Introduction

WHAT IS PROPORTIONAL THINKING?

Proportional thinking is based on recognizing and forming multiplicative comparisons between quantities. It involves thinking of numbers in relative terms rather than absolute terms. For example, when comparing 4 to 10, thinking of 10 as $2\frac{1}{2}$ fours rather than as 6 more than 4 is proportional thinking. Similarly, deciding that a price increase from $2 to $4 (a $2 increase) is a more dramatic change than an increase from $90 to $100 (a $10 increase), because the first price was doubled and the second was not nearly doubled, is thinking proportionally.

Another way to make sense of proportional thinking is to think of it as unitizing. Proportional thinking involves viewing one measurement (or amount) as so many units of another. It might be thinking of a set of 10 fingers as 2 units of 5 fingers, or it might be thinking of 1 m as 100 units of 1 cm.

Proportional thinking often requires transferring the use of a multiplicative relationship from one pair of numbers to another pair of numbers. For example, if you know that 3 identical items cost $12 and want to know how many 6 will cost, you transfer the multiplicative relationship between 3 and 6 to a multiplicative relationship between 12 and some other number. Or you transfer the multiplicative relationship between 3 and 12 to the multiplicative relationship between 6 and another number. You are, in essence, determining equivalent ratios. In fact, in the National Council of Teachers of Mathematics (NCTM) resource *Developing Essential Understanding of Ratios, Proportions & Proportional Reasoning, Grades 6–8*, the big idea in proportional reasoning is described as a recognition that "when two quantities are related proportionally, the ratio of one quantity to the other is invariant as the numerical values of both quantities change by the same factor" (Lobato, Ellis, Charles, & Zbiek, 2010, p. 11).

When Do We Use It?

Proportional thinking, also often referred to as proportional reasoning, is used in everyday life in many ways. Just a few examples are listed on the next page:

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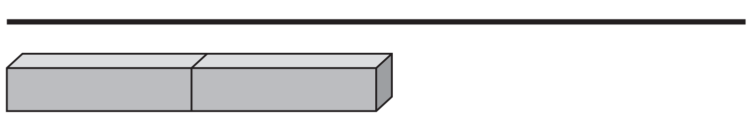
- *Exchanging coins:* Every time you exchange quarters for nickels, or nickels for quarters, you change the unit of measure and, therefore, the number of units required. Because the exchange is always 5 nickels for 1 quarter, determining the number of nickels for so many quarters or vice versa is a use of proportional reasoning.
- *Changing measurement units:* Every time you change feet to inches or yards to feet or inches to yards, you change the unit of measure and, therefore, the number of units required. Changing a measurement from one unit to another is a use of proportional reasoning.
- *Calculating a best buy:* Every time you try to decide if so many gallons at one price is a better buy than a different number of gallons at a different price, you use proportional reasoning.
- *Planning a trip:* Every time you decide how many hours it will take for a trip based on an average speed, you use proportional reasoning.
- *Maps:* Every time you use a map with a given scale ratio to determine an actual distance, you use proportional reasoning.
- *Cooking:* Every time you adjust a recipe based on the number of people you want to feed or you figure out how many $\frac{1}{3}$ cup measures it would take to measure $\frac{1}{2}$ cup, you use proportional reasoning.

Where Is It in the Math Curriculum?

Although proportional reasoning is not formally mentioned as a topic in the Common Core math curriculum until 6th grade, its roots appear much earlier. Because proportional reasoning involves thinking of one number as a multiple of another (e.g., thinking of 6 as 2 threes or as 3 twos), it is applied as students begin to work in 3rd grade with simple multiplication and division. But proportional thinking actually begins even earlier than that.

For example, when students in earlier grades think about why they get to 50 quickly when they skip count by 10, but slowly when they skip count by 2, or when they think about why the whole line shown below is probably 4 rods long based on how far 2 rods extend along that line, they are building proportional reasoning.



Some work in place value can also be thought of in terms of proportional thinking. For example, realizing that 300 ones must be 3 hundreds since 100 ones is 1 hundred is an example of proportional thinking.

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As will be illustrated throughout this resource, work in a wide range of areas involves proportional reasoning: work in probability, work in creating and interpreting graphs with scales, all work with fractions, work with multiplication and division, some work with patterns, some work with solving equations, and some work involving linear relationships.

ESSENTIAL UNDERSTANDINGS RELATED TO PROPORTIONAL THINKING

The list below outlines some important understandings underlying proportional reasoning and pre-proportional reasoning that are useful and that should be addressed throughout the grades:

- It is often useful to think of one amount as so many units of another amount, for example, 1 dollar as 4 quarters, 7 days as 1 week, 20 eggs as $1\frac{2}{3}$ dozen eggs, etc.
- If you use a bigger unit, you need fewer of them to express a quantity. For example, it takes only 10 tens to make 100, but it takes 20 fives. Or, it only takes 1 yard to measure 3 feet. Or, 1 is 5 fifths, but only 2 halves.

  Another way to rephrase this idea is the following: Any amount can be a small amount of a big unit or a big amount of a small unit. For example, 5 is half of a 10, but only a quarter of a 20.
- If units are related, you can use that relationship to predict how many of one unit you will have if you know how many there are of the other. For example, since 4 quarters make 1 dollar, you can predict that 12 quarters makes 3 dollars.
- Any two numbers can be compared multiplicatively, even if one is less than the other. For example, just as 6 is two 3s, 3 is half of a 6.
- How far apart two numbers are additively is unrelated to how far apart they are multiplicatively. For example, the pair of numbers 3 and 6 and the pair of numbers 100 and 200 have the same multiplicative relationship, even though the first numbers are only 3 apart and the second are 100 apart.
- Using a fraction, a decimal, or a percent is a form of multiplicative comparison. For example, the reason $\frac{2}{3} = \frac{4}{6} = \frac{6}{9}$ is because in each case the numerator is $\frac{2}{3}$ of the denominator, or the denominator is $\frac{3}{2}$ of the numerator. For example, the decimal 0.231 is a way to compare 231 to 1000. For example, the percent 42% is a way to compare 42 to 100.

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These ideas emerge and re-emerge through this resource, through the grades, in the sections on underlying ideas, as well as in the sections listing Good Questions to Ask.

FOCUSING ON THE CCSSM STANDARDS FOR MATHEMATICAL PRACTICE

The CCSSM Standards for Mathematical Practice derive from the processes of the National Council of Teachers of Mathematics (NCTM, 2000) and the strands of mathematical proficiency from *Adding It Up* (National Research Council, 2001). The standards for mathematical practice describe the mathematical environment in which it is intended that the Common Core State Standards for Mathematics are learned. These standards for mathematical practice are meant to influence the instructional stance that teachers take when presenting tasks to help students grasp the content standards. The standards for mathematical practice are addressed in this resource both in the underlying ideas presented for each topic and in the types of Good Questions suggested.

Listed below are just a few examples of attention to each standard for mathematical practice in this resource.

1. ***Make sense of problems and persevere in solving them.*** Students in Grade 3 are encouraged to begin to think of multiplication in terms of a change of unit to help make sense of certain problems (page 37). Students in Grade 7 need to make sense of a problem that asks them to figure out which dimension change has the most effect on the surface area of a prism (page 90).

2. ***Reason abstractly and quantitatively.*** Proportional thinking is all about reasoning, so there are many reasoning opportunities presented in this resource. For example, in Grade 2, students reason about the relationship between 60 – 40 and 6 – 4 when they are stimulated to recognize the fact that they can describe the first situation as essentially the second one with a simple change of unit (from tens to ones) (page 28). Grade 5 students reason abstractly and quantitatively as they compare the growth rates of two patterns (page 54) and as they consider the value of digits in a place value situation (page 55). Using double number lines helps Grade 6 students make sense of how percents of numbers other than 100 relate to those numbers (pages 72–73).

3. ***Construct viable arguments and critique the reasoning of others.*** Based on work with counters, students in Kindergarten discover that it is not possible to create two equal groups when working with certain numbers of counters (pages 11–12)

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and students in Grade 1 see that when halves of objects are bigger, so are the whole objects, or vice versa (page 22). Students in Grade 6 are asked to create an argument to predict why certain percent situations are not possible (page 74).

4. ***Model with mathematics.*** In Grade 1, students are modeling with mathematics when they measure half a distance and predict the whole distance (page 21). In Grade 7, students use mathematics to model probability situations (page 91). In Grade 8, students use dilations to create similar shapes (pages 98–99) and they use lines of good fit to model real-life situations (pages 102–103).

5. ***Use appropriate tools strategically.*** In Kindergarten, students use counters to explain principles (page 11). The 100-chart is a useful tool in Grade 1 to help students become familiar with multiples of 10 (page 18). Base-ten blocks are useful at many levels for many purposes, but one purpose is to help Grade 2 students see that counting larger subgroups is an efficient way to count (page 26). In Grade 4, using appropriate tools helps students understand various ways to compare two ratios or fractions (pages 46–47), and in Grade 7, 100-grids or double number lines help students calculate percents (page 81).

6. ***Attend to precision.*** Students in Grade 3 must attend to precision when they try to model a number as trains of another number, using Cuisenaire rods (pages 35–36). Students in Grade 7 consider precision when using strategies to get a sense of the size of π (page 87).

7. ***Look for and make use of structure.*** Students in Grade 1 start to recognize that when counting equal groups of objects, there are always two ways to count—either the number of groups or the number of items; the structure of every situation involving equal groups—multiplication—leads to this conclusion (pages 15–16). Students in Grade 3 might begin to notice that halving one number and doubling another leads to the same result because of what multiplication means (page 39). Students in Grade 7 begin to realize that if one variable is proportional to another, it is because the equation is of the form $y = mx$ and the graph is a line that goes through the origin (page 85).

8. ***Look for and express regularity in repeated reasoning.*** Students in Grade 2 might notice that there are two reasonable definitions for the notion of even number: either a number made up of a lot of twos or a number made up of two of the same whole number (page 23). Students in Grade 7 are expected to use tables of values involving proportional variables to see that 0 always matches 0, that 1 always matches r, where r is a unit rate of some sort, and that the y-values increase by r as the x-values increase by 1 (pages 83–85).

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FOCUSING ON THE NCTM PRINCIPLES TO ACTIONS

Recently, the National Council of Teachers of Mathematics released *Principles to Actions: Ensuring Mathematical Success for All* (NCTM, 2014), articulating a vision for the conditions, structures, and policies that are critical to move mathematics education forward. Included are eight teaching practices, some of which focus on the tasks that teachers set and others on the pedagogical approach of the teacher. This resource directly supports many of those suggested practices.

For example, beyond the standards themselves, the underlying ideas articulated in this resource establish mathematical goals to focus learning and frequently use and connect mathematical representations. The tasks suggested as Good Questions are purposeful and promote reasoning and problem solving and meaningful discourse. The underlying ideas often require complex and non-algorithmic thinking.

FOCUSING ON THE CCSSM STANDARDS FOR MATHEMATICAL CONTENT

By its very organizational structure, this resource focuses on the mathematical content standards related to proportional thinking in the Common Core State Standards for Mathematics.

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KINDERGARTEN

Counting and Comparing Equal Groups

Counting and Cardinality	CCSSM K.CC
Count to tell the number of objects.	
5. Count to answer "how many?" questions about as many as 20 things arranged in a line, a rectangular array, or a circle, or as many as 10 things in a scattered configuration; given a number from 1–20, count out that many objects.	
Compare numbers.	
6. Identify whether the number of objects in one group is greater than, less than, or equal to the number of objects in another group, e.g., by using matching and counting strategies.	

IMPORTANT UNDERLYING IDEAS

- ***Representing amounts as groups of other amounts.*** Much of the work we do with number in Kindergarten focuses students on counting amounts that are provided and representing requested amounts. Sometimes, instead of asking to see 4 counters or 10 counters, etc., we could ask to see two 2s or two 5s, encouraging students to think in units. Once the student has represented, for example, two 5s, we could then let him or her count the items to realize that 10 is what two 5s is.

- ***Comparing different numbers of groups of the same size or the same number of different-sized groups.*** As young students compare quantities, we might ask whether two 5s is more or less than three 5s, whether four 4s is more or less than three 4s, or whether four 2s is more or less than four 3s, without having them determine the actual totals each time, but rather encouraging them to continue to think in groups.

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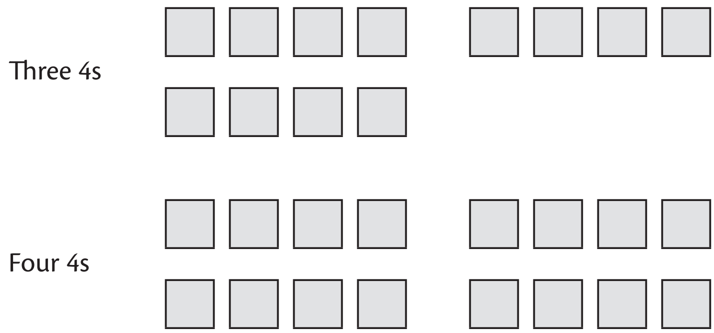
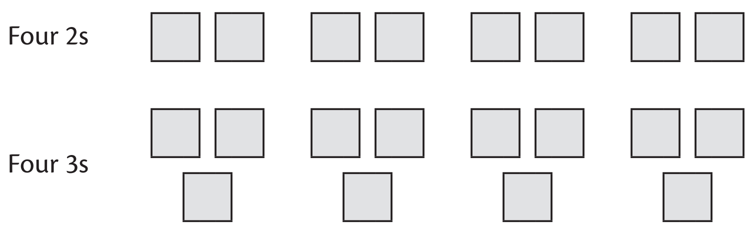
For example, three 4s is less than four 4s because there are extra items—the extra group of 4 after the three 4s have been matched with the first three 4s among the four 4s.



It is stronger reasoning for students to realize that four groups of 4 is more than three groups of 4 just because there are more groups of the same amount, rather than needing to depend on figuring out that 16 is more than 12. Later, to decide if 16 is more than 12, students might use the grouping idea to work backward. This is an example of the mathematical practice standard of reasoning abstractly and quantitatively.

Similarly, we want students to realize that four 2s is less than four 3s because there is an extra bit in each group. The actual numbers should not be the focus, but, rather, the reasoning about the extra in each group. Essentially we are matching two items among the three in each of the four groups of three to the two items in each of the four groups of two.



Later, when students are comparing 8 to 12, one of the rationales for deciding that 12 is more than 8 could be that 12 is four 3s whereas 8 is only four 2s.

Comparing groups in this way ties into the notion that the same number of a bigger unit (vs. a smaller unit) is a bigger measure. That is why 2 nickels is less than 2 dimes or 2 short sticks make a length shorter than 2 long sticks.

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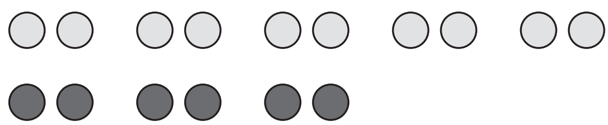
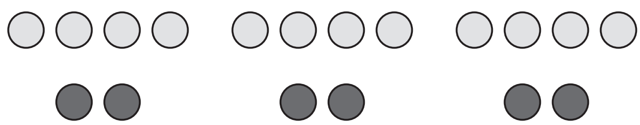
Good Questions to Ask

- ***Ask:*** Show 5 counters. Next, show two 5s, then three 5s. [Encourage students to show two 5s by grouping 5 counters twice rather than counting out 10 individual items.]
- Tell students that Susie is holding 2 books in each of her hands and Liam is holding 3 books in each hand. ***Ask:*** Who is holding more books? Do you have to count to be sure? [We want students to realize that they could count, but they do not have to—each group is bigger, so the total is more.]
- ***Ask:*** What different things does this picture show about one amount being more than another amount? [It could be that five groups of 2 is more than three groups of 2 or that 10 is more than 6.]



- ***Ask:*** What different things does this picture show about one amount being more than another amount?



Decomposing Numbers into Equal Groups

Operations and Algebraic Thinking	CCSSM K.OA
Understand addition as putting together and adding to, and understand subtraction as taking apart and taking from.	
3. Decompose numbers less than or equal to 10 into pairs in more than one way, e.g., by using objects or drawings, and record each decomposition by a drawing or equation (e.g., 5 = 2 + 3 and 5 = 4 + 1).	

IMPORTANT UNDERLYING IDEAS

➢ ***Creating equal groups.*** As students decompose numbers 10 and under, make sure to draw attention to situations where the decomposition leads to equal groups. For example, 10 = 5 + 5, 8 = 4 + 4, etc. Ensure that students realize that only certain numbers can be shown this way when using counters, that is, 2, 4, 6, 8, 10, . . . , but not 1, 3, 5, 7, ...

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Students might also be led to notice that when each of the equal-sized groups is increased by 1, the total is increased by 2, not 1.

Good Questions to Ask

- ***Ask:*** I showed a number as two equal groups. Tell me some numbers I might have been showing. Tell me a number I was not showing.
- ***Ask:*** Draw a picture or use your counters to show that 8 is two 4s.
- ***Ask:*** Show two equal groups with your counters. Write an equation to show what you did. Now add 1 to each of your groups. What equation do you write now? How did the numbers change from the first equation?

Comparing Measurements Relatively

Measurement and Data	CCSSM K.MD
Describe and compare measurable attributes.	
2. Directly compare two objects with a measurable attribute in common, to see which object has "more of"/"less of" the attribute, and describe the difference. For example, directly compare the heights of two children and describe one child as taller/shorter.	

IMPORTANT UNDERLYING IDEAS

➢ ***Relating lengths to each other in relative terms.*** Although students at this level do not yet use units to describe or compare measurements, they could be encouraged, when comparing two lengths that are "foldable," such as strings, to see if the longer length is more than two of or four of the shorter length.

For example, the longer line at the left below is less than two of the shorter one because, when it is folded on itself, it is shorter than the short line.

But the longer line at the left below is more than two of the shorter one because, when it is folded on itself, it extends beyond the short line.

This type of comparison helps students look at relative differences.

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We can help students think more relatively than absolutely by showing that the absolute amount longer may not be relevant. For example, Line 2 below is not much longer than Line 1, but it is still more than two of Line 1. Line 4 is a lot longer than Line 3, but it is less than two of Line 3.

Good Questions to Ask

- Provide two pieces of yarn of different colors, where one piece is more than double the length of the other. ***Ask:*** Which color string do you think is the longer one? Do you think it is more than two of the other color?
- Provide two pieces of yarn of different colors, where one piece is a little more than four times as long as the other. ***Ask:*** Fold the long string in half (so the ends meet) and then do that again. Is the folded string longer or shorter than the short string you have? How many of the short string do you think would fit into the long one? Why?
- Provide a long string, some yarn, and scissors. ***Ask:*** Cut a short string that is close to half the length of the long string.

Summary

Although work in proportional thinking is still very informal at this level, there is an opportunity for teachers to help children think in terms of one amount being units or groups of another, a concept that underlies proportional thinking. Approaches can include modeling and recognizing equal groups, comparing amounts by comparing the equal numbers of groups or the equal-sized groups that make them up, and thinking of one length relative to another length.

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