MAKING CONNECTIONS BETWEEN SEQUENCES AND MATHEMATICAL MODELS

Almost all secondary mathematics curricula include linear and exponential models. More and more programs also include arithmetic and geometric sequences, as well. These topics are linked, but students often do not make the connections. Further, the notation used for sequences may hinder students from making connections with their study of models. For example, $u_{n+1} = u_n + d$ at first appears to have very little relation to $y = mx + b$; nevertheless, aside from issues of domain, the functions are mathematically equivalent. The use of spreadsheets can mitigate this confusion with notation while helping students make the desired connections. This article describes a three-phase, five-day project that teachers can use to help students make connections between linear and exponential models and between arithmetic and geometric sequences.

As they work on this project, students also gain a better grasp of the "add-add" property of linear functions and the "add-multiply" property of exponential functions. Additionally, they develop a better sense of slope, y-intercept, domain, and recursion. Prerequisite work includes the study of linear and exponential functions and the study of arithmetic and geometric sequences. Some familiarity with spreadsheets is also expected but not essential.

The use of spreadsheets can mitigate students' confusion with notation

PRELIMINARIES

Before beginning the project, lead the students through a review of linear and exponential models, using real-world applications. During the review, have students identify and discuss their selection of independent and dependent variables, the domain of the function, their interpretations of slope for linear models, rate of increase or decrease for exponential models, and y-intercept. Then review topics from arithmetic and geometric sequences. These reviews should take approximately two days.

The next step is to ensure that the students can create and use spreadsheets. Depending on the level of the students, this review may take from one to three days. Before proceeding, students should be able to identify rows and columns; work with labels, numbers, and formulas; format cells; fill formulas down; recognize relative cell position; and create x-y graphs.

After this introductory review, the students work in pairs on the following activity:

- Devise a spreadsheet, like that shown in **figure 1,** that shows the value of a $10 000 investment over a period of twenty years if the investment earns—
  a) 4 percent per year,
  b) 7 percent per year,
  c) 10 percent per year,
  d) 13 percent per year.
- Graph all four functions on the same axes, using the year as the x-value. (See **fig. 2.**)

YEAR	AMT@4%	AMT@7%	AMT@10%	AMT@13%
0	$10 000	$10 000	$10 000	$10 000
1	$10 400	$10 700	$11 000	$11 300
2	$10 816	$11 449	$12 100	$12 769
3	$11 249	$12 250	$13 310	$14 429
4	$11 699	$13 108	$14 641	$16 305
5	$12 167	$14 026	$16 105	$18 424
6	$12 653	$15 007	$17 716	$20 820
7	$13 159	$16 058	$19 487	$23 526
8	$13 686	$17 182	$21 436	$26 584
9	$14 233	$18 385	$23 579	$30 040
10	$14 802	$19 672	$25 937	$33 946
11	$15 395	$21 049	$28 531	$38 359
12	$16 010	$22 522	$31 384	$43 345
13	$16 651	$24 098	$34 523	$48 980
14	$17 317	$25 785	$37 975	$55 348
15	$18 009	$27 590	$41 772	$62 543
16	$18 730	$29 522	$45 950	$70 673
17	$19 479	$31 588	$50 545	$79 861
18	$20 258	$33 799	$55 599	$90 243
19	$21 068	$36 165	$61 159	$101 974
20	$21 911	$38 697	$67 275	$115 231

Fig. 1
Spreadsheet showing value of $10 000 investment at different interest rates

"Sharing Teaching Ideas" offers practical tips on teaching topics related to the secondary school mathematics curriculum. We hope to include classroom-tested approaches that offer new slants on familiar subjects for the beginning and the experienced teacher. Of particular interest are alternative forms of classroom assessment. See the masthead page for details on submitting manuscripts for review.

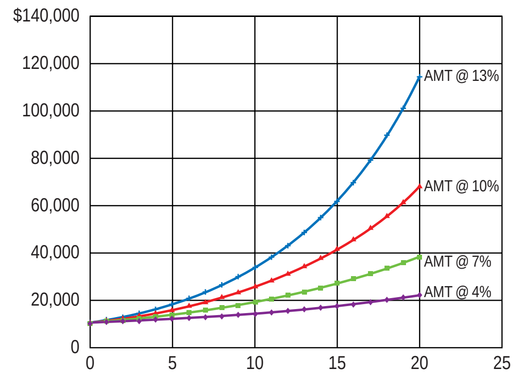

Fig. 2
Graph of all four functions

- Print the results, and show the formulas used to determine the values. The formulas are $A_{new} = A_{old} + 1$ and, for 4 percent growth, for example, $B_{new} = B_{old} \cdot 1.04$.

After the students demonstrate mastery of these ideas, you are ready to begin the project.

PHASE 1: ARITHMETIC SEQUENCES AND LINEAR FUNCTIONS

Day 1

On the first day, students work in pairs on two scenarios that concern arithmetic sequences. The prompts that follow the scenarios lead the students through recursive models.

1. A fifty-gallon bathtub is empty. You turn the faucet, and the tub fills at the rate of 3.4 gallons a minute.
2. Suppose that you have $80 saved in your drawer at home. You have no other income, and you need to pay your little brother $6.50 each week as a bribe.

- Create a spreadsheet for each scenario. For each, put the independent variable in column A and the dependent variable in column B. For all columns, use a recursive formula that can be filled down. In other words, use a formula that generates the correct number in the cell on the basis of the number immediately above it.
- What recursive formula is used for column A? ($A_{new} = A_{old} + 1$.)
- What recursive formula is used for column B? ($B_{new} = B_{old} + 3.4$ for scenario 1; $B_{new} = B_{old}$ – $6.50 for scenario 2.)
- What type of sequence are these? Why? (arithmetic; constant difference)
- What is the domain for these functions? Why? ($0 \le x \le 14$ so that the tub does not overflow; $0 \le x \le 12$ so that money does not become negative. The differences inherent in these scenarios can be explored; the first concerns continuous variables, and the second concerns discrete variables.)
- Use your spreadsheet to graph these functions.
- What shape is the function? (linear)

For homework, students write a paragraph in which they explain why the graphs are the particular shape that they are. The answers should be based on sound mathematical reasoning.

Day 2

At the beginning of the next class, students share their answers. If necessary, emphasize that the functions are linear because of the constant slope; that is, each set increase in time—which is an additive change—produces a set change—again, an additive change—in the dependent variable.

Then pose the same problems but with one difference. This time, in column B, the students are to use an explicit formula; that is, they should use column A as input rather than the previous response from column B. Students should spend about thirty minutes working on the problem and sharing their results with the rest of the class. The formulas are $B = 3.4A$ and $B = 80 - 6.5A$, respectively.

For homework, students explore the connections between arithmetic sequences and linear models. Are they always the same? Are both useful? This assignment is the first part of a paper that should include examples of spreadsheets and graphs. To support their answers, students should include one problem assigned during their study or review of linear models and its solution using an arithmetic sequence, as well as one problem assigned during their study or review of arithmetic sequences and its solution using a linear model.

Students explore the connections between arithmetic series and linear models

PHASE 2: GEOMETRIC SEQUENCES AND EXPONENTIAL FUNCTIONS

Day 3

To help students understand the connections between phases 1 and 2, you should approach this topic in a similar manner. Phase 2 begins with two scenarios, one that increases and one that decreases. These scenarios concern geometric sequences.

1. Suppose that you invest $1500 in the bond market. The investment grows at the rate of 7.6 percent per year.
2. Suppose that you hold a superball 200 centimeters above the ground. You let go of the ball,

and it bounces many times. On each bounce, it returns to a height that is 80 percent of the height from which it started.

- Create a spreadsheet for each scenario. In each, use column A for the independent variable and column B for the dependent variable. For all columns, use a recursive formula that can be filled down. In other words, use a formula that generates the correct number in the cell on the basis of the number immediately above it.
- What recursive formula is used for column A? ($A_{new} = A_{old} + 1$.)
- What recursive formula is used for column B? ($B_{new} = B_{old} * 1.076$ in the first scenario; $B_{new} = B_{old} * .8$ in the second scenario.)
- What type of sequence did you create in column B? Why?
(Geometric; for each new term, the old one is multiplied by a constant.)
- What is the domain for these functions? Why? ($0 \leq x < \infty$; in theory, at least, these functions could continue forever.)
- Use your spreadsheet to graph the function.
- What shape is each function?
(They are exponential curves, the first increasing and the second decreasing.)

Students should see how linear and exponential functions can be built directly from arithmetic and geometric sequences

For homework, students write a paragraph that explains why the graphs are the particular shape that they are. The answers should be based on sound mathematical reasoning.

Day 4

At the beginning of the next class, students share their answers. If necessary, emphasize that the functions are exponential because for an additive increase in the independent variable, a multiplicative change occurs in the dependent variable.

As in phase 1, pose the same scenarios. This time, however, the students are to use an explicit formula for column B; that is, they should use column A as input rather than the previous response in column B. Students should spend about thirty minutes working on the problem and sharing their results with the rest of the class. The formulas are $B = 1500 * 1.076 \wedge A$ for the first scenario and $B = 200 * .8 \wedge A$ for the second scenario. For homework, students explore the connections between geometric sequences and exponential models. Are they always the same? Are both useful? This assignment is the second half of the paper assigned previously. To support their answers, students should include one problem assigned during their study or review of exponential models and its solution using a geometric sequence, as well as one problem assigned during their study or review of geometric sequences and its solution using an exponential model. The final paper should include spreadsheets and graphs.

PHASE 3: ENSURING THAT THE CONNECTIONS ARE MADE

Day 5

Have students share major ideas and examples from their papers. The major ideas should already be internalized, but take class time to lead the students through the notation with which they have studied arithmetic and geometric sequences and linear and exponential functions. Use an example to show that the constant difference in an arithmetic sequence corresponds directly to the slope of the linear function and that the initial value of the sequence is related to, if not the same as, the y-intercept. Also use an example to show that the constant ratio in a geometric sequence corresponds directly to the base of the exponential function and that the initial value of the sequence is related to, if not the same as, the constant a in the form $y = a \bullet bx$. Note that in both examples, if the first value used for the independent variable is 0, the initial sequence value also represents the y-intercept of the function.

Students should also see how linear and exponential functions can be built directly from arithmetic and geometric sequences, respectively. For the bribe problem,

$y_0 = 80,$
$y_1 = 80 - 6.5,$
$y_2 = (80 - 6.5) - 6.5 = 80 - 6.5 \bullet 2,$
$y_3 = ((80 - 6.5) - 6.5) - 6.5 = 80 - 6.5 \bullet 3,$
$y_x = 80 - 6.5x.$

Similarly, for the superball problem,

$y_0 = 200,$
$y_1 = 200 \bullet 0.8,$
$y_2 = (200 \bullet 0.8) \bullet 0.8 = 200 \bullet 0.8^2,$
$y_3 = ((200 \bullet 0.8) \bullet 0.8) \bullet 0.8 = 200 \bullet 0.8^3,$
$y_x = 200 \bullet 0.8^x.$

CONCLUSIONS

Spreadsheets, which are unencumbered by a surplus of notation, allow students to explore the relationships between arithmetic sequences and linear models and between geometric sequences and exponential models. Additionally, with your guidance through this project, students can use spreadsheets to better understand domain, recursive and explicit formulas, slope, and the connection between the initial value of a sequence and the y-intercept of a mathematical model.

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