property of equality is applied? How do the graph pairs change when a property of number and operation is applied?

In the classroom (first-year algebra)

As students work, groups produce graph pairs as in figure 2.1.

First Method	Second Method
Symbolic Manipulation $6x + 3 = 12 + 3x$ **Graphical Approach**  This is the first graph pair.	**Symbolic Manipulation** $6x + 3 = 12 + 3x$ **Graphical Approach** 
Symbolic Manipulation $6x + 3 = 12 + 3x$ $3(2x + 1) = 3(4 + x)$ **Graphical Approach**  Graph pairs stay the same.	**Symbolic Manipulation** $6x + 3 = 12 + 3x$ $3x + 3 = 12$ **Graphical Approach**  Slopes change.

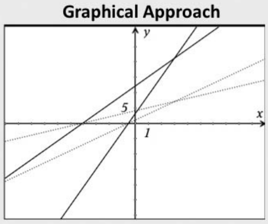
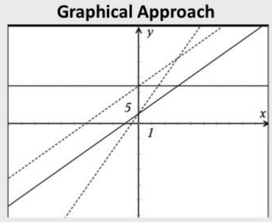
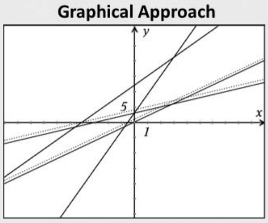
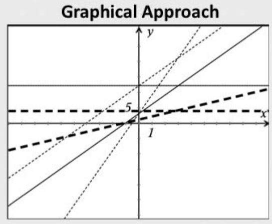
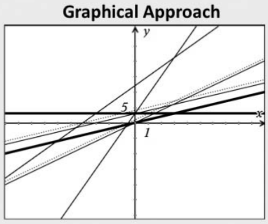
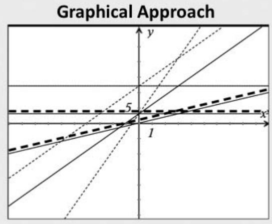
First Method	Second Method
Symbolic Manipulation $6x + 3 = 12 + 3x$ $3(2x + 1) = 3(4 + x)$ $2x + 1 = 4 + x$ **Graphical Approach**  Slope and y-intercepts change; x-intercepts stay the same.	**Symbolic Manipulation** $6x + 3 = 12 + 3x$ $3x + 3 = 12$ $3(x + 1) = 12$ **Graphical Approach**  Graph pairs stay the same.
Symbolic Manipulation $6x + 3 = 12 + 3x$ $3(2x + 1) = 3(4 + x)$ $2x + 1 = 4 + x$ $2x = 3 + x$ **Graphical Approach**  Graph pair shifts down.	**Symbolic Manipulation** $6x + 3 = 12 + 3x$ $3x + 3 = 12$ $3(x + 1) = 12$ $x + 1 = 4$ **Graphical Approach**  Slope and y-intercepts change; x-intercepts stay the same.
Symbolic Manipulation $6x + 3 = 12 + 3x$ $3(2x + 1) = 3(4 + x)$ $2x + 1 = 4 + x$ $2x = 3 + x$ $x = 3$ **Graphical Approach**  Slopes and y-intercepts change.	**Symbolic Manipulation** $6x + 3 = 12 + 3x$ $3x + 3 = 12$ $3(x + 1) = 12$ $x + 1 = 4$ $x = 3$ **Graphical Approach**  Graph pair shifts down.

Fig. 2.1. Graphs for two methods of solving $6x + 3 = 12 + 3x$

Students then label their symbolic steps (see fig. 2.2).

Symbolic Manipulation	Reason
$6x + 3 = 12 + 3x$	
$3(2x + 1) = 3(4 + x)$	Distributive property of multiplication over addition
$2x + 1 = 4 + x$	Multiplication property of equality
$2x = 3 + x$	Addition property of equality
$x = 3$	Addition property of equality

Fig. 2.2 Symbolic steps labeled

Student groups describe the differences in the graph pairs using phrases such as "but they don't change" for the properties of number and operations and "they slide and tilt" for the properties of equality. The teacher prompts groups to justify their observations with questions such as "Why should applying the properties of number and operations not change the graphs?" and "Why should applying the properties of equality change the graphs in the way that it does change them?"

In the whole-class discussion, students talk about how the properties of number and operations "change the way the expression looks but not its value." They describe the properties of equality in terms of "adding so many x's means the coefficient of x changes, so the slopes change, and we have a new graph pair."

The teacher capitalizes on the opportunity to engage students in a discussion about the difference between applying properties of number and operation and applying properties of equality. Students develop explanations of how each application of properties of number and operation provides ways to rewrite an expression to form an equivalent expression—an expression that has the same value for any given value of x in this case. They also observe how each application of a property of equality produces a graph pair that represents an equivalent equation—an equation with the same solution set as the equation to which the property was applied.

In the following dialogue, to emphasize the meaning of solution of an equation, the teacher focuses students on what does not vary.

Teacher:	We have talked about some of the similarities and differences in the graph pairs. What is the same about all the graph pairs?
Fredo:	Their intersection points line up.
Teacher:	What does that mean?
Henry:	The same x-value is a solution for all the equations.
Teacher:	Suppose you had a graph pair whose intersection points did not line up. What would that mean?
Gina:	It's impossible!
Henry:	We had a time when they didn't, but it was wrong.
Teacher:	Wrong?
Henry:	Yes, we made a mistake when we added negative one to both sides.
Gina:	Is that it? If the intersection points don't line up, does that mean there was a mistake?

The teacher later encourages students to make other observations and conjectures: "You noticed that adding a number on both sides shifts the graph pair up or down. How does adding a multiple of x affect the graph pair?" Not letting students stop after they merely report what they can see in the graph pairs, the teacher asks them to justify why these particular changes in the graph pairs reflect what happens with the symbols. Students connect symbolic moves back to what they know about slope and intercept, noting, for example, that adding a multiple of x corresponds to creating two functions with new slopes but the same y-intercepts. The teacher might follow this by asking how multiplying by a constant changes the graph pair and then inviting students to reason symbolically and graphically to justify their observations.

Table 2.1
Key Elements, Reasoning Habits, and Special Technology Notes and Issues in Example 2.1

Key Elements of Mathematics
Meaningful use of symbols
Mindful manipulation
Reasoned solving
Linking expressions and functions
Using multiple representations of functions
Analyzing the effects of parameters
Key Reasoning Habits
Analyzing a problem
seeking patterns and relationships
Implementing a strategy
making purposeful use of procedures
Seeking and using connections
working across different contexts and different representations
Reflecting on a solution
reconciling different approaches
generalizing a solution
Special Technology Notes and Issues in This Example
Labeling the graphs is helpful if screen space allows.

Students with experience in using graphs not only to solve linear equations but also to understand the role of equivalent equations, properties of equality, and properties of inequality in solving linear equations have a powerful tool to solve a variety of other equations—including equations that are difficult or impossible to solve by hand. The procedure might be described simply as "graph two functions based on the members of the equation and find where their graphs intersect." However, to solve equations successfully, students must reason about the functions involved. The video clip Algebra Solution, available at www.facebook.com/group.php?gid=265232006604&v=wall, illustrates a variation on this example that capitalizes on recently developed split-screen options and linked representations.

Example 2.2 illustrates both the power of technology-based methods and the need to reason about functions to make sense of technology results and solutions. Students enter this task knowing at least four ways to solve equations such as $x^3 = 7$ using a computer algebra system (CAS): graph with intersection (see fig. 2.3a), guess-and-test with numerical values (fig. 2.3b), direct solve (fig. 2.3c), and zooming with a table (fig. 2.3d). [Similar methods with a non-CAS calculator are possible, but the direct-solve method would be replaced with a numerical method, sometimes designated by "nSolve" and entered in this case as **nSolve(*x*^3=7,*x*)**.] These four methods involve four different ways to represent the functions: graphical, numeric, symbolic, and tabular. As students generalize the four technology-based methods to new kinds of equations, they see how reasoning about functions helps them overcome the limitations of the by-hand procedure and the limitations of their technology.

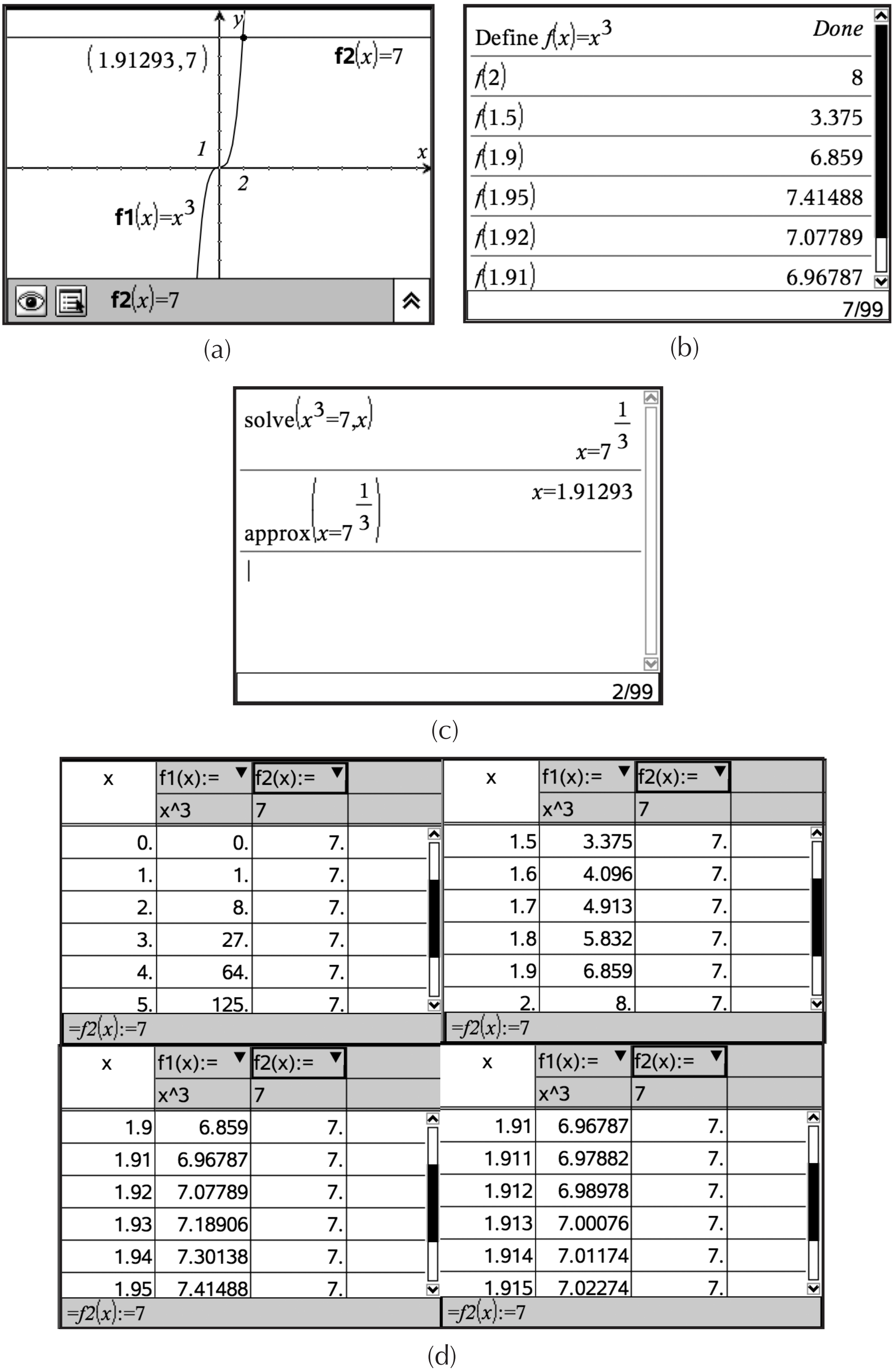


Fig. 2.3. Four CAS-based strategies for solving $x^3 = 7$ for x over the real numbers

Example 2.2. Beyond By-Hand Solutions

Task

Apply each of the four strategies—direct solve, graphical intersection, table zoom, and numerical approximation—to solve the following equation for x over the real numbers: $\ln x = 5 \sin x$.

In the classroom (second-year algebra, precalculus, trigonometry)

Students begin using the different methods and are drawn to the direct solve command (in approximation mode) for its efficiency and easy-to-read numerical results. A few students notice the message, “More solutions may exist,” but continue to produce graphs and coordinates of intersection points. Their screen images appear in figure 2.4.

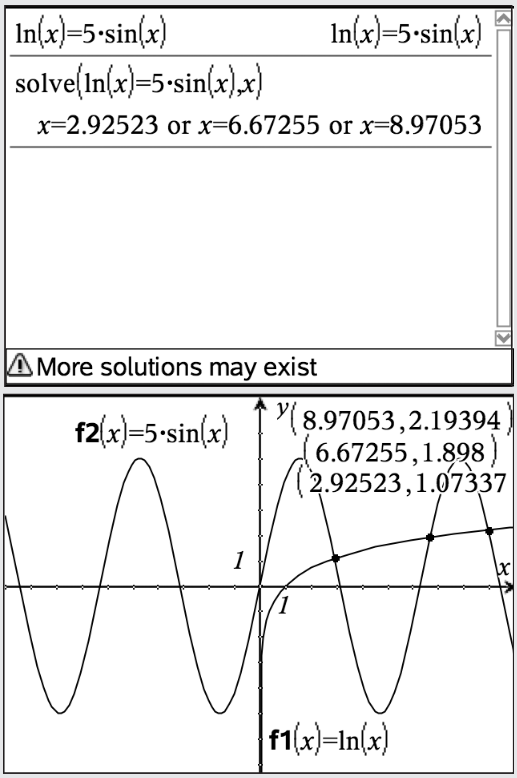


Fig. 2.4. Results for direct-solve and graphical methods

Teacher: How sure are you that you have found all the solutions?

Louisa: We have three.

Nate: But graphs continue to the right. Let’s see. [*scrolls over*] Oh! There are more (fig. 2.5).

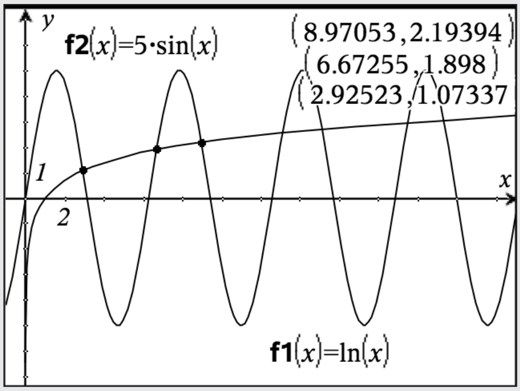


Fig. 2.5. More results for graphical method

Teacher: How many more are there?

Louisa: Infinitely many.

Teacher: How do you know?

Louisa: Sine functions and logarithmic functions continue forever.

Nate: But, wait, the logarithmic function increases.

Teacher: Does it eventually become higher than the other graph?

Louisa: I don't know. Let's see. [*starts scrolling*] Yes, it does go on forever (fig. 2.6).

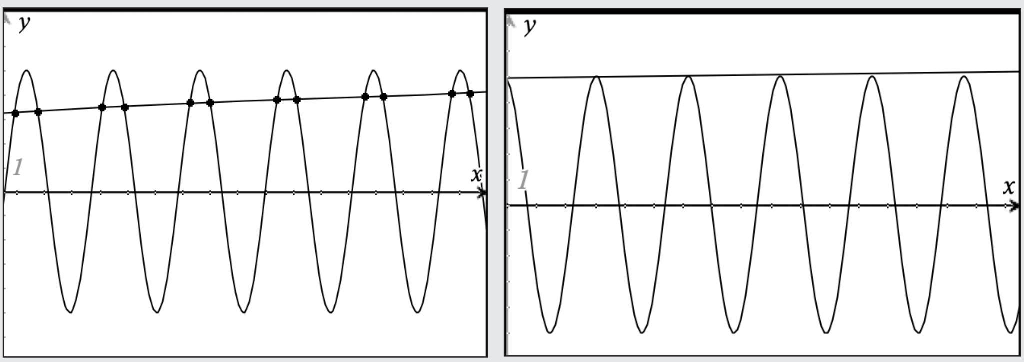


Fig. 2.6. Continuing to scroll

Dylan: No, wait, it can't go on forever. The value of 5 sin x is never greater than 5. The value of ln(x) can be more than 5. For example, look at this:

solve$(\ln(x)=1000,x)$	$x=e^{1000}$
approx$(x=e^{1000})$	$x=1.97007\text{E}434$

Jake: That's a *huge* number!

Louisa: So, there aren't infinitely many solutions.

Table 2.2
Key Elements, Reasoning Habits, and Special Technology Notes and Issues in Example 2.2

Key Elements of Mathematics
Mindful manipulation
Reasoned solving
Linking expressions and functions
Using multiple representations of functions

Key Reasoning Habits
Analyzing a problem
 seeking patterns and relationships
 looking for hidden structure
Implementing a strategy
 making purposeful use of procedures
 making logical deductions
Seeking and using connections
 working across different contexts and different representations
Reflecting on a solution
 considering the reasonableness of a solution
 revisiting initial assumptions
 generalizing a solution

Special Technology Notes and Issues in This Example
Radian mode should be used.

As suggested in example 2.1, the use of graphical and tabular methods allows students to solve a wider repertoire of equations than would be possible with only by-hand symbolic methods. Reasoning with technology requires not simply having different methods but orchestrating their use not only to produce solutions but also to explore the nature and number of solutions. In the case of the equation $\ln x = 5 \sin x$, students make strategic use of different methods. They also reason about the properties of the functions they graph to conclude that the number of solutions is finite. Tasks like this bring out a blend of strategic competence and conceptual reasoning. The teacher could have extended the reasoning opportunity by asking students to determine the exact number of solutions to the equation. Further challenges might ask students to determine the value(s) of A in $\ln x = A \sin x$ such that there are exactly ten solutions or whether there can be an odd number of solutions for an equation of the form $\ln x = A \sin x$.

Generalizing across Families of Functions

Within any one function family, each member is completely identifiable by its unique combination of parameters. For example, slope 7 and y-intercept –19.6 are enough to identify a unique linear function. Different symbolic forms for members within a family convey different information about the