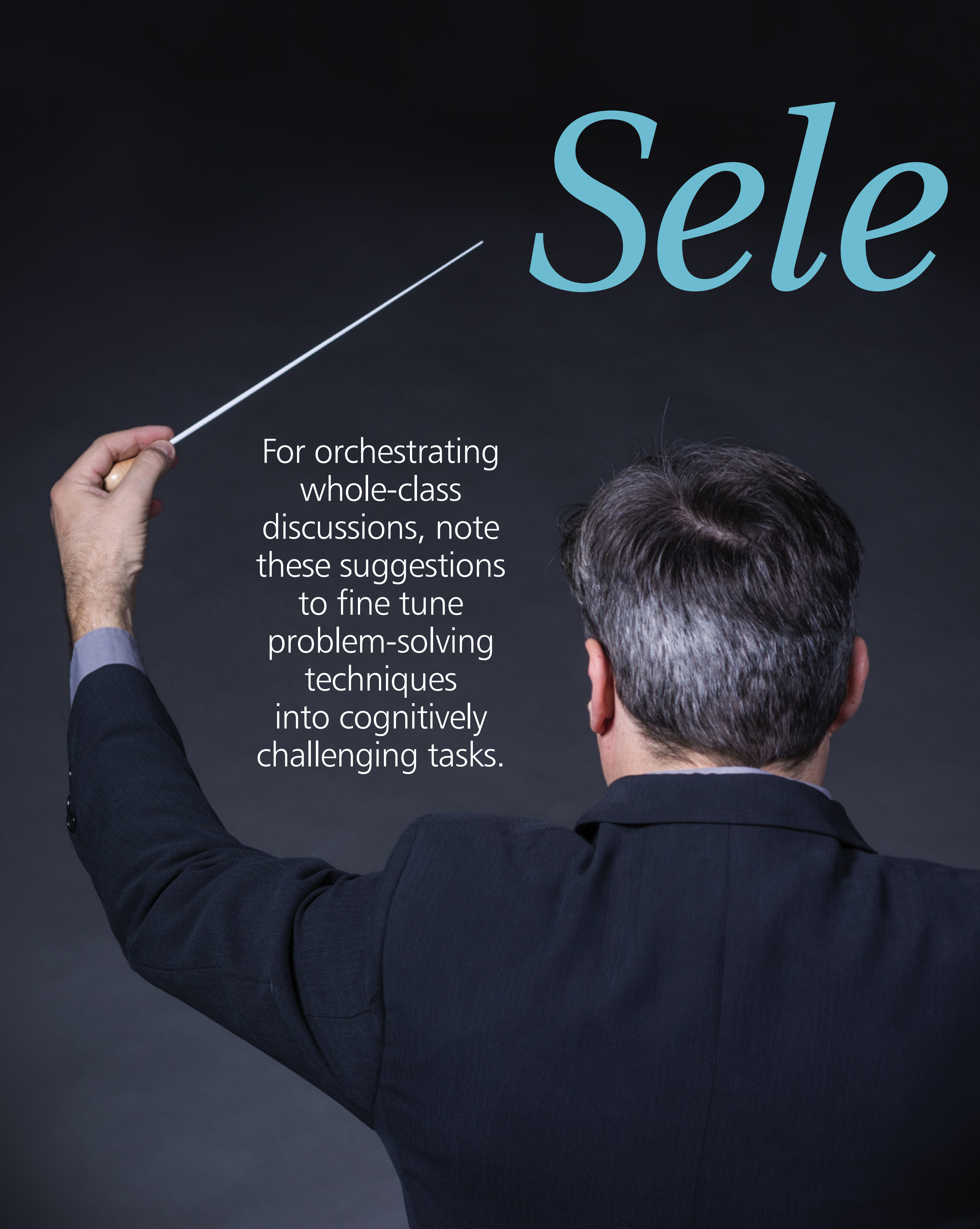

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For orchestrating whole-class discussions, note these suggestions to fine tune problem-solving techniques into cognitively challenging tasks.

cting and Sequencing

Students' Solution Strategies

Erin M. Meikle

Ms. Snyder has just presented a challenging task to her fifth-grade students. They know something about subtracting fractions but have never solved problems with unlike denominators. Snyder gives her students time to complete the mathematical task. She waits for them to come up with their own solution strategies and then hopes to orchestrate a whole-class discussion around those strategies that will best help students understand the key concepts underlying subtracting fractions with unlike denominators. The students present a variety of strategies, so Snyder now faces the challenge of first choosing a few of the strategies to discuss and then sequencing them in a productive way. How does she make these decisions?

To ensure that their whole-class discussions are more than just a time to share different solution strategies (Stein et al. 2008), many teachers are using the Five Practices model to guide them in orchestrating whole-class mathematics discussions that focus on understanding:

1. **Anticipate** an array of possible student-generated solution strategies to a mathematical task before implementing the lesson.
2. **Monitor** students' work as they grapple with the task.
3. **Select** a subset of the student-generated solution strategies to be shared and discussed during the whole-class discussion.
4. **Sequence** the selected student-generated solution strategies in a coherent way.
5. **Connect** the solution strategies in ways that will highlight important mathematical ideas.

HEMEROSKOPION/THINKSTOCK

FIGURE 1

PSTs familiarized themselves with the Five Practices model before solving a subtraction-of-fractions task. The instructor gave them a mathematical learning goal and its underlying concepts.

The mathematical task: $\frac{5}{6}-\frac{3}{5}$. The mathematical learning goal for this task:

Students will understand why the common denominator procedure for subtraction of fractions works. In particular, students will understand these three underlying concepts:

1. The meaning of the numerator: The numerator is the number of pieces of size $1/n$.
2. The meaning of the denominator: The denominator represents the size of the pieces and tells you what the size is.
3. The need for same-size pieces—To be compared, two fractions must be expressed in terms of pieces of the same size.

These five practices are intended to make orchestrating whole-class discussions around student-generated solution strategies more manageable, but they are still difficult to enact.

Although the selecting and sequencing practices seem especially critical, they might be difficult to learn, so I studied them in more detail. I wanted to better understand the thinking behind the selecting and sequencing decisions that teachers make so that I could help prepare future teachers. During one semester of a mathematics methods course for elementary school teachers, I worked with twenty-three preservice teachers (PSTs) on a set of selecting and sequencing solution strategies activities; twenty-two of the PSTs agreed to let me use their written work.

Throughout the activities, PSTs were asked *how* they would select and sequence solution strategies in ways that promote the mathematical learning goal of the lesson and to explain *why* they would select and sequence the solution strategies in this way. The PSTs read and discussed the Five Practices article (Smith et al. 2009) to become familiar with the Five Practices model before (a) considering the mathematical task and learning goal (see **fig. 1**) and then (b) thinking about *how* and *why* they would select and sequence the anticipated solution strategies to this task. Because PSTs generally have difficulty generating multiple solution strategies to mathematical tasks (Borko and Livingston 1989), I furnished the anticipated solution strategies (see **fig. 2**).

FIGURE 2

Here are six anticipated solution strategies to the mathematical task in figure 1, $\frac{5}{6}-\frac{3}{5}$.

(a)

I can't subtract 3/5 from 5/6 because they don't have the same denominator. I have to find a common denominator. How about 60? I know that 5 and 6 both go into 60 and an equivalent fraction for 60 is 50/60. So, now I can write

$$\frac{50}{60}-\frac{36}{60}=\frac{14}{60}$$

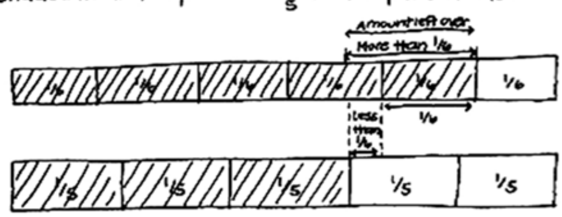
(b)

$$\frac{5}{6}-\frac{3}{5}=\frac{2}{1}=2$$

This doesn't seem correct because $\frac{5}{6}$ and $\frac{3}{5}$ are both less than 1. I don't understand how the answer could be 2 but the work makes sense to me.

(c)

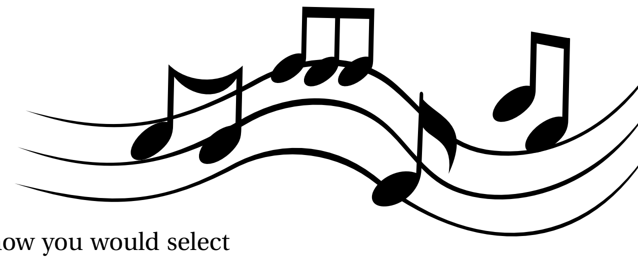
For $\frac{5}{6}$, there are 5 parts out of 6 parts. For $\frac{3}{5}$, there are 3 parts out of 5 parts. I drew a picture of $\frac{5}{6}$ with 5 parts shaded in and a picture of $\frac{3}{5}$ with 3 parts shaded in.



If I take away $\frac{3}{5}$ from $\frac{5}{6}$, then there's 1 part less than $\frac{1}{6}$ left and one part a little more than $\frac{1}{6}$ left. For the part that is less than $\frac{1}{6}$, there is about $\frac{1}{2}$ of $\frac{1}{6}$ which equals $\frac{1}{12}$. The other part is about $\frac{1}{6}=\frac{2}{12}$.

$$\frac{2}{12}+\frac{1}{12}=\frac{3}{12}$$

This means that $\frac{5}{6}-\frac{3}{5}$ is about $\frac{3}{12}$.



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The PSTs' rationales for the ways they selected and sequenced solution strategies could be grouped into three categories. The purpose of this article is to describe these three categories and to show how one of them has the potential to translate into a more productive whole-class discussion than the others. Before you read any further, take a moment to solve the task in **figure 1** and carefully read the mathematical learning goal and its underlying concepts. Then, think about how you would select and sequence the solution strategies in **figure 2** for a whole-class discussion. Why would you select and sequence them in this way?

Why did PSTs select particular solution strategies?

PSTs selected solution strategies for pedagogical reasons (pedagogical moves) and two mathematical reasons (*mathematical procedures*

(d)

I didn't know how to subtract $\frac{3}{5}$ from $\frac{5}{6}$ because the denominator in $\frac{3}{5}$ is 5 and the denominator in $\frac{5}{6}$ is 6. I thought about how to write these fractions differently with the same denominators. I knew that 5 and 6 both go into 30 so I decided to write equivalent fractions with a denominator of 30.

$$\frac{5}{6} \times \frac{5}{5} = \frac{25}{30} \quad \text{and} \quad \frac{3}{5} \times \frac{6}{6} = \frac{18}{30}$$

$\frac{5}{5}$ and $\frac{6}{6}$ are both equal to 1 so multiplying by these does not change the numbers.

$$\frac{25}{30} - \frac{18}{30} = \frac{7}{30}$$

(e)

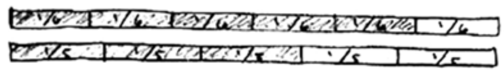
I thought of $\frac{5}{6}$ as a pizza broken up into 6 pieces and only 1 piece was eaten. To subtract $\frac{3}{5}$ from $\frac{5}{6}$, I drew a picture of the pizza and broke up the 5 remaining pieces into fifths. A fifth of 5 is one piece of pizza which means $\frac{3}{5}$ is 1+1+1 or 3 pieces of pizza. I am left with 3 pieces (3 fifths) of pizza.

▨ = portion of pizza not eaten

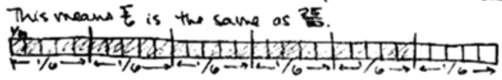
☐ = portion of pizza that was eaten

(f)

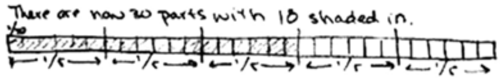
I am taking away $\frac{3}{5}$ from $\frac{5}{6}$. For $\frac{5}{6}$, I drew a rectangle with 6 parts and shaded in 5 of them. For $\frac{3}{5}$, I drew a rectangle and shaded in 3 of them.



Since the two fractions do not have the same denominator, I need to represent each fraction with a rectangle that is cut into the same number of parts. A common denominator for 5 and 6 is 30. I divided each of the 6 parts in $\frac{5}{6}$ into 5 parts to show the common denominator of 30. There are now 30 parts with 25 shaded in. This means $\frac{5}{6}$ is the same as $\frac{25}{30}$.



Next, I divided each of the 5 parts in $\frac{3}{5}$ into 6 parts to show the common denominator of 30. There are now 30 parts with 18 shaded in.



$$\frac{5}{6} - \frac{3}{5} = \frac{25}{30} - \frac{18}{30} = \frac{7}{30}$$

and *underlying concepts)*. In this section, I will explain each category and present an example from the PSTs' rationales for each category. These examples are from PSTs' rationales for selecting solution strategies from **figure 2**. Each rationale *could be classified into more than one category* because PSTs gave more than one reason for choosing a solution strategy.

Category 1: Pedagogical moves

The first category of rationales for their selections is called *pedagogical moves*. Considering pedagogical moves when planning discussions is always important for teachers to do, and in fact all the PSTs justified their selections that were related to pedagogical moves. However, some students provided rationales that primarily focused on pedagogical moves and failed to include deeper consideration of the mathematical learning goals. For example, consider part of Francine's rationale:

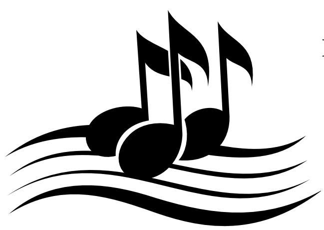
> I will show solution strategies *b*, *c*, *f*, and *d*. I want to make sure students are getting a visual representation of the problem as well as being able to see the problem in a systematic manner. . . .
>
> By showing a misconception that they can just do the operation straight across without changing, they will understand that a change must be made to the fractions.

Francine's rationale focuses mostly on superficial features of the solution strategies that she found attractive, such as a misconception in one strategy and a visual representation in another strategy. She appears to be selecting so she can implement particular pedagogical considerations, like moving from a category with a misconception to a correct strategy. But she is not using the mathematical learning goal to help determine the direction of the whole-class discussion. Surface-level features, such as visual representations, seem to be driving her selection decisions.

Category 2: Mathematical procedures

The second category for selecting rationales is *mathematical procedures*, which gets a little closer to analyzing solution strategies for the potential to promote the mathematical learning goal. Mathematical procedures rationales address important practices related to a learning goal but do not directly address the underlying concepts related to the mathematical learning goal. All the PSTs provided rationales that were related to mathematical procedures. For example, consider part of Eduardo's rationale:

> If I was teaching this class, I would choose solution strategies *a*, *b*, *d*, and *f*. Solution strategy *a:* They knew that the original denominators, 5 and 6, are both multiples of 60. Though they did not articulate how they found the equivalent fractions with 60 as the denominator, they seem to understand that the common denominator is essential. . . . Solution strategy *b*: I would want them to share with the class why common denominators are so important. Solution strategy *d*: This group was very similar to solution strategy *a*. I think it is important to contrast an answer with the denominator of 30 with a denominator of 60. . . . Solution strategy *f:* They found equivalent fractions by dividing their sixths by 5 and their fifths by 6.



Eduardo highlights mathematical procedures related to subtraction of fractions, such as finding a common denominator and finding equivalent fractions. But he does not analyze the solution strategies in terms

To fill an auditorium with beautiful music, a conductor cues different instruments. So, too, a teacher must cue student presentations of various solution strategies to fill a classroom with beautiful mathermatics.

of their potential to reveal the three underlying concepts of the mathematical learning goal in **figure 1**.

Category 3: Underlying concepts

The third category for selecting rationales is underlying concepts, which directly address the underlying concepts related to the mathematical learning goal. Of the PSTs, 54.5 percent provided selecting rationales that were related to underlying concepts. Dominic's rationale seems to have the most potential to highlight the key mathematics concepts of the task. Consider part of Dominic's rationale:

> I would select *b* because . . . it is important to make sure students know that the denominator represents the piece size and the numerator represents the number of pieces. I would select *d* because it addresses the fact 5/6 and 3/5 do not have the same-size pieces . . . The students subtracted the number of pieces they got from 18/30 from the number of pieces they got from 25/30 to get 7/30. This strategy is correct and addresses finding a common denominator and the fact that you are subtracting the number of pieces you have and not the size of the group, like strategy *b* did. I would show strategy *f*, because . . . this would really help them see that the denominator represents the size of the pieces and the numerator represents the number of pieces. These students represented 5/6 by cutting a whole into 6 pieces and shading 5 pieces in and 3/5 by cutting a whole into 5 pieces and shading 3 of them in. When they realized the piece sizes were different, they cut the whole of sizes 1/5 pieces into 6 parts for each piece so that both wholes had pieces of size 1/30.

Dominic's rationale attends to the underlying concepts of the mathematical learning goal. First, he notes how each solution strategy selected has the potential to promote the first two underlying concepts: the meaning of the numerator and the meaning of the denominator. Second, he mentions the third underlying concept (the necessity of same-size pieces for subtracting two fractions) when he explains why he selected solution strategies *d* and *f*. In particular, he explains how the visual representation in solution strategy *f* could help illustrate that the piece sizes are different. Based on their rationales, it seems like a whole-class discussion led by Dominic has the potential to be more productive than the others because the underlying concepts seem to be driving Dominic's selecting decisions. Because Dominic has clearly analyzed the solution strategies for the underlying concepts, he might be more likely to orchestrate a whole-class discussion during which the underlying concepts of the mathematical learning goal are made public to the entire class.

Practical suggestions for selecting and sequencing

Francine, Eduardo, and Dominic had similar selections of solution strategies—they all selected solution strategies *b*, *d*, and *f* from **figure 2.** But their rationales for their selected solution strategies varied to the extent that they were driven by the mathematical learning goal. Smith and Stein (2011) recommend that the first step to engaging in the Five Practices is identifying a mathematical learning goal and a mathematical task that aligns with this goal. Then, anticipate an array of solution strategies that students might generate to the task. Once the mathematical learning goal, mathematical task, and anticipated solution strategies are in

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TABLE 1

Uncover underlying concepts of a mathematical learning goal that is related to a mathematical procedure by working out the procedure in detail, thinking at each step about what knowledge is needed to understand why the step works the way it does.

Learning goals

Nonunpacked mathematical learning goal	Unpacked mathematical learning goal
Students will understand why the standard algorithm for addition of multidigit whole numbers works according to the joining meaning of addition. Example: $\begin{array}{r} {\scriptstyle 1} \\ 57 \\ +\ \underline{36} \\ 93 \end{array}$	Students will understand *why* the standard algorithm for addition of multidigit whole numbers works according to the joining meaning of addition. a. Addition can be interpreted as joining two or more quantities to find a missing whole. b. The answer in the ones column represents the number of ones that we have after joining all the ones in the ones column and after we have exchanged 10 ones for 1 ten. c. The little 1 above the tens column represents the one group of ten that was exchanged for 10 ones. d. The answer in the tens column represents the number of tens that we have after joining all the tens in the tens column.

place, planning for selecting and sequencing solution strategies can begin. The remainder of this article presents two suggestions to develop a plan and rationale for selecting and sequencing solution strategies in ways that will direct the whole-class discussion toward the mathematical learning goal.

Keep the mathematical learning goal front and center when selecting

A mathematical learning goal should describe *what* a teacher would like students to understand by the end of a lesson. "Describing learning goals precisely requires unpacking them into component goals or subgoals" (Hiebert et al. 2007). **Table 1** shows an example of an unpacked mathematical learning goal and one that is not.

One way of uncovering the underlying concepts that lie behind a mathematical learning goal that is related to a mathematical procedure would be to work out the procedure in detail and think at each step about what knowledge is needed to understand why that step works the way it does. For example, underlying concept c of the unpacked mathematical learning goal in **table 1** could be uncovered by performing the standard algorithm for addition and thinking about why the "little one" is written above the tens column.

After the underlying concepts of the mathematical learning goal have been identified, analyze the solution strategies first and foremost for their ability to reveal the underlying concepts of the mathematical learning goal. Analyzing the solution strategies might be easier if the underlying concepts have been clearly specified ahead of time so that what one needs to look for in the solution strategies becomes evident. If a solution strategy does not seem to have the potential to promote any of the underlying concepts of the mathematical learning goal, then do not select this solution strategy for the whole-class discussion. Keeping the mathematical learning goal front and center will help focus a teacher's attention on selecting solution strategies for their mathematical potential and not only to fulfill a pedagogically based sequencing strategy (such as starting with a misconception and building up to a correct solution strategy). Make sure to double check that the compilation of selected solution strategies has the potential to attend to *all* the underlying concepts of the mathematical learning goal and not just *some* of the underlying concepts.

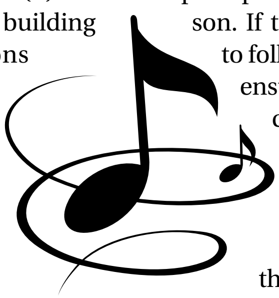
Compare and contrast to help formulate a sequence

To construct a rationale for *sequencing* solution strategies that would be grouped under the *underlying concepts* category, a suggestion is to analyze the selected solution strategies for similarities and differences that are related to the underlying concepts of the mathematical learning goal. Dominic makes an underlying concepts connection when he says in reference to solution strategy d,

> This solution is correct and addresses finding a common denominator and the fact that you are subtracting the number of pieces you have and not the size of the group like solution strategy b did.

He is connecting solution strategies b and d by contrasting how solution strategy d attends to the meaning of the numerator and denominator concepts and solution strategy b does not. It would make sense for Dominic to place solution strategies b and d next to each other in the sequence so that this connection can be highlighted to the class. Consequently, after analyzing solution strategies for underlying concepts connections, it makes sense to cluster solution strategies with connections together in the sequence. Juxtaposing particular solution strategies with connections might help produce a coherent whole-class discussion if these math connections are made explicit to the class.

Pedagogical considerations are not irrelevant when considering how to sequence students' solution strategies. They can be useful but are best considered after decisions have been made on the basis of the mathematical potential of the strategies. A teacher can finalize the sequence by using such pedagogical moves as (a) starting with a misconception and building up to a correct solution strategy or (b) starting with a visual representation and building up to more abstract representations (Stein et al. 2008). These pedagogical moves should be used only after the solution strategies have been *selected* based on the underlying concepts of the mathematical learning goal and have been analyzed for similarities or differences related to the underlying concepts.



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To select and sequence solution strategies successfully during a whole-class discussion, teachers need to plan ahead. It is difficult to (a) select solution strategies that will promote the learning goal, (b) compare and contrast solution strategies, and (c) use pedagogical considerations to formulate an impromptu sequence in this order during a lesson. If teachers plan ahead, it makes sense to follow these suggestions in this order to ensure that no parts of the mathematical learning goal are missed and so that connections can be made easily between or among solution strategies. If you are teaching a lesson to a mathematical task for the first time and do not know anyone else who has taught the lesson, then it might be difficult to anticipate students' solution strategies, and you might have to manage these suggestions on the spot. Teachers should not expect to select and sequence solution strategies perfectly the first time they teach a lesson. To improve for the next time they teach the lesson, teachers can keep a care-

→ reflect and discuss

"Selecting and Sequencing Students' Solution Strategies"

Reflective teaching is a process of self-observation and self-evaluation. It means looking at your classroom practice, thinking about what you do and why you do it, and then evaluating whether it works. By collecting information about what goes on in our classrooms, and then analyzing and evaluating this information, we identify and explore our own practices and underlying beliefs.

The following questions related to "Selecting and Sequencing Students' Solution Strategies," by Erin M. Meikle, are suggested prompts to aid you in reflecting on the article and on how the author's idea might benefit your own classroom practice. You are encouraged to reflect on the article independently as well as discuss it with your colleagues.

- Look carefully at the selecting rationales created by Francine, Eduardo, and Dominic. How would you revise the rationales created by Francine and Eduardo to make them attend to the underlying concepts of the mathematical learning goal?
- Choose a mathematical procedure that you will be teaching in your class and a task that you will be using to teach that procedure. To identify the underlying concepts of the mathematical learning goal, work out each step of the procedure and think about what knowledge is needed to understand why each step of the procedure works. Be as specific as possible.
- Brainstorm with your colleagues to determine an array of solution strategies to the task. Analyze the solution strategies to determine which ones align with the underlying concepts of the mathematical learning goal. How did the level of specificity of the underlying concepts of the mathematical learning goal help you analyze the solution strategies?

We invite you to tell us how you used Reflect and Discuss as part of your professional development. The Editorial Panel appreciates the interest and values the views of those who take the time to send us their comments. Submit letters to Teaching Children Mathematics *at* ***tcm@nctm.org***. *Please include Readers Exchange in the subject line. Because of space limitations, letters and rejoinders from authors beyond the 250-word limit may be subject to abridgment. Letters are also edited for style and content.*

ful record of students' solution strategies that come to the surface during the lesson and then use these to develop a plan for selecting and sequencing solution strategies the next time they teach the lesson.

Helping students achieve ambitious learning goals

The underlying concepts of the mathematical learning goal must be clear to the teacher for the teacher to help make these underlying concepts clear to the students. These underlying concepts must guide all decision making for a whole-class discussion—*selecting*, *sequencing*, and *connecting*. If teachers consider only pedagogical and mathematical procedures when selecting solution strategies, then they might fail to make explicit the main mathematical learning goal during the whole-class discussion. Consequently, keeping the mathematical learning goals front and center is important to guide *selecting*, *sequencing*, and *connecting* decisions.

Unpacking learning goals for underlying concepts is difficult work, as are then selecting and sequencing solution strategies in ways that align with a mathematical learning goal. But this kind of work is crucial in creating whole-class discussions, which help students achieve ambitious learning goals. As teachers move toward meeting the Common Core State Standards for Mathematics (CCSSI 2010), adopting practices like these could give students richer opportunities to meet these more ambitious learning goals.

Let's chat!

On the second Wednesday of each month, *TCM* hosts a lively discussion with authors and *TCM* readers about an important topic in our field. This month, we will discuss this feature article by Erin M. Meikle on **November 9, 2016,** at **9:00 p.m. ET**. Follow along using #TCMchat.

Unable to participate in the live chat? Follow us on Twitter@TCM_at_NCTM and watch for a link to the recap.

Common Core Connections

3.NF.1
5.NF.1

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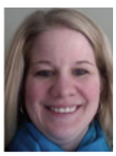

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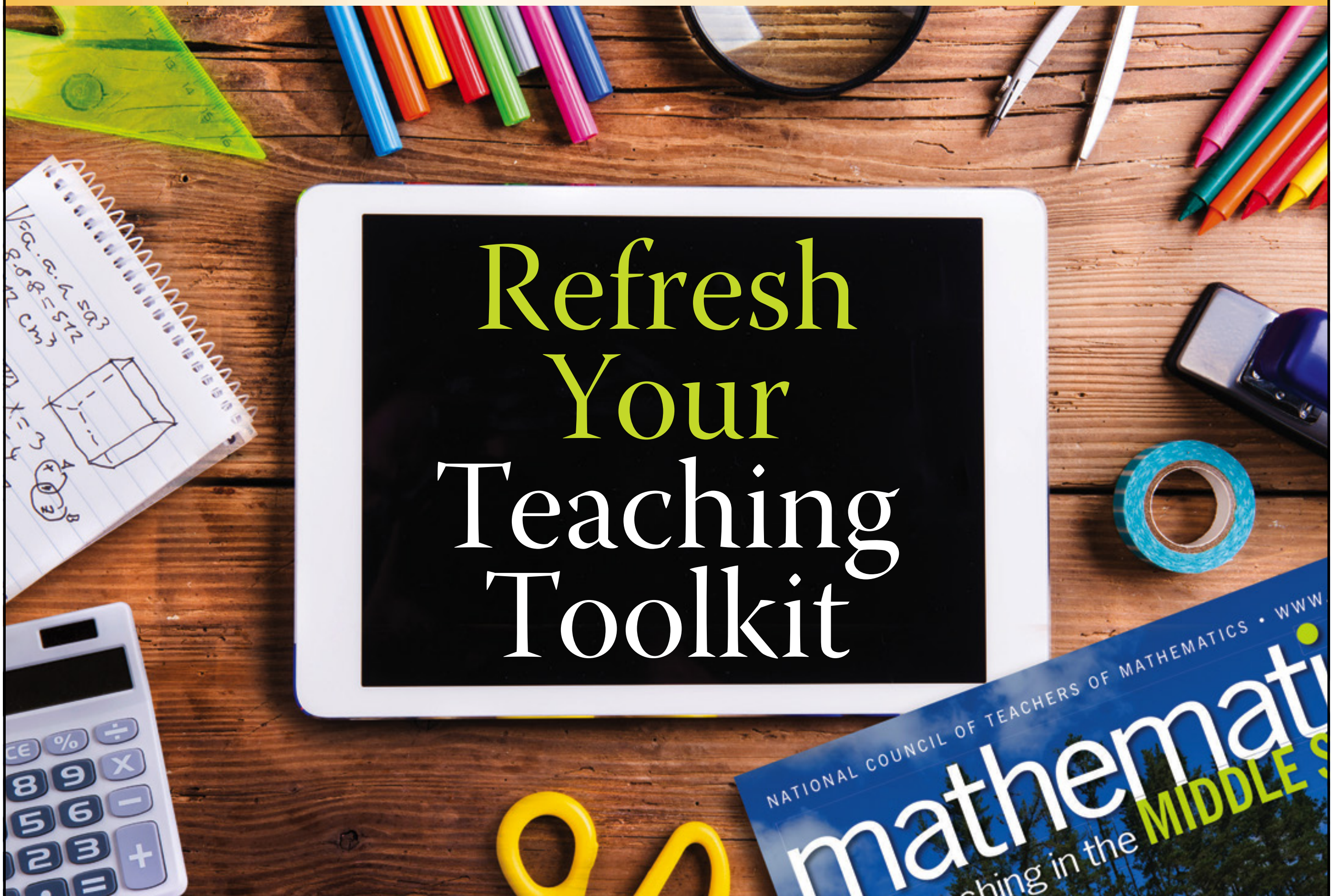

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