CHAPTER

1

Different aspects of size

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From early on, children are interested in questions of size: Who is the tallest child in the class? Who gets the biggest piece of cake? Straightforward though these questions may sound, it is not always clear what we mean when we ask, "How big is this object?" or "Which of these two objects is bigger?" In fact, there are many ways to look at size and size comparisons. Is this object longer than another? Taller? Wider? Does it cover more area? Does it contain more volume? We are sometimes surprised to discover that, looked at one way, this object is bigger, but looked at another way, it's smaller.

In this first chapter of *Measuring Space in One, Two, and Three Dimensions*, we watch children investigate different aspects of size. Some explorations start with a question about measuring: How would you measure a puddle? How big is your foot? If we wanted to know how big this part of the rug is, how would we figure it out? Other explorations grow out of the challenge to compare objects, for example, by ordering a set of rectangles or containers or by comparing the size of two boxes.

Through the children's explorations, we can examine these aspects of size ourselves and begin considering exactly what children must learn about them. What do the children in these cases bring to their inquiries? What do they discover? And what have they yet to learn? As you read this chapter, identify one place in each case where a child has worked through, or is in the process of working through, an issue about measurement. What is that issue?

case 1

Many methods for measurement

Mary

GRADES 1 AND 2, MAY

I teach a first- and second-grade combination class. Over the past year, we have studied measurement as it integrates with social studies or science. Recently I ran across a good measurement problem to be used for assessment. I decided to try it to determine how many different ideas my students have about possible ways to measure. This was the question: How would you measure a puddle?[1]

Teacher: First of all, what is a puddle?

Lakeah: It's kind of like mud.

Kristina: It's like a little lake on the ground.

Noriko: It's like when it rains, but not enough to flood.

Josh: A source of leftover rain.

Teacher: I want you to look at the things on this table [*a sponge, a cup, a ruler, a tape measure, yarn, cubes, and a spoon*]. Do they give you any ideas about how you might go about measuring a puddle?

Jenny: Do you mean how deep it is, or how far it is around?

Teacher: You could think about measuring the puddle in both of those ways. Is there anyone who already has an idea about how to measure a puddle?

Kristina: Cubes. I would lay them down in a row across the puddle and count them from tip to tip.

After the children shared a few more ideas orally, I wanted them to put their ideas on paper. I told them they should use words, pictures, or numbers to explain their thinking. I also reminded them to organize their work.

When the children were ready, we came together to discuss their ideas. We sat in a circle with the sponge, cup, tape measure, and other measuring tools placed in the middle. Sue, Jonathan, and Jenny said they would measure around the puddle.

Sue: I would put inch sticks around the puddle.

Jenny: I would use a tape measure because it bends.

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[1] From J. Westley, *Puddle Questions: Assessing Mathematical Thinking* (Mountain View, Calif.: Creative Publications, 1994).

Jonathan presented his drawing (fig. 1.1). He told us he would measure heel-to-toe around the puddle. Then he would compare the length of his foot to a “real” foot (as designated by a ruler) and find out how big the puddle was. He said he would probably use a tape measure to figure this out.

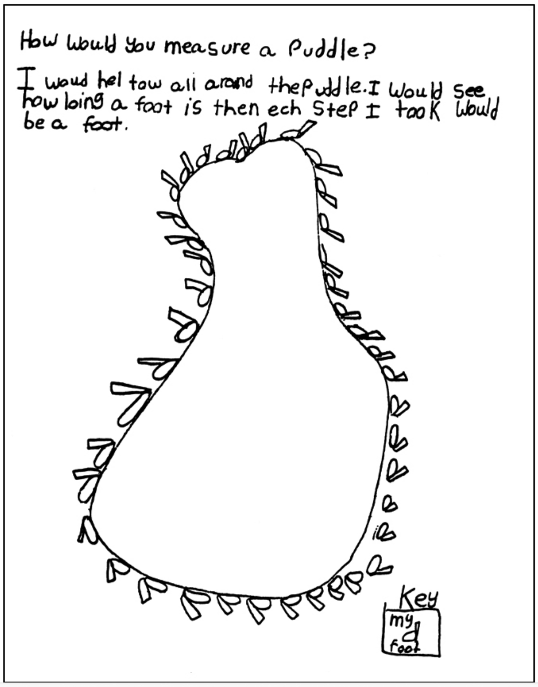


Fig. 1.1. Jonathan shows how he would measure a puddle.

Teacher: Is there another way to measure the puddle besides going around the outside?

Noriko: Measure the height.

Teacher: How?

Noriko used her hands and a yardstick to show how she would find out how deep the water was. We talked about how to figure out the water level of a body of water. Kyle shared a similar idea in his work (fig. 1.2), using a metric ruler to find the puddle’s depth.

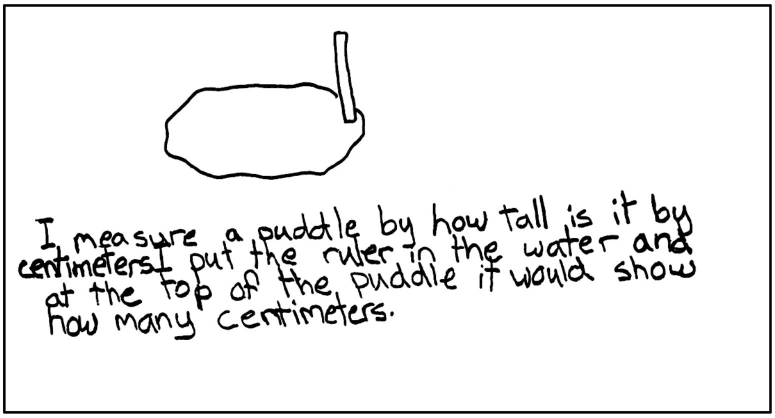


Fig. 1.2. Kyle's work focuses on finding the depth of a puddle ("how tall is it?").

We were ready for another measurement strategy, and Josh volunteered his technique.

Josh: How big it is.

Teacher: Explain "big."

Josh: The length of it. Take some string, put it across the puddle, and then hold it up to a ruler.

Sean: I did length, but I did cubes. I would put cubes along the side of it and since the cubes are inches, just count the cubes.

Chloe offered yet another measurement strategy.

Chloe: Weigh it.

Teacher: How?

Chloe: Put the water in a cup.

Teacher: What if it doesn't fit in a cup?

Chloe: Get a bigger cup.

Allison: But the cup will make it weigh, too.

Teacher: What can we do about the cup?

Kristina: We should use a really light cup.

Teacher: Any other ideas about what we could do about the cup?

Jenny: I would weigh the cup, then the water and the cup, and take away the weight of the cup. [*Wow! Big idea.*]

Noriko: But I don't think it will work, because some of the puddle will still be on the ground.

Chloe: I would use a sponge to get every drop.

In Chloe's written work (fig. 1.3), she demonstrates her flexibility by listing five measurement strategies.

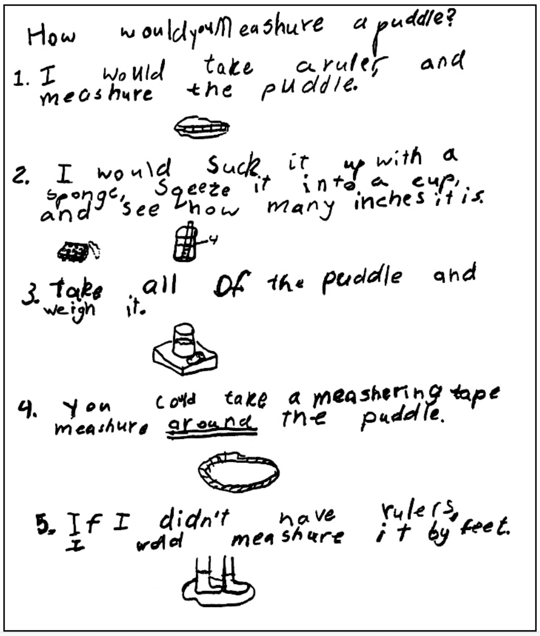


Fig. 1.3. Chloe's ways to measure a puddle address several aspects of size.

I was very pleased with the variety and depth of ideas the children had about measurement. We spent some time talking about when you would want to use a straight stick and when something that curves (tape measure, yarn) would work better to measure a distance. This was a discussion we'd had before, and I was pleased to see that the idea was included in their thought processes.

I was particularly intrigued with the discussion about weight. No one brought up the idea of volume: seeing how many times a cup could be filled with puddle water. Instead, they just wanted a bigger cup to hold it all. I would like to try some volume activities in the future.

case 2

My foot is nine miles

Barbara

KINDERGARTEN, FEBRUARY

The children in my class often speak of the bigness of things. When asked to share their ideas about why they are calling something big, they apparently have not yet developed the language they need to support their exclamations of how big or how small things look.

Lately the children have been tracing their feet and cutting out the resulting paper foot shape. As they work in the rug area, there is a constant buzz of conversation. Because of their varying skills in tracing and cutting, the feet they produce vary greatly in size, from feet that would fit a giant to feet that would fit a mouse! There are also many examples of more accurate sizes in between. The vast differences in these paper feet have stimulated the children to discuss size (bigness and smallness). When I ask why they think their foot is big or small, they say—

> “It looks big."
>
> “I wear a size 7."
>
> “Look at it."
>
> “Mine is big; his is small."
>
> “Big. Me big."

I decide to use their interest in their feet and in “bigness” to begin some activities with measurement. We have not formally explored measurement concepts, so I am curious to find out how the children will approach a measurement activity. I mention that we will spend some time measuring the children’s traced feet. They are pretty excited about this. We gather on the rug, and I explain that I will trace one foot for each of them. Then they can use anything they want in the room to measure their foot, however they want to do it. When I mention the measuring part, a few children speak of needing a measuring tape.

Teacher: What’s a measuring tape?

Tammy: What you measure stuff with.

Teacher: You measure stuff with a measuring tape. But what is it?

Tammy: It’s a tool that measures things so you know if it is big or small.

Ellen: I have one at home. My dad used it to measure me.

Teacher: I don’t have a measuring tape at school.

Ellen: I can bring mine in tomorrow.

Teacher: Let’s try to use things in the classroom to measure our feet.

One by one, I trace each child's foot, and then gather the children for a quick discussion. I ask, "What does it mean to measure your foot?"

I'm met with total silence from the whole group, along with some very confused and disengaged looks. I hold up one of their papers with a traced foot on it.

Teacher: There are lots of ways to measure things. Ellen has started to measure her foot by going from the top to the bottom [*with my hand, I motion along the length of the foot*]. There are lots of other ways to find out how big your foot is. For example, you can . . .

Suddenly, there is a burst of excitement in the room. Several children have ideas.

Rocky: You can go around the sides.

Abigail: You can do a side.

Michael: I know. You can look at the inside.

Teacher: There are lots of different ways. Give it a try.

At this point, the children start to move around the room with more purpose and energy. I think, "Whoa! What just happened?" As I continue to circulate, I notice several interesting things about measuring.

Ellen has confidently placed cubes on her traced foot, from the middle of her heel up to the middle toe, with cubes hanging over the edges of the shape at each end. I ask her how big her foot measures. She counts her cubes and says, "Ten inches." I leave Ellen to record her work (fig. 1.4). When I return later, she says, "Actually, it is nine." When I ask why, she reports, "I had to take one away. It was too big. The cubes went too far."

Fig. 1.4. Ellen measured her traced foot with cubes placed end to end.

Yusi uses cubes in a similar way, but his cubes make a more erratic line, with many cubes extending beyond the toe and heel. He notices this on his own and says, “Oops, not here.” When I ask why, Yusi motions to the area on the paper that is not his foot and says again, “Not here.” He fixes the cubes. When I ask him how big his foot measures, he says, “Ten feet.” English is not his first language, and I think that his use of the word feet is not the measurement term but refers to his foot.

Oscar also places his cubes inside his traced foot, up the middle, but his cubes reach the traced lines exactly, not spilling over at all. When I approach him, he offers, “I found out nine foots.” He has used nine cubes.

I ask Oscar, “How big is your foot?”

He says, “My foot is nine miles.” He points to his real foot and says, “Big. Bigger.”

Michael has covered his entire traced foot with wooden inch cubes, not attending to the outline. I ask, “How big is your foot, Michael?”

He says, “Sixteen pounds.” This number matches the sixteen blocks he used to cover up the foot shape.

Maletu also fills in his foot, but uses pompom balls. The basket has pompom balls of varying sizes. He uses six of the biggest ones and one smaller one. They do not completely fill the space, but they are spread out all over the place: in the foot, outside the foot, all around.

Teacher: How big is your foot?

Maletu: I grow.

Teacher: Yes, you are a big boy. What did you find out about your foot?

Maletu: Here is my foot.

Teacher: I see that you used pompom balls. How many did you use?

Maletu: I count seven.

Teacher: How big is your foot?

Maletu: My shoe.

Rocky brings his paper over to me. I did not see him do the actual measuring, and from his paper I cannot tell what he did, but I can read what he wrote: “My foot is 2 feet across.” He tells me that he used the game pieces. I still have no idea what he is talking about or how he has measured. I convince him to do it again so I can watch.

Rocky gets out some of our smallest geoboards and places them around the outside of his foot. The placement is not tidy—the boards overlap a lot and are crooked—but they do sort of fit around the perimeter. He uses six boards to fit all the way around. On his paper Rocky has drawn only five of the boards, but writes that his foot is two feet across. When he finally explains all this to me, he points to the two geoboards that go up the side of his foot and calls that the “two feet across” part.

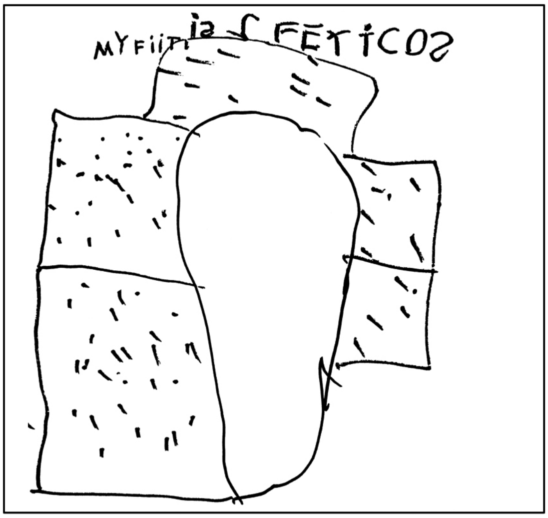


Fig. 1.5. Rocky surrounded his foot with plastic squares and called this measure “2 feet across,” referring to the two squares along the length.

Needless to say, I am impressed, surprised, and stimulated by all the work. Many of the children do have ideas about measuring. But what was going on for them? They showed excitement about measuring, and they shared some conventional ideas. Initially, they were puzzled when given the chance to measure, but they were able to offer ideas when given an example. They used terms such as inches, feet, miles, and pounds. Some of them “fixed” their work to stay within boundaries. Most of them used only one kind of object in their measuring. It is a good starting place. I’ll be taking into account what I learned today as we continue our work on measurement throughout the year.

case 3

Sizing up the meeting area

Isabelle

GRADE 2, FEBRUARY

There is a rectangular carpeted section of the floor in my second-grade classroom we call the "meeting area," and we use it in a variety of ways. Sometimes we sit in a circle, such as during Morning Meeting. At other times, such as Read-Aloud Time, the children sit in informal rows. When we're counting to the "daily number," keeping track of how many days we've been in school, the children sit in a horseshoe facing the number line. At other times of the day, children use the area to read with a partner or work with a small group.

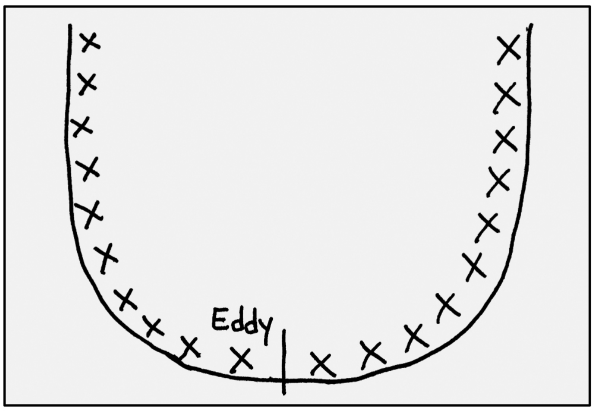
We often struggle to fit all twenty second graders into the meeting area in a way that affords enough "personal space." It looks plenty big until we're all trying to sit there together. One February day, I decided the meeting area would pose an interesting context for some work with area and measurement. When the children were gathered, I reminded them about the various ways we use this area. I told them I often wonder about the size of the space, and I invited them to think about that with me. I asked, "If we wanted to know how big this part of the rug is, how would we figure it out?"

Keith: I think it's 20 feet.

Teacher: How do you think about that, when you say 20 feet?

Keith: Because from Eddy all the way up to the number line, well, there's half of the class right here [*he motions from Eddy halfway around the horseshoe*], and there's half of the class right here on the other side [*now he motions from Eddy halfway around in the other direction*].

I quickly drew Keith's idea on chart paper.



Teacher: So are you saying that you're using the people in the horseshoe to measure?

Keith: Yes.

Teacher: Any other ideas about how we could find out how big this area of the rug is?

Fiona: Well, I think it might be like two yardsticks.

Teacher: Could you say a little bit more?

Fiona: I think there could be like one yardstick right here and another yardstick over there [*she points across the horseshoe*]. And then that would be two yardsticks. And then going this way [*she draws a line in the air perpendicular to the first*], there would be two yardsticks.

At this point, I went to get Fiona a meterstick with inches marked on one side, explaining that I didn't have a yardstick but this would work just as well. I asked her if she could show us what she meant.

Fiona: I mean, put the stick from the middle over to there. [*She lays the stick across the horseshoe, reaching from one edge into the empty center.*]

Teacher: I wonder how that's helping you figure out how big this part of the rug is. You said two metersticks, and you've put that meterstick down one time. Show us what two metersticks would be.

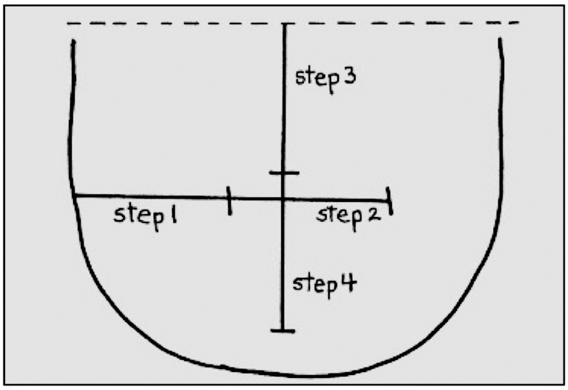
Fiona used her finger to mark the spot where the meterstick ended, then moved the beginning of the meterstick to the spot being held by her finger.

Teacher: And then you said something about two more metersticks?

Fiona: I meant two more metersticks going this way. [*Again, she draws a line in the air perpendicular to the meterstick on the floor.*]

Teacher: Could you take the meterstick and show us what you mean?

Fiona moved the meterstick so one end was at the number line, laying it across the horseshoe in the other direction. Again she marked the ending spot with one finger, picked up the meterstick, and moved it along so it started at the spot held by her finger. I drew another diagram on the chart paper to represent what Fiona had done.



Teacher: Are you finished?

Fiona: Yeah, but maybe one more meterstick, like three metersticks going this way and three metersticks going that way.

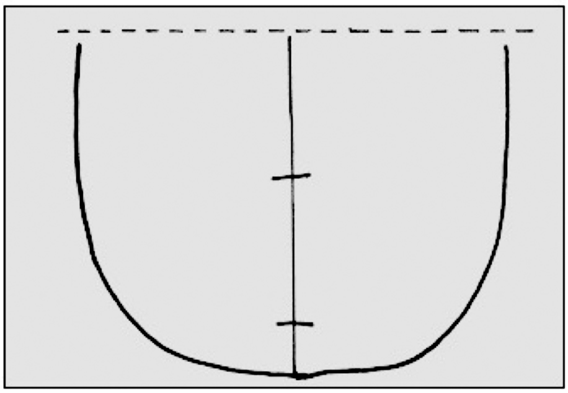
Teacher: But you didn't actually put the metersticks down three times. Are you just estimating?

Fiona: Yeah.

When I asked for any other ideas, Roger raised his hand. Roger has a learning disability that makes it difficult for him to organize and express his thoughts clearly. He also has a difficult time sequencing numbers and letters visually. As a result, I often need to listen extra hard and to help him clarify his ideas.

Roger: [*He places one end of the stick near the number line.*] See, it would be right here. Then you could move it down right here [*he marks the end of the stick and moves it another length*]. And then there would be a little piece left at the end. Then you could count it.

I drew a sketch of Roger's idea:



Teacher: So you're noticing the piece that didn't get measured, and to you it doesn't seem like a whole meterstick?

Roger: Then you could add on the little piece, and I know how much it is altogether.

Teacher: What would you say it is altogether?

Roger: Almost 100.

Teacher: Almost 100 what?

Roger: Meteors.

Teacher: Meters?

Roger: Yeah.

Teacher: And how do you get 100?

Roger: Two big ones almost make 100, plus a little one at the end.

Teacher: Where does the 100 come from? What are you noticing that makes you say 100? Is it those little numbers on the stick?

Roger: Yeah, I count another 83 up.

Two things were getting in Roger's way here. First, the numbers on the ends of the meterstick were covered by metal hanging plates. Second, he was looking at the numbers upside down. Thus he read 38 as 83.

Teacher: I think you're looking at that number upside down. Look over at the other side of the stick—now, what's that number?

Roger: 38.

Roger seemed concerned with making Fiona's method more accurate. I decided to check in with the rest of the class to see if anyone could paraphrase what Roger had said so far. Tracy volunteered.

Tracy: I think he's saying it's 38 inches across the meterstick. That's all I heard.

Teacher: That's 38 centimeters. Roger, can you help Tracy with the rest?

Roger: Because another 38 after 100.

Teacher: Roger, are you saying that there would be a little left over, so you're estimating it's 100?

Roger: Yeah, and you can use the numbers to count up the rest.

Teacher: So it sounds like you want to use the numbers on the meterstick, and add them together to find how many centimeters it is?

Roger: Yeah.

Roger had struggled to organize his thoughts, and much of what he said had come out in bits and pieces. I sensed that most of his classmates had been struggling to grasp what he was thinking. I also had to restrain myself, helping him just enough to get his thoughts out while protecting his ownership of those ideas at the same time.

Teacher: I think we have enough listening energy left for one more person to share their thinking.

Gabrielle: 20 feet.

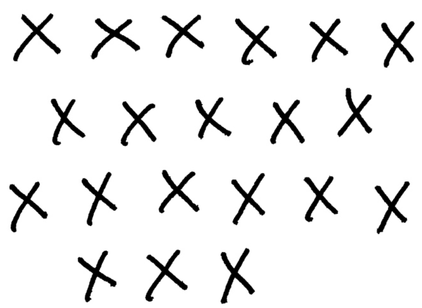
Teacher: Where do you get 20 feet?

Gabrielle: Because it's not about the horseshoe. Do you know how we sit in rows when you read? Well, we take up all the rug by making three rows.

Teacher: So you're noticing that when we sit on the rug in three rows, we seem to fill that space with 20 people.

Gabrielle: Yeah, and sometimes we make a fourth row. Sometimes the fourth row doesn't have a lot of people. It just has 4 or something.

I drew the four rows to illustrate Gabrielle's thinking:



Gabrielle was recalling that depending on how the children arrange themselves, we have from five to seven people in the first three rows. Occasionally we have a fourth row that doesn't seem quite full. There was no response to Gabrielle's idea, other than some wiggling bodies. Finally, Tanya raised her hand.

Tanya: What Gabrielle is saying . . . I don't know how that could help us figure out how big this part of the rug is.

Gabrielle: Because there's 20 people on the rug, and we fill up the whole rug, and so it might be 20 feet.

Teacher: So you're saying every person would stand for a foot? What do you think about that Tanya?

Tanya: I don't know what I think about it.

Teacher: We're going to leave it there for now. Let's take another look at the four different ways that people had to think about it.

We stopped there and wound up the discussion by looking back over my sketches on the charts.

I've become interested in what children pay attention to when they set out to solve a problem. Keith seemed to be looking at the boundaries of the space. Fiona and Roger looked at the space inside the boundaries and seemed to think about each dimension (what I'll call the length and the width) separately. Gabrielle was using the way we fill a space as a measurement of its size. I think it's interesting that Gabrielle's idea drew the only negative reactions. Tanya was able to express her confusion, but judging from the nodding heads that accompanied her comments, Tanya wasn't the only one who didn't know what to think about Gabrielle's idea.

case 4

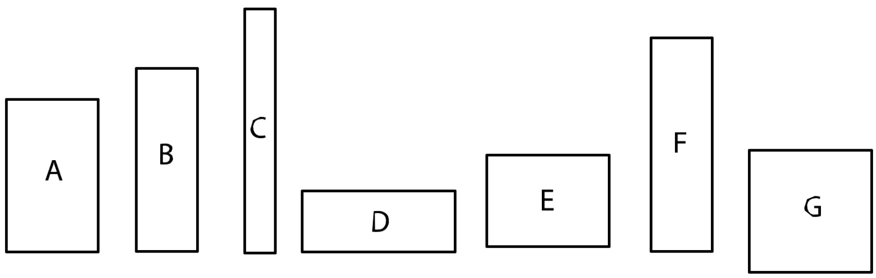
Rectangles and chocolate bars

Olivia

GRADE 3, JANUARY

My job this year is to help my colleagues implement Investigations in Number, Data, and Space, a curriculum new to our school. This means that I periodically work through a single set of lessons with several different classes. In January, I presented the activity “Ordering Rectangles.”[2] Although I worked with three different third-grade classrooms, the conversations were strikingly similar, so I’ll write them up as though I worked with a single class.

For this activity, pairs of students are given a set of seven rectangles, with dimensions running from 1 to 8 inches. The rectangles are lettered for identification purposes. Partners work together to put the rectangles in order, from the biggest to the smallest. These terms are not defined for them. Students are left to determine for themselves what is meant by “biggest.”



Day One, The Ordering Rectangles Activity

I began the activity by telling the students to cut out their rectangles and order them from biggest to smallest, discussing the order with their partner. When both partners agreed, they were to paste the rectangles in this order on a piece of construction paper. I explained that when everyone was finished, the class would come together to share what each pair had found. The students eagerly set to work. From my observations, partners worked well together and relatively quickly agreed on an order for the rectangles.

After this first step, the students were anxious to come together to share their work. All were eager to find out which pairs had “gotten it right.” I explained that I was not going to give an answer, and that later we would be working together on another activity to help us think about the size of the rectangles. Saying this seemed to make the students feel less competitive and more relaxed about sharing their answers.

[2] “Ordering Rectangles” From TERC, *Shapes, Blocks, and Symmetry,* a grade 2 unit of *Investigations in Number, Data, and Space* (Glenview, Ill.: Pearson Education, 2008).

As the students came to me, either in pairs or individually, I wrote their sequences on chart paper. As I had seen previously when doing this activity with other classes, all the students had characterized C as the biggest rectangle.

Teacher: Everyone picked C as the biggest rectangle. What is it about C that made you call it the biggest?

David: It's the longest rectangle.

I asked whether other students agreed with David's description of C and received many affirmative responses. However, I wondered what they meant by the term longest. Was it really length that they were paying attention to? I turned rectangle C on its side.

Teacher: If I turn C like this [*sideways*], is it still the longest rectangle? [*Many murmured yeses and nodding heads.*] If it were turned on its side, would you still call it the biggest rectangle?

The response to this question was much less definitive. Many students looked confused. I could hear replies of "I don't think it would be the biggest anymore."

Teacher: It sounds like many of you don't think C would be the biggest if it is turned sideways. So what is it about C that makes it the biggest if it is standing up like this? [*I turned it back so that the letter C had the proper orientation.*]

Yasmine: If it's that way, it's the tallest.

Yasmine's use of the word tallest to describe C resulted in looks of relief from much of the class. Many chimed in their agreement. In fact, from the sequences I had recorded, clearly most students had taken height as the determining factor for "bigness."

Teacher: It sounds like one way to tell which is the biggest is to look at how tall the rectangles are.

There were many yeses and nods. When I asked the class what else they observed about the sequences I had recorded, a number of hands went up. I called on Kateria.

Kateria: All of us had C, F, and B as our first three rectangles.

When I asked her why she thought that that was true, she replied that they were the three tallest rectangles. I asked about other observations.

Roberto: More of us put the order as C, F, B, A, D, G, E than any other way [fig. 1.6].

Roberto was correct. Interestingly, this was the most common order predicted by the Investigations authors. Students who picked this order had changed the orientation of rectangles D and E (as the side-turned letters show) and had used height as the number one factor in determining size. (As I write this case, I am aware of missed opportunities for pushing my students' thinking. I don't recall asking why they decided that A was "bigger" than D when both rectangles have the same height or why G was "bigger" than E where the same holds true.

In fact, two groups had placed D before A. Had I questioned the students about how they made these choices, I could have triggered an acknowledgment of the need to consider width as well as height when determining overall size.)

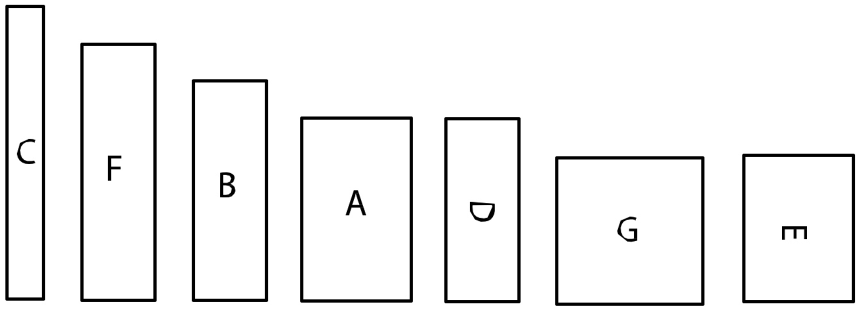


Fig. 1.6. This is the “biggest to smallest” order that most students chose, with rectangles D and E turned on end.

Time was running short. When I asked for more observations, someone noted that C, F, B, A, G, E, D was the second most popular order chosen (fig. 1.7). Students who sequenced their rectangles this way had maintained the original orientation (with the letters D and E upright), but still made height the main factor in determining size.

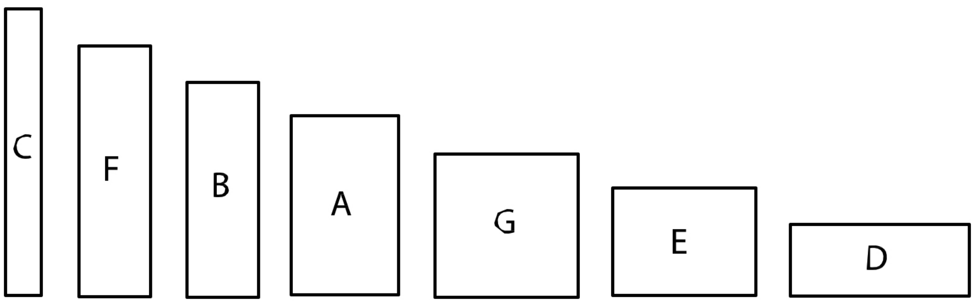


Fig. 1.7. When rectangles D and E are not turned on end, ordering by height results in a different “biggest to smallest” order.

Our time was up for the day. I told the students that the next day, we would look more deeply into the question of what makes one rectangle bigger than another.

As I thought about my experiences doing the “Ordering Rectangles” activity with these classes, I was struck by how remarkably similar the results and discussions had been. In all three classrooms, students had looked at the “tallness” of each rectangle as the way to determine relative size. The two sequences illustrated here were the most commonly selected, depending only on how the students had oriented D and E. What made height so compelling to all these students?

I pondered this question before moving on to the "Covering Rectangles" activity the next day, and a couple of thoughts came to mind. First, most students in these classes had not been introduced to, or had not explored, the concept of area. While a couple of students in one class had made reference to the "space inside" a four-sided figure, I don't remember that any other students ever raised the idea of the space inside the rectangles. My second thought was about our society's fascination with height. When we talk about which student in a class is the biggest, for example, we typically base this on height. Doesn't it make sense then that young students often think of big and tall as synonymous?

Day Two, The Covering Rectangles Activity

The follow-up activity for the second day is "Covering Rectangles," which challenges students to think about the idea of size in a deeper way. Now the students are asked to imagine that the same rectangles (A–G) are chocolate bars. The question of which candy bars would have the most chocolate encourages the students to focus on more than just the height of the rectangles.

Before introducing this activity, I wanted the students to think about other ways that they might define "biggest." I asked once again about their unanimous selection of C as the biggest rectangle.

Shamika: Well, C is the tallest rectangle, so I think that it must be the biggest.

I asked the students if they thought the tallest thing would always be the biggest. My question was met with a number of confused looks and some murmured expressions of uncertainty.

Teacher: What if two people walked into this room, and one of them was tall and thin and the other person was shorter but very wide. Which person would you say was bigger?

Jason: The shorter person could weigh more, so maybe you could say that, that person was the biggest.

Brent: I think Jason's right. Maybe the tallest doesn't always mean biggest.

I told the class that we were going to look at the rectangles again to see if there was another way to think about their size. I explained that when mathematicians talk about the size of rectangles and other shapes, they sometimes think about the space inside of these shapes.

Teacher: Mathematicians call the space inside of these rectangles the area of the shapes. One way for us to think about the area of these rectangles is to imagine that they are candy bars and to think about how much chocolate each bar would have.

I said that when I was a little girl, every time my grandfather came to visit, he would bring my sister and me each a giant-size chocolate bar. When we took the wrappers off these bars (which the students agreed were rectangles), we could see that they were made up of many small squares of chocolate. I suggested that the color tiles we use for other activities are similar to those small squares of chocolate, and I asked the class how we could use the tiles to figure out how much "chocolate" each rectangle/candy bar contained.[3]

Marisol: We could put the tiles on the rectangles to see how many we need to cover each one.

The other students thought Marisol's idea was a good one. They were anxious to begin. However, before they started the activity, I was curious to see if talking about size in this way had already changed their perception of which bar was the "biggest."

Teacher: We are going to do what Marisol suggested, but first do any of you have a prediction about which candy bar will have the most chocolate? You all thought C was the biggest rectangle before. What do you think now?

The response to my question was almost immediate. So many students had answers that I can't attribute them to specific individuals:

"I think that G will have the most chocolate."

"I think it's G, too."

"I think A is the biggest candy bar."

"I think it's A or G."

The conversation went on like this. Most students focused on G or A with a couple of Es thrown in as well. I was surprised at the rapidity and seeming painlessness of this shift in their thinking. They didn't seem at all concerned about their prior conclusion that C was the "biggest" rectangle.

Teacher: It's interesting that so many of you are thinking that G or A will have the most chocolate. Before, you all said that C was the biggest rectangle. Why do you think that G or A will have the most chocolate and not C?

Francisco: Well, I think so because G and A are both fatter than C.

Other students agreed with Francisco. I was struck by the fact that so many of them were no longer looking at only the height of the rectangles. I would have liked to ask more questions. However, it was clear that the students were getting restless. It was time to start the activity.

The students eagerly went to work with their partners from the "Ordering Rectangles" activity. They covered the rectangles with the color tiles and carefully recorded the number of tiles needed to cover each one. They talked with their partners about how their results would change the order of their rectangles. They discussed which chocolate bar they would most like

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[3] Of course, the amount of chocolate would actually be measured in volume, but this was a complication my third graders didn't need to consider. If the bars are all the same thickness, the one that covers more area also contains more chocolate. The idea of comparing the amount of chocolate helped the children think about the problem.

to have. Xiomara wrote that she would like G because “it is the biggest chocolate bar and I like chocolate bars a lot.” Yasmine, who does not like chocolate, wrote that she would want C “because it is the smallest one.”

When the students came together to share what they had found, all agreed that G was the candy bar with the most chocolate and that C, the one they had initially called the biggest, was in fact the opposite. They agreed that while B and E “looked different,” each contained the same amount of chocolate. A, which many students had thought would be the biggest, was identified as having one less square than G.

One by one, volunteers got up and placed a rectangle to show the new order they had found: G, A, F, B and E, D, and finally C (fig. 1.8).

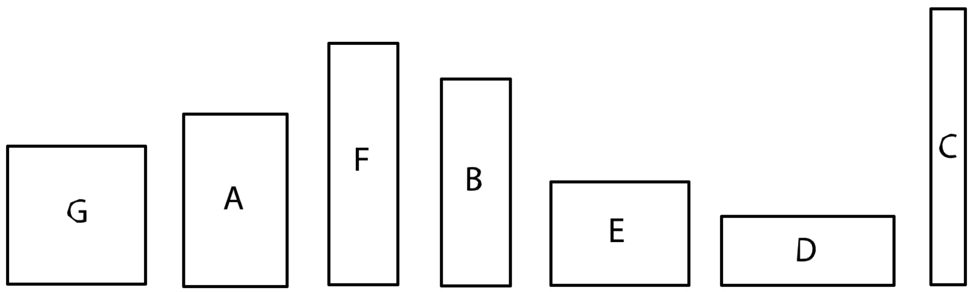


Fig. 1.8. When students used color tiles to determine the area of the seven rectangles, they arrived at this “biggest to smallest” order.

In writing this case and reflecting on the events and discussions that took place, I am left with a number of questions. If my memory serves me, some students used interchangeably the terms “the biggest rectangle” and “the one with the most chocolate.” Did they all see these two things as the same? Or was C still the “biggest” rectangle in the minds of some students, even though it didn’t contain the most chocolate? I’m sure that some (and possibly many) of the students did experience a shift in their thinking about what we mean by “biggest.” However, others probably did not. What could I have done to make this experience more meaningful to all students?

When I did these same activities in a different class earlier in the year, Anita raised her hand and stated that she could prove in another way that C was not the biggest one. She had noticed that C was half the width of F. “If I cut F in half the long way and put one half on top of the other, F will be much longer than C.” Anita’s observation sparked a lot of interest among her classmates and led to an exploration in which students cut up the other rectangles and reconfigured them in an attempt to prove that they were all bigger than C.

No student in my current group made a similar observation. I am wondering now if I should have tried to push their thinking in this direction by encouraging them to think of other ways to prove the relative sizes of the rectangles. Anita’s observation was very powerful. If a similar observation had come from me, the teacher, rather than from one of the students, would it have had the same power and meaning? I think the answer to that question is no.

case 5

Which box is bigger? Which box holds more?

Beverly

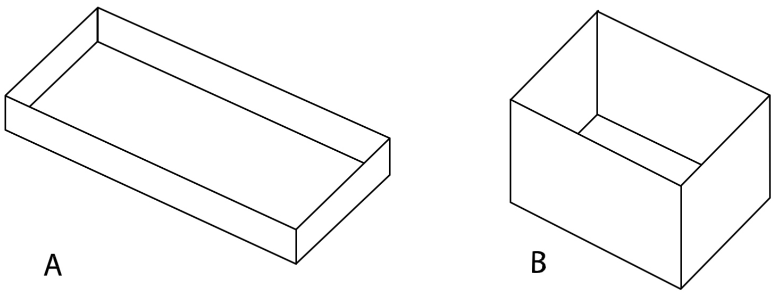
KINDERGARTEN, MAY

As part of math workshop this week, I put out two empty boxes for children to consider as they worked with me in a small group. I had thought a lot about the dimensions before I assembled the two boxes. When I taped cardboard together to form the faces of each box, I purposely constructed them without tops. My plan was to offer children one-inch cubes if they wanted to try filling the boxes, so I made the dimensions as follows:

Box A: 10 inches long, 4 inches wide, and 1 inch deep

Box B: 6 inches long, 4 inches wide, and 3 inches deep

The children were very curious as I set the empty boxes on the table.



Jared: What are we going to do with these?

Teacher: What can boxes be used for?

Melinda: You can put things in them, like jewelry.

Teacher: Which box do you think is bigger?

Five of the six children pointed to Box B. "This one!" they chorused.

Jared: [*picking up Box A*] This one is longer.

Teacher: How do you know which one is bigger?

Melinda: That one [*pointing to A*] is longer. That one [*pointing to B*] is taller.

Alex: They are bigger in different ways.

Teacher: Which one holds more?

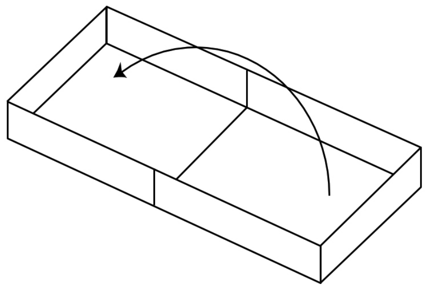
Amanda: [*pointing to A*] This might.

Jared and Alex also pointed to Box A, but Ivan said he thought Box B held more. Megan also pointed to Box B. Melinda remained quiet.

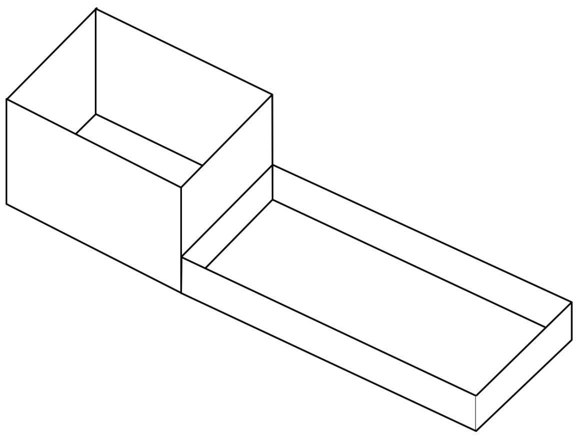
Alex: Actually, they carry the same amount.

Melinda: Actually, they do. If this one [Box A] were folded up, it would be the same.

I was intrigued by Melinda's declaration. As she spoke, she tried to motion with her hands how you could "fold up" Box A in half.



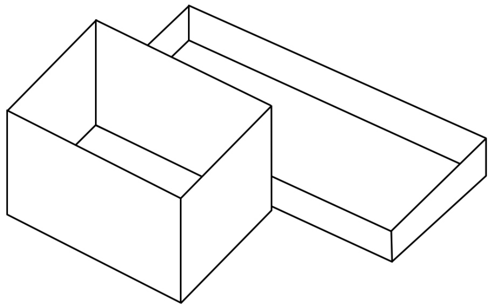
Jared moved the boxes so that their 4-inch sides were aligned.



Jared: These [sides] are the same.

Ivan: But it's not the same length.

Ivan then moved the boxes so that their longest sides were touching. He aligned the corners and motioned that Box A extended beyond Box B.



Ivan: See!

Teacher: How could we find out which box holds more?

Ivan: By just putting them in [*he points to the basket of one-inch cubes on the table*].

The children quickly became very busy putting cubes into the two boxes. Without much direction, they teamed up so three children were working on each box. Melinda, Alex, and Megan worked on Box A.

Melinda: I noticed something. This row has all green.

Indeed, as the children worked, first putting the cubes randomly into the box, they began to fit them together, forming an array. One row did have all green cubes.

Meanwhile, Jared, Amanda, and Ivan were working on Box B. They were piling the cubes in, whereas the other group was carefully placing one cube at a time so that no gaps occurred.

Jared: This is so much!

As Jared spoke, he looked at the other box. Without saying another word, he began to reconfigure the cubes so they filled the space exactly and formed arrays like the ones in Box A.

Megan: I think that one [Box B] is going to hold more.

Jared: I think they'll hold the same amount.

Megan: Maybe we can count.

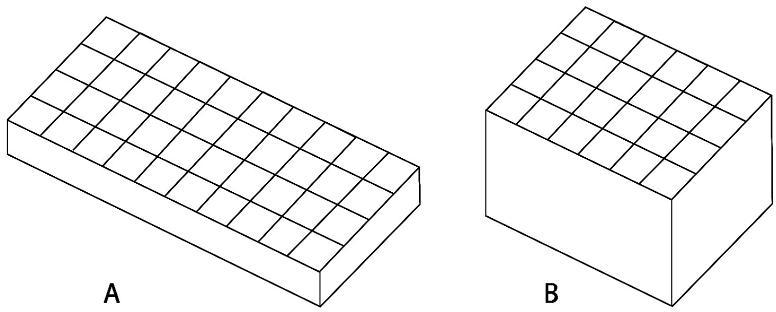
These little exchanges continued while the children were filling the boxes. As the group with Box A filled in one layer, they started a second layer above the edges of the box.

Melinda: We can pile them up.

Alex: No. That's too many.

Jared: Anyway, we need more [for Box B].

Melinda did not insist on her plan, acknowledging Alex's suggestion that she was putting too many blocks in their box. Instead, she gave Jared the extra blocks for the other group's box. Within a few moments, the two boxes were filled exactly to their brims.



Melinda: I think ours [Box A] has more.

Teacher: Why?

Melinda: ’Cause I can see more. [*She strokes her hand along the top of the cubes in Box A.*] This one [Box B] only has 3 in a row up.

Ivan: [*drawing his finger across a row of four cubes in Box B*] No, 4.

Melinda: Three in a row, up in the box.

Melinda was referring to the height or depth of Box B as she said “up in the box.” Ivan was looking at the four cubes that went across the width of Box B. Alex took a cube and placed it along the outside edge of Box B. Then he stacked four cubes up the outside, removed one, and placed his hand flat across his stack and the top of the box, saying, “See, three!”

Ivan seemed to need to repeat this procedure for himself; he took three additional blocks and stacked them up against Box B, too.

Ivan: This [depth of Box B] has 3. That [depth of Box A] has 1.

Teacher: How can we find out which box holds more cubes?

Jared: We can dump them out.

Alex: We can take out one and test it. See, this one [Box B] has 6 across. I’m six!

Megan: Ours [Box A] has 10.

Amanda: We have 40 [in Box A]. I counted.

Jared: [*looking at Box B*] We have 24. I can’t count the inside.

Alex: Count 3. If 3 down inside, then 3 in the middle. These are all pairs of 3. [*He puts his finger on top of one of the cubes in Box B.*]

Melinda: Let’s count.

Jared: 3 plus 3 is 6, 6 plus 6 is 12, 12 plus 12 is . . . I don’t know, 14?

As Jared was calculating aloud, Alex counted the cubes in the top layer of Box B.

Alex: It’s 24, then 2 left inside.

Melinda: We’ll dump them out.

Alex: No, we don't have to.

Melinda: I know! 1, 2, 3 [*she taps one block three times with her finger, then moves to a second block*], 4, 5, 6.

As Melinda tried her strategy for counting, the other children chimed in. She began to lose count because some children said the numbers faster than she could. I offered my help and followed her procedure until we reached 72.

Jared: Wow! That one [Box A] only has 40.

Melinda: Yeah, but if you folded it up, it would be the same.

Returning to her earlier idea, Melinda drew an imaginary line with her finger across Box A where she would fold it if she could. Her finger showed me that she saw she could divide the box in half. I saw 20 cubes on each side of her imaginary line.

Teacher: So, which box holds more?

Jared: This one [Box B]. It has 72.

Megan: But 40 is more on the top.

Melinda: I think they are the same.

This investigation with boxes elicited more conversation and thinking than I ever anticipated. How are these young children able to think and reason about the size of these boxes? When we first started talking, Alex declared that the boxes "are bigger in different ways." He is able to pay attention to different dimensions as he compares the two boxes, and the other children seem to follow this thinking.

When the children began to fill the boxes, I was amazed at the level of sophistication I saw. What prompted them to fill the boxes in neat rows, leaving no empty spaces? We certainly did not talk about this, and yet Jared shifted his group at one point to get them working more like the group that was filling Box A. Then there was the interchange about the depth of Box B. How do the children make sense of this? It is very interesting that they need to build an example outside the box. Also, they seem to be using something they can see about Box A in order to make sense of what they cannot see in Box B.

Finally, I think about what it means to count the actual cubes, and how that does or does not affect their thinking about which box holds more. Megan knows that there are 40 cubes on the top of Box A, and that the 40 she can see in Box A is more than what she can see on the top of Box B. Is this a beginning look at area? What does Melinda see as she speaks of folding up Box A, saying that it would then be the same as Box B? These are such complex ideas to consider. They are not easy questions, or ideas, for these children—or for me.