Part 1

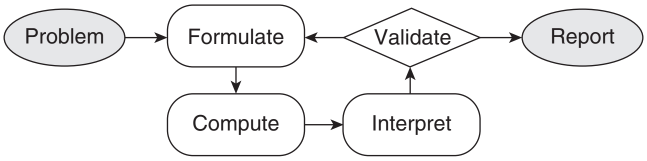
Modeling

Modeling

Conceptual Category Overview

Mathematical modeling involves using mathematics (and statistics) to better understand problems that arise in everyday life or the workplace. Modeling is at the very heart of mathematics, showing its importance and relevance as a human activity, thus providing motivation for the study of mathematics. Unfortunately, the word *model* is used in many different ways. For example, we may create a model of a mathematical idea or concept, such as using algebra tiles to represent a trinomial. This, however, is not *mathematical modeling* in the sense in which it is intended in this conceptual category. Here, we refer to a mathematical model as a representation of a real-world situation in order to draw conclusions about that situation using mathematical techniques. For example, a function may be built to represent the relationship between the number of widgets a factory builds and the resulting profit. This function can then be analyzed to determine what number of widgets will result in maximal profit.

Recall that Modeling is also listed as a Standard for Mathematical Practice that carries across all grades, K–12. Thus, students of all grades should be involved in using mathematics to better understand the world in which they live. Middle school students should be comfortable with creating an equation or algebraic rule, a geometric figure, or other mathematical representation of a real-world situation. However, Modeling is singled out as a conceptual category at the high school level, meaning it is also a curricular goal in and of itself. This implies that it should be more explicitly and intensively addressed. The modeling process may be more extensive, as in the *modeling cycle* presented in the Common Core State Standards for Mathematics, shown below. Note that the cycle begins with a real-world problem to be solved and ends with conclusions about the problem based on mathematics (and/or statistics). The intent is that students will be engaged in carrying out this cycle for themselves, rather than following a script provided by the teacher.



A brief overview of the six parts of the cycle follows:

(1) Problem: Identify a real-world situation that requires some sort of in-depth analysis to reach a conclusion, along with pertinent variables and assumptions made about the situation.
(2) Formulate: Formulate a mathematical representation of the problem, such as equations, functions, geometric figures, or statistical models.
(3) Compute: Use your model to draw conclusions.
(4) Interpret: Conclude what your solution means within the context of the problem.
(5) Validate: Consider whether the problem reasonably answers the question. If necessary, go back and revisit your variables and assumptions and make any other adjustments to your model that might be needed.
(6) Report: When the problem has been answered, report out the findings including justification for how the answer was derived.

Note that the cycle may need to be repeated more than once, and at any point in the cycle, backtracking to the previous stages of the cycle may be necessary.

Modeling activities may occur at different levels of intensity, ranging from a problem that can be addressed in a single class period to those that may take multiple days or even weeks. The following "Sample Unit Planning Page" outlines an entire unit focused on analyzing the mathematics behind "The Pit and the Pendulum," a story written by Edgar Allan Poe. Mathematical modeling should be a regular part of the high school mathematics classroom at some level, and all students should have the opportunity to engage in extended modeling activities at some point within a course. Students should be comfortable with mathematical modeling and what a mathematical model is.

The use of technology is an expected part of mathematical modeling, given that real-world data are often messy, resulting in models that may not be amenable to paper-and-pencil techniques. Use of spreadsheets, graphing utilities, and other tools assist students in devising appropriate models and data displays that may be useful in analyzing a situation.

Unlike other conceptual categories, Modeling does not have its own standards. Rather, this is a content emphasis to be incorporated across the curriculum. Standards within other conceptual categories that are marked with a star symbol (★) provide particular opportunities to engage with modeling. However, extended modeling activities typically address multiple standards, possibly across multiple domains or conceptual categories, requiring the students to make connections among multiple concepts.

The following chapters, which address the other conceptual categories for high school mathematics, include discussions of how modeling can be incorporated into each standard.

Sample PLANNING PAGE

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Modeling is best interpreted not as a collection of isolated topics but rather in relation to other standards. The following is an example of a potential unit exploration that integrates the modeling cycle found in the Common Core State Standards for Mathematics. *The Interactive Mathematics Program*'s unit "The Pit and the Pendulum" published by It's About Time, Inc. will be used to help illustrate the conceptual category of modeling in the CCSS-M. This unit, which involves content standards from the conceptual categories of Statistics and Probability and Functions, was chosen because of its ability to not only integrate modeling with mathematics but also because of its ability to integrate related standards in the Common Core.

Integrated Standards:

S.ID.A.4: Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve.

S.IC.B.5: Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant.

F.BF.B.3: Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.

Standards for Mathematical Practice:

SFMP 1. Make sense of problems, and persevere in solving them.

The investigation begins with students exploring factors such as the pendulum's length, the bob's weight, the angle of release, and other potential factors that may affect the period length of a pendulum. Students will need to create models to predict the period length of a 30-foot pendulum.

SFMP 2. Use quantitative reasoning.

The understanding of chance variability, standard deviation, the normal distribution, and probability is used to make decisions on factors influencing the period length of the pendulum. Later in the unit, students identify parameters and their graphical affects to determine the best model fit.

SFMP 3. Construct viable arguments, and critique the reasoning of others.

Students will make arguments for the consideration of certain factors affecting the period length of a pendulum. Additionally, students will justify the effects of different parameter shifts in functions through simulation and the use of trial and error.

SFMP 4. Model with mathematics.

Students will use graphical and analytical models to determine the best-fit equation to their collected data. Students will model and explore possible affecting factors to the change of period length with tools from the class.

Sample PLANNING PAGE (Continued)

SFMP 5. Use appropriate tools strategically.
Students will change the weight of the pendulum with washers, the length of the pendulum by cutting string, angle of release with a protractor, and other potential factors based on availability of supplies. Students will use software and other technological devices to explore parameter changes in specific functions and determine differences in distributions through simulation.

SFMP 6. Attend to precision.
Measurement from multiple students will be needed to reduce variability in the situation and highlight the idea of measurement variation. Students will attend to the precision of their solutions by comparing their predictions with the actual 30-foot pendulum.

SFMP 7. Look for and make use of structure.
Students will use the structure of the normal distribution or the ideas of randomization to justify the accuracy of the measurements and calculations.

SFMP 8. Look for and express regularity in repeated reasoning.
Students will connect shifts in parameters of a function to different graphical representations using repeated regularity in repeated reasoning. Students will express this through developing a model of best fit with their chosen function.

Goal:

Link classroom mathematics and statistics to everyday life, work, and decision-making. Choose and use appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions.

Planning:

Materials: NCTM Core Math Tools to analyze data; Desmos to analyze and explore parameters in functions; washers, string, protractors, meter sticks, and timers to complete pendulum investigation; "The Pit and the Pendulum*" short story (abridged), provided in It's About Time's supplementary materials.

Sample Modeling Unit:

The overarching unit problem centers on the situation described in Edgar Allan Poe's short story, "The Pit and the Pendulum." In this story, a pendulum with a blade on it slowly descends toward a prisoner tied up on a table directly below the pendulum. The prisoner tries to use the blade on the pendulum to cut his ties and escape. In order to decide whether this approach is really feasible, students need to understand factors that contribute to the period length of a pendulum and how it would affect the prisoner's ability to escape. To make conclusions about these two overarching ideas, students need to understand chance variability and use models for prediction. Most interestingly, this unit exemplifies the modeling cycle within the CCSS-M.

The investigation begins with the class reading an abridged version of the short story and formulating a general question to be answered, "Does the hero of the story have time to carry out his escape plan?" (Problem, Modeling Cycle Part 1). To better understand the problem, the students gather data about factors that may affect the period length of the pendulum, such as the pendulum's length, the weight of the bob, the angle of release, and other potential factors. Analytical modeling of the bob weight can be presented and used for justification in addition to data collection. The central overarching problem is a direct modeling link to a physical situation in which students must make conclusions based on mathematics and statistics.

* Fendel D., Resek, D., Alper, L, and Fraser, S.(2015). *Interactive Mathematics Program, Year 1*. "The Pit and the Pendulum." Mount Kisco, NY: It's About Time, Inc. https://mathimp.org/curriculum/samples.html

Sample PLANNING PAGE

Students begin the second part of the Modeling Cycle (Formulate) by identifying variables in the swing of a pendulum and exploring which variables have a significant impact on the period. The data shows that the length of the pendulum is the most significant factor affecting the period length of the pendulum. Students then create a model by creating and selecting graphical, tabular, algebraic, and statistical representations that describe the relationship between the length of the pendulum and the period length. Students explore possible models through transforming linear, quadratic, and square root parent functions to determine the best fit to their recorded data. These explorations can be further modeled through graphing utilities, spreadsheets, computer software, and applets.

Students draw conclusions as a result of their investigations at multiple times, the third step of the Modeling Cycle (Compute). Early in the unit, many questions arise because of measurement variability and natural variability. The understanding of chance variability, standard deviation, the normal distribution, and probability is used to make decisions on factors influencing the period length of the pendulum. Later in the unit, students identify parameters in certain functions and their relationships to transformations of their graphical representation.

The fourth portion of the Modeling Cycle (Interpret) requires students to interpret the results of mathematics in terms of the original situation through a descriptive model or function and evaluate the feasibility of the answer. Students re-explore the overarching problem and make predictions for a 30-foot pendulum. The use of inappropriate models causes students to extrapolate unreasonable predictions. Students must grapple with the mathematics previously discovered while simultaneously analyzing their models. Students complete the fifth portion of the Modeling Cycle (Validate) almost simultaneously by validating their predictions with an actual 30-foot pendulum. Students are asked to either improve their model or determine if it is acceptable.

The culmination of the unit is presented as a portfolio in which students report on their conclusions and the reasoning behind them, corresponding to the sixth and final phase of the Modeling Cycle (Report). Students present their best work throughout the unit and explain how they arrived at a solution for the problem. Specific attention in the portfolio is placed on the two overarching mathematical ideas of using chance variability and modeling functions. Students also share other thoughts related to content integration and where they need to improve in their portfolios.

Notes

Sample PLANNING PAGE (Continued)

Questions/Prompts:

(More elaboration on questioning and prompts to develop student learning for this unit can be found in the teacher's manual for the Pit and Pendulum Unit, produced by It's About Time).

Experimental Modeling:

- How did you measure the period of your pendulum?
- What might affect the period of a pendulum?
- Did you get different period for the same pendulum? Why?
- How certain are you that the variable affects the period?
- How might you test if a factor actually changes the period?

Modeling Parameters of Functions:

- How does the parent function relate to the new function?
- When you change the parameters of the model, how does this affect the graphed function?
- Why does the function shift in the opposite direction from translations in geometry?
- How can you verify that the function is moving as you say?
- How did you use your equation to predict for the 30-foot pendulum?
- What function did you find to fit the data? Why did you pick this function?
- Do you think the relationship is linear? How much of a difference would this make in predicting a 30-foot pendulum?
- How could you represent the shifts you have described in function notation?

Differentiating Instruction:

Struggling Students:

The initial experiment for this unit can be difficult for students. They often attempt to explore multiple factors simultaneously without setting a control for comparison. Allowing time for students to grapple with data that have come from multiple changes can help them understand the need for control groups and the need to change only one variable at a time.

Once students have started collecting useful data from experiments, teachers need to look for justifications based on centrality and spread. Students often describe differences purely on measures of center, and they need to be directed to also address the issue of spread.

Multiple standards can be addressed at different grade levels with student data of pendulum periods. Students will later need to re-experiment with the known factor that contributes to the pendulum's period by measuring the periods of different length pendulums. Ensure students have short and long pendulum lengths to help deal with appropriate model fitting. Students will be asked to extrapolate their models for a 30-foot pendulum, but the use of measurements closer to this length can help students foresee possible issues with the linear model.

Extensions:

There are multiple standards that may be addressed during this unit. Specifically, students may use simulation to determine if differences in parameters exist. Students may justify arguments based on centrality, spread, and the overlap of the data sets' distributions. Students having a deep understanding for the experimental nature of this unit opens up multiple avenues for statistical experiments and projects. Direct extensions also exist in the sciences related to potential and kinetic energy of a pendulum. Connections may also be made to vectors and the direction of a pendulum swinging. In addition to energy in science, velocity and momentum may be explored using vectors and the weight of the bobber.

Reflection Questions: Modeling

1. In what ways may you talk about mathematical modeling so that students differentiate it from creating a model of a mathematical idea?

2. What ideas do you have for incorporating modeling tasks of different time demands, including those taking several days, into a course that you are currently teaching? (Remember, the starred standards have connections to modeling.)

3. Extended modeling tasks can take significant classroom time. How might the sample unit plan in this chapter be considered time-effective by addressing multiple standards?

4. How might modeling incorporate both Content Standards and Standards for Mathematical Practice?