Perspectives on Modeling in School Mathematics

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The notion that concrete objects can be used to facilitate children's understanding of mathematical concepts and procedures is well established (Piaget 1962; National Council of Teachers of Mathematics [NCTM] 2001). Concrete objects or manipulatives (such as blocks, chips, base-ten cubes, geoboards, or algebra tiles) are used, particularly in kindergarten–grade 8, to *model mathematics.* With the rapid development of technology, virtual manipulatives (apps) have become widely used in K–12 to model mathematics and have been shown to have a positive impact on students' mathematics learning (Moyer, Niezgoda, and Stanley 2005; Bolyard and Moyer-Packenham 2012; Moyer-Packenham forthcoming; and chapters 3 and 4 in this volume). Although it might seem equivalent, *mathematical modeling* is different. Mathematical modeling can be described as "using mathematics or statistics to describe (i.e., model) a real world situation and deduce additional information about the situation by mathematical or statistical computation and analysis" (Common Core Standards Writing Team 2013, p. 5). *Mathematical modeling* and *modeling mathematics* are not the same, and so it is unfortunate that the same root word of "model" appears in two distinct constructs.

The distinction between models of mathematics and mathematical modeling is not always clear in U.S. standards documents and in the mathematics education literature. More specifically, the Common Core State Standards for Mathematics (CCSSM) uses the terms *model* and *modeling* to mean both *modeling mathematics* and *mathematical modeling* without clarifying the difference in meaning (National Governors Association Center for Best Practices and Council of Chief State School Officers [NGA Center and CCSSO] 2010). In many cases, CCSSM and the Progressions documents discuss models in terms of modeling mathematics—that is, using concrete representations such as rectangular arrays for multiplication. In other cases, such as in the high school conceptual category and the K–12 Standard for Mathematical Practice 4 (MP.4: Model with mathematics), the word *model* is used to refer to mathematical modeling, as described above, linking classroom mathematics to something from everyday life that is not inherently mathematical.

Modeling Mathematics

Simply stated, modeling mathematics refers to using representations of mathematics to communicate mathematical concepts or ideas. A key feature of modeling mathematics is that the process *begins in the mathematical world*, rather than the real world. According to van de Walle (2007), "A *model for a mathematical concept* refers to any object, picture, or drawing that represents the concept or onto which the relationship for that concept can be imposed" (p. 31). Examples of modeling a mathematical concept can be found in the CCSSM overview for first grade and is illustrated in figure 1.1:

> Students develop strategies for adding and subtracting whole numbers based on their prior work with small numbers. They use a variety of **models, including discrete objects and length-based models (e.g., cubes connected to form lengths), to model** add-to, take-from, put-together, take-apart, and compare situations to develop meaning for the operations of addition and subtraction, and to develop strategies to solve arithmetic problems with these operations. (NGA Center and CCSSO 2010, p. 13, bold added)

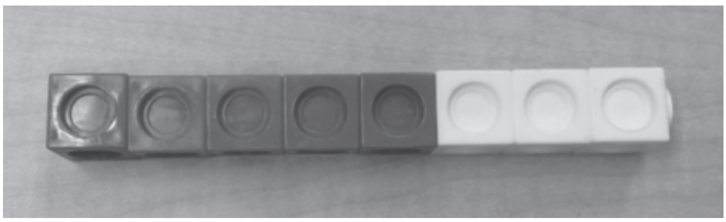

Fig. 1.1. A concrete model of 5 + 3 = 8 represented with manipulatives

In CCSSM, it is suggested that students use physical objects, such as cubes, to represent quantities and to model addition or subtraction operations using those objects. Other examples of modeling mathematics in CCSSM include adding and subtracting numbers up to 1,000 using concrete models or drawings (grade 2); using area models to represent the distributive property in mathematical reasoning (grade 3); and solving real-world problems involving multiplication of fractions and mixed numbers by, for example, using fraction models or equations to represent the problem (grade 5).

Models for mathematical concepts support students in exploring and communicating mathematical ideas (van de Walle 2007). Lesh, Post, and Behr (1987) describe five "representations" for concepts, two of which are manipulative models and pictures (see fig. 1.2 for a version of these representations from van de Walle [2007]). Models for concepts can be written symbols, oral language, and real-world situations. Today the representational set has been extended to include dynamic computer apps.

When Lesh and colleagues addressed contextualizing mathematics in "real-world" situations, they were referring to modeling the mathematics in a situation rather than mathematical modeling. Consider the following example:

> Show a 6th grader one-fourth of a real pizza, and then ask, "If I eat this much pizza, and then one-third of another pizza [of the same size], how much will I have eaten altogether?" (Lesh, Post, and Behr 1987, p. 37)

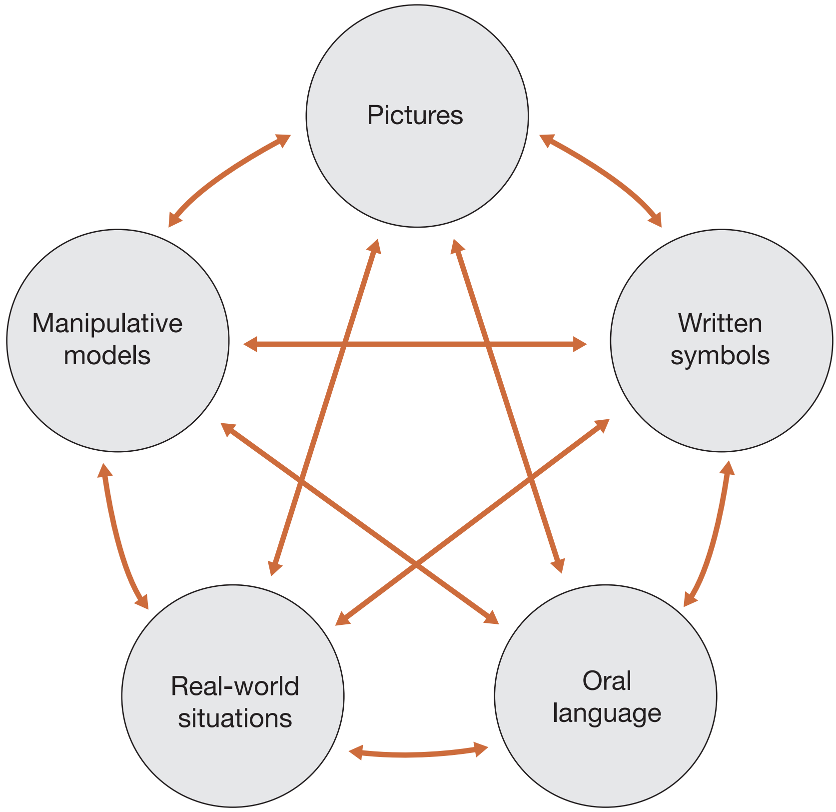

Fig. 1.2. Five representations of mathematical ideas (van de Walle 2007)

This question is essentially not about pizza and could have been about an orange, a cookie, or a cherry pie. The pizza is being used to represent a unit or whole. The problem is about number but contextualized with an object from the real world. That it is pizza is irrelevant; one need not know anything about pizza to solve the problem, and, as is always the case in modeling mathematics, the representation begins in the *mathematical* world rather than the *real* world.

Mathematical Modeling

Mathematical modeling, in contrast to modeling mathematics, links mathematics and authentic real-world questions. Mathematical modeling is essential for applied mathematicians and for professionals in disciplines as varied as biology, engineering, finance, computer science, and the social sciences. Mathematicians, and professional modelers in particular, must deal with a variety of real-world problems where the main task is to translate a problem into a mathematical form. This translation is the essence of mathematical modeling—namely, clarifying the problem, identifying variables, making approximations, and reporting out on the conclusions (Edwards and Hamson 2007). Applied mathematics uses mathematics to understand, evaluate, or predict something relative to the world *outside of mathematics* (Pollak 2003):

> What distinguishes [mathematical] modeling from other forms of applications of mathematics are (1) *explicit* attention at the beginning of the *process* of getting from the problem outside of mathematics to its mathematical formulation, and (2) an explicit reconciliation between the mathematics and the real-world situation at the end . . . the results have to be both mathematically correct and reasonable in the real-world context. (Pollak 2003, p. 649)

What makes a mathematical modeling task stand out from other "real-world" applications is the cyclic nature of getting the problem from outside of mathematics (i.e., the task was not inherently mathematical or mathematized for the student), mathematizing it, and then checking the model back against reality (see fig. 1.3). In other words, the basic reason to model with mathematics is to understand reality, or something about the real world (Common Core Standards Writing Team 2013).

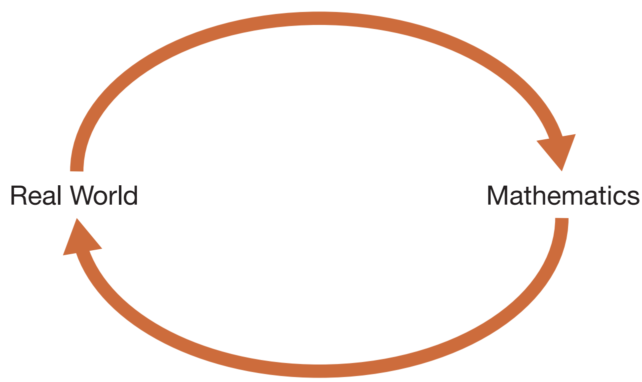

Fig. 1.3. The cycle of connecting the real world and mathematics

Henry Pollak, a mathematician and strong advocate of incorporating applications into the mathematics curriculum at all levels of education, has argued that all students must learn mathematical modeling in order to use mathematics in their daily lives, as citizens, and in the workforce (Pollak 2003 and the foreword to this volume). Blum and Borromeo Ferri (2009) had added that mathematical modeling can also support the learning of mathematics in terms of motivation, comprehension, and retention, and in terms of demonstrating what mathematics is and how it can be used. See chapters 8 and 9 in this volume for more on this topic.

Genesis of Mathematical Modeling in School Mathematics

According to Kaiser (2016), applications of mathematics and mathematical modeling already played an important role in school mathematics in the nineteenth century in Europe and North America. Felix Klein, German mathematician and mathematics educator, introduced more applications to school mathematics in Germany and other parts of Europe through the development of an innovative curriculum that integrated applications of mathematics into upper-level school mathematics instruction. This development was strongly influenced by growing technological enterprises, especially in engineering. Klein argued for a balance between applications and modeling and pure mathematics in mathematics instruction. This is important as the balance between pure and applied mathematics in the curriculum has never existed, according to Burkhardt (2006), because the dominant intellectual influences on school mathematics have come from pure mathematicians.

Kaiser (2016) argued that a major shift occurred as a result of the famous 1968 symposium, "Why to Teach Mathematics So as to Be Useful" (Freudenthal 1968; Pollak 1968). Why and how to include applications and mathematical modeling in mathematics education has been a focus of mathematics education research ever since. According to Kaiser, analyses of the modeling discussions from the beginning of the last century until the 1980s yielded two main perspectives—the pragmatic and the humanistic. The pragmatic perspective focuses on utilitarian goals, emphasizing the ability of learners to apply mathematics to solve practical problems (cf. Pollak 1968, 2003, 2012). The scientific-humanistic perspective emphasizes the ability of the learners to create relationships between mathematics and reality (Freudenthal 1968).

We can trace the evolution of modeling in the United States by examining the standards documents that have appeared over the last twenty-five years. NCTM's (1989) *Standards* document repeatedly called for more real-world applications of mathematics. For example, in the Mathematics as Problem Solving standards, the "real world" is mentioned in every grade band, as in this example from K–grade 4: "Students should have many experiences in creating problems from real-world activities, from organized data, and from equations" (p. 23). In fact, of the twenty unique mentions of the "real world" in the document, fourteen of those were related to mathematical modeling. Through a search of a PDF file of the document, the word "model" was uniquely mentioned seventy-two times. Over half of those mentions (n=38) pertained to mathematical modeling, while the remaining mentions related to modeling mathematics or were ambiguous. Mathematical modeling was primarily discussed in grades 9–12 in the Mathematics as Problem Solving, Mathematical Connections, and Trigonometry standards.

In *Principles and Standards for School Mathematics* (NCTM 2000), there were 112 unique mentions of the word "model," but more than half of them (n=65) referenced modeling mathematics. This is not surprising given that the Representation Standard, which called for creating and using representations to organize, record, and communicate mathematical ideas, was introduced. One representation standard seemed to be implicitly devoted to mathematical modeling: "Use representations to model and interpret physical, social, and mathematical phenomena" (NCTM 2000, p. 70). In this standard, the authors acknowledged that the word *model* had many different meanings:

> . . . model is used to refer to physical materials with which students work in school—manipulative models. . . . Yet another usage treats the term as if it were roughly synonymous with representation. The term mathematical model, which is the focus of this context, means a mathematical representation of the elements and relationships in an idealized version of a complex phenomenon. Mathematical models can be used to clarify and interpret the phenomenon and to solve problems. (NCTM 2000, p. 70)

Within these standards most references to mathematical modeling appeared in Algebra, while other references were included in Representation, Data Analysis and Probability, and Geometry.

Finally, the emphasis on mathematical modeling in CCSSM represents an evolutionary step from previous standards documents (e.g., NCTM 1989, 2000), where the focus on modeling was neither as explicit nor as detailed (Zbiek and Conner 2006). Through the use of stars (★) in the high school standards, CCSSM authors argued that modeling is best interpreted in relation to other standards. It is important to note that although SMP4 is intended to cut across K–12, there are no stars in the K–8 content standards to support teachers' identification of opportunities to teach mathematical modeling. However, in the High School Modeling Progressions document, the authors argued that SMP4, Model with Mathematics, "focuses on [mathematical] modeling and modeling draws on and develops all eight [Standards for Mathematical Practice]" (p. 8). This integration of SMPs considers modeling as a capstone, they argued, and helps explain why modeling with mathematics and statistics is so challenging.

Important Features of Mathematical Modeling

Developing definitions of particular terms in mathematics education is a noted challenge. For example, Kieran and Wagner (1989) reported that after a four-day conference on research in the teaching and learning of school algebra, no clear and succinct definition of algebra was ever agreed upon, even though attempts were made to come to a consensus. Similarly, there is no single agreed-upon definition of mathematical modeling; instead, there are definitions or descriptions put forth by individual authors or assumptions of shared understandings. Here is a sample of published descriptions of mathematical modeling:

- [The application of mathematics involves] the representation of our so-called "real world" in mathematical terms so that we may gain a more precise understanding of its significant properties, and which may hopefully allow for some form of prediction of future events. This has been described in the term "mathematical modeling." (McClone 1976, pp. 1–2)
- Overall, modeling is seen as a creative process in making sense of the real world to describe, control, or optimize aspects of a situation; interpret results; and make modifications to the model if it is not adequate for the situation. (Kaiser 2016)

Looking across these and other proffered descriptions of mathematical modeling (Pollak 2012), we identify commonalities that are themes across this volume's chapters. In particular, mathematical modeling authentically connects to the real world; it is used to explain phenomena in the real world and/or make predictions about future behavior of a system in the real world; it requires creativity and making choices, assumptions, and decisions; it is an iterative process; and there can be multiple approaches and answers. In the paragraphs that follow, we elaborate on these features of mathematical modeling.

Mathematical modeling ***authentically*** *connects to the real world, starting with ill-defined, often messy real-world problems with no unique correct answer.* The modeler's investigation begins with a question about a real-world phenomenon—these questions are typically messy, lacking definition, and contain uncertainties and multiple complicating factors. The questions are not typical "textbook" questions with a single, known-in-advance, correct approach and answer. In mathematical modeling, the modeler does research and brainstorms toward formulating and defining the problem. An early goal of this process is to articulate what the model will predict or explain about the real world (Bliss, Fowler, and Galluzo 2014).

Mathematical modeling is used to explain phenomena in the real world and/or make predictions about the future behavior of a system in the real world. Of equal importance with the fact that mathematical modeling starts in the real world is the reason why it starts at all. The process of mathematical modeling is intended to help the modeler understand or predict something about the real world and to develop theories and explanations that provide insight and understanding of the original real-world situation. Its unrivaled success as an explanatory and predictive tool is what makes mathematical modeling ubiquitous for scientists, engineers, mathematicians, social scientists, economists, and others across a variety of disciplines.

Mathematical modeling requires the modeler to be creative and make choices, assumptions, and decisions. In order to move from a complicated real-world question to a mathematical model that can be analyzed, mathematical modelers must make a variety of choices, assumptions, and decisions. They must choose what aspect of the situation to focus on, ignore the aspects that they assume are of secondary importance, and decide how to formulate the real-world situation mathematically. In other words, modelers need to decide what is important and how to piece it all together (Bliss, Fowler, and Galluzo 2014). These types of choices, assumptions, and decisions are not arbitrary but are guided by knowledge of both the real world and mathematics and must be made repeatedly throughout the process. This creative process is what makes engaging with mathematical modeling challenging, but also interesting and fun!

Mathematical modeling is an iterative process. The mathematical modeler is attempting to answer questions about the real world using mathematics. These questions are a priori not evidentially mathematical in nature. Through choices, assumptions, and decisions, modelers restrict their inquiry to that of a system that can be converted into mathematical terms. This implies that the modelers must return to the real world during their investigation and compare their mathematical insights and predictions with the actual real-world system. A mismatch between these insights and predictions and the behavior of the real world drives the modeler forward, leading to revised choices, assumptions, and decisions; further mathematical analysis; and additional comparisons. This back-and-forth activity between the real world and the mathematical world drives the iterative process of mathematical modeling. The modeling process therefore *begins* and *ends* in the real world (Edwards and Hamson 2007).

There are multiple paths open to the mathematical modeler and no one clear, unique approach or answer. The real world allows for many areas of investigation, and the investigation of the real world via mathematical modeling allows for many avenues along which those mathematical investigations may proceed. The choices, decisions, and assumptions made by the modeler, by necessity, lead to different, not necessarily equivalent, mathematical models of a given phenomena. Consequently, in a mathematical modeling investigation there are multiple paths to a solution. When different people look at the same modeling task, they can have diverse perspectives into the task's resolution. As a result, there can be several different, yet valid alternative solutions, which should be described as "a solution" rather than "the solution" (Bliss, Fowler, and Galluzo 2014). The ultimate arbiter of the validity and usefulness of a mathematical model is the real world. A mathematical model may be judged by the accuracy of its predictions, the power of its explanations, or the simplicity of its implementation.

Mathematical Models and Mathematical Modeling Cycles

Above we considered descriptions of the *mathematical modeling process*. Here we consider various cycles that represent that process and briefly discuss the products of this process: *mathematical models*. Among the more succinct and useful definitions of *mathematical model* is the one recently offered by Bliss, Fowler, and Galluzo (2014), writing on behalf of the Society for Industrial and Applied Mathematics (SIAM). Bliss and colleagues stated: "A mathematical model is a representation of a system or scenario that is used to gain qualitative and/or quantitative understanding of some real-world problem and to predict future behavior" (p. 3). More colloquially, Pollak, in the introduction to the *Mathematical Modeling Handbook* (Gould, Murray, and Sanfratello 2012), explored these ideas by first describing the wide range of areas where mathematical modeling and mathematical models are used and then stated:

> Whether the problem is huge or little, the process of "interaction" between the mathematics and the real world is the same: the real situation usually has so many facets that you can't take everything into account, so you decide which aspects are most important and keep those. At this point, you have an idealized version of the real-world situation, which you can then translate into mathematical terms. What do you have now? A *mathematical model* of the idealized question. You then apply your mathematical instincts and knowledge to the model, and gain interesting insights, examples, approximations, theorems, and algorithms. You translate all this back into the real-world situation, and you hope to have a theory for the idealized question. But, you have to check back: are the results practical, the answers reasonable, the consequences acceptable? If so, great! If not, take another look at the choices you made at the beginning, and try again. This entire process is what is called *mathematical modeling*. (Pollak 2012, p. viii)

According to Pollak, a major difference between mathematical modeling and problem solving is that problem solving either does not refer to the real world at all, or, if it does, it usually begins with the idealized real-world situation in mathematical terms and ends with a mathematical result. In contrast, modeling begins in the "unedited" world, and after engaging in problem formulation and problem solving, the modeler moves back into the real world where the results are considered against the original context.

Many mathematics educators have attempted to capture the essential components of the mathematical process through mathematical modeling cycles. Just as there is no one agreed-upon definition of mathematical modeling, there is no one agreed-upon modeling cycle. Rather, the cycles are attempts by their authors to capture the essence of a creative, dynamic process. This is one of the challenges of teaching and learning mathematical modeling: the lack of unanimity about the essence and the vision of the modeling process and the inherent complexity of the process itself (Perrenet and Zwaneveld 2012).

Just as the modeling process itself may vary, so also do diagrams representing the process. The reader is cautioned to remember that such cycles are incomplete representations of the process and should be taken as guides rather than as rules or procedures to follow linearly. A sample of various mathematical modeling cycle diagrams appears in figures 1.4–1.7. The particular modeling cycle shown in figure 1.6 is elaborated further in chapter 6 of this volume.

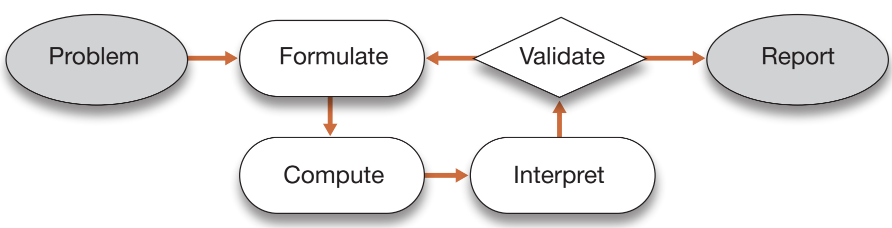

Fig. 1.4. The mathematical modeling cycle from CCSSM (NGA Center and CCSSO 2010)

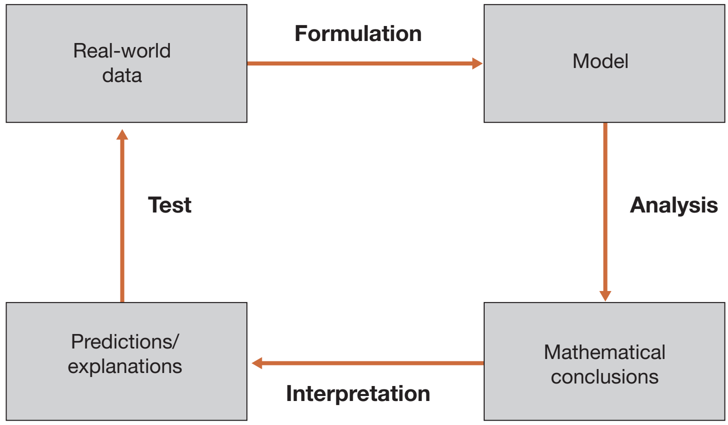

Fig. 1.5. The modeling process portrayed as a closed system (Dossey et al. 2002, p. 114)

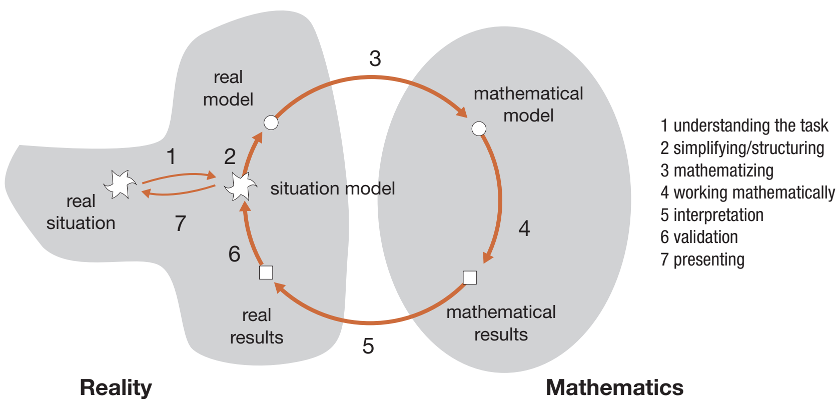

Fig. 1.6. The modeling process from Blum (2011, p. 18)

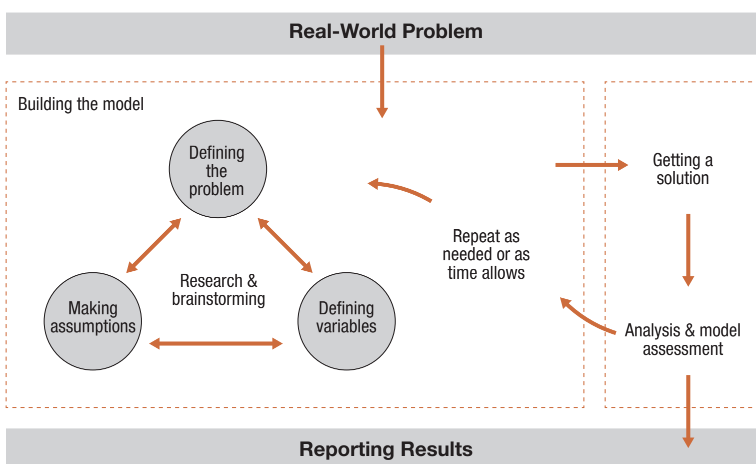

Fig. 1.7. Bliss and colleagues' (2014) overview of the modeling process (p. 6)

There are similarities and differences across various authors' modeling cycles. In addition, mathematical modeling cycles often have features that overlap with cycles used to describe processes in other STEM disciplines. For example, when we look at the engineering design cycle or the software development cycle, we see the same basic structure found in our various modeling cycles. In each case, the practitioner defines and attempts to understand the problem, develops something (a model, a prototype, a program), tests and refines that something, and iterates as necessary (see fig. 1.8). The two major differences between these various processes is the output of each process and the type of problems each process is intended to solve. In the case of engineering design, the output is a prototype or a physical design, and the problem is a design problem. In computer software development, the output is computer software or a program, and the problem is a design problem specific to computer science. In mathematical modeling, the output is a mathematical model, and the problem is one of gaining insight or predictive ability into a real-world phenomenon. Understanding these differences is an essential part of being able to successfully teach each of these important, yet different, approaches to problem solving. (See chapter 14 in this volume for more information on the engineering design process.)

Given the breadth and complexity of problems in the real world for which mathematical modeling is used, and given the multitude of models that may be constructed for a given situation, the neophyte mathematical modeler may rightly feel overwhelmed. Fortunately, various authors, including the authors of CCSSM (see NGA Center and CCSSO 2010, p. 73) and those in this volume, have provided classification and enumeration of commonly encountered types of mathematical models (e.g., descriptive, analytic, stochastic, and deterministic). The reader is especially referred to chapter 2 in this volume for a detailed discussion of the variety and types of common mathematical models.

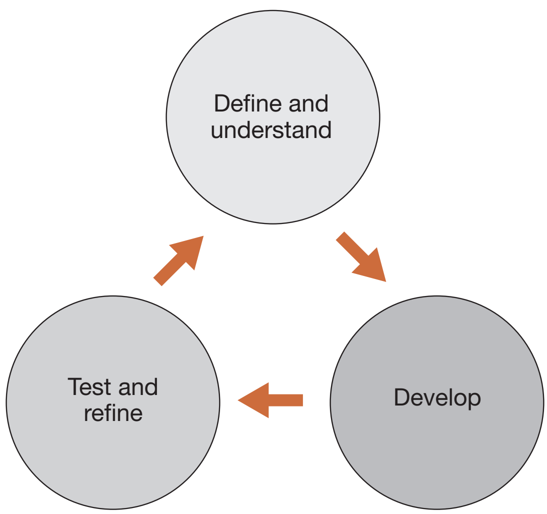

Fig. 1.8. Basic skeleton structure of STEM research and development cycles

Why Mathematical Modeling?

The broad and growing utility of mathematical modeling in an increasing variety of disciplines increases the importance of its inclusion in school mathematics. Perhaps even more compelling than the emphasis of CCSSM on mathematical modeling as evidence for its growing importance is the emphasis on mathematical modeling in the Next Generation Science Standards (NGSS Lead States 2013). Released in 2013, the NGSS serve as the blueprint for K–12 science education across the United States. Similar to the CCSSM's Standards for Mathematical Practice, the NGSS defines a set of eight Science and Engineering Practices (SEP). Of particular interest to readers of this volume are SEP2 (Developing and using models) and SEP5 (Using mathematics and computational thinking). While SEP2 is broader than CCSSM mathematical practice standard 4 (MP.4), encompassing both mathematical modeling and other relevant forms of modeling in science, the importance of mathematical modeling and mathematical models is emphasized in the description of SEP2. Similarly, the description of SEP5 will appear familiar to readers of CCSSM:

> In both science and engineering, mathematics and computation are fundamental tools for representing physical variables and their relationships. They are used for a range of tasks such as constructing simulations; statistically analyzing data; and recognizing, expressing, and applying quantitative relationships.

In NGSS Appendix L: *Connections to the Common Core State Standards for Mathematics* (NGSS Lead States 2013), one finds three CCSSM mathematical practice standards identified as integral to science:

- MP.2: Reason abstractly and quantitatively.
- MP.4: Model with mathematics.
- MP.5: Use appropriate tools strategically.

Again we see the notion of mathematical modeling (MP.4) as integral to the practice of science. It should be emphasized that the authors of the NGSS promoted genuine integration of mathematics into the science classroom. That is, they advocated for the design of tasks that do more than simply include science, engineering, and mathematics as elements within a single task; rather, they advocated for the development of truly integrated tasks that allow students to experience and understand the interplay of science, engineering, and mathematics as it is genuinely practiced. Such a call has implications for the development of mathematics curricula as much as it does for science curricula. The importance of mathematical modeling in the wider world and the clear recognition of the importance of mathematical modeling in K–12 outside of the mathematics classroom highlight the need for innovative curricula that truly integrate traditionally isolated subjects across the STEM disciplines.

Summary

This chapter identified five features of mathematical modeling, but there remains a need for further clarification. For example, these five features manifest differently. Must all five be present for a school mathematics task to be considered mathematical modeling? Which features are most central, which might be secondary, and why? Are features missing from this set? These questions point to other important directions of future inquiry. For instance, while some chapters in this volume (e.g., chapters 16 and 25) explore selected aspects of modeling, more work is needed to further unpack modeling sub-competencies. Can these sub-competencies be developed individually, or do they lose something when they are isolated?

Related to these lines of inquiry is the question of what constitutes mathematical modeling across K–12, particularly in the early grades. As with the broader body of research on mathematical modeling, only a handful of the chapters here focus specifically on the elementary grades. What does mathematical modeling with young children look like? Answers will, of course, depend largely on the definition of modeling used. For instance, what does it mean to "explain phenomena in the real world" in an elementary classroom? Does what counts as a meaningful or adequate explanation of a real-world phenomenon vary depending on the mathematical sophistication of the modeler? Would asking younger students to mathematize real-world situations without having them actually develop mathematical models support the development of important modeling skills that can be leveraged later in the upper grades?

More work is needed to decompose the practice of mathematical modeling so that it can be taught in authentic ways that simulate the work of professionals who actually engage in mathematical modeling. The field needs to better understand the kinds of knowledge necessary for teachers to develop "deep disciplinary understandings" (Ball 1993, p. 373) of mathematical modeling. Developing this knowledge can support the teaching of mathematical modeling so that it is "intellectually honest" (Bruner 1960) and represents, in some form, what mathematical modelers actually do.

Last, we note that as mathematical modeling becomes more prominent in K–12 mathematics, more attention needs to be paid to how mathematical modeling, knowledge of mathematical modeling, and the teaching and learning of mathematical modeling interact with the world. Mathematical modeling is the central point of intersection between mathematics and the natural and, increasingly, the social world. As such, the teaching and learning of mathematical modeling requires attention to issues of diversity and equity. Access to mathematical modeling education is a significant starting point in this direction. Beyond the issue of access, awareness can be raised regarding how mathematics can be used toward social progress (Borba and Skovsmose 1997). This will require cognizance of and deep conversations about the array of issues presented in this chapter and across this volume in terms of the mathematical and nonmathematical nature of the contexts being explored.

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