IRRATIONAL NUMBERS ON THE NUMBER LINE: PERFECTLY PLACED

How do your students react to such numbers as $2 \pm \sqrt{3}$ or $1 + \sqrt{3}/2$? If your students are like mine, they regard such numbers as unfortunate "answers" rather than numbers with a precise location on the number line. Some of my students do not easily distinguish between $2 + \sqrt{3}$ and $2\sqrt{3}$. I devised the following low-tech, hands-on activity to improve my students' understanding of these irrational numbers. Each student creates a number line from adding-machine tape and uses a square and a precisely folded triangle as the only measuring devices. The activity can be introduced in any algebra or geometry class after the Pythagorean theorem has been presented.

I give each student a length of adding-machine tape that is two to two-and-one-half feet long. Then I give each student a 5 1/2-by-8 1/2-inch sheet of colored paper. Because it contrasts with white adding-machine tape, the colored paper helps students measure more accurately. Copier paper works well, but construction paper works less well because the creases are unforgiving. Fold the 5 1/2-by-8 1/2-inch sheet in half lengthwise, and tear it on the crease. Take one of the two pieces, fold it in half lengthwise, and reopen it. That fold is shown in **figure 1**. Place point A, as shown in the figure, on the crease so that the fold line has one endpoint at point *B* of the rectangle. Unfold it, and tear or cut off the right triangle. By using physical manipulations, can you convince yourself and your students that these simple folds created a right triangle with sides in the ratio $1 : \sqrt{3} : 2$?

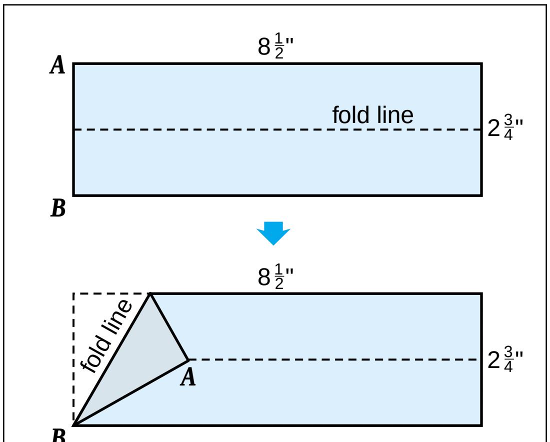

Fig. 1

The simplest explanation for students who are unfamiliar with the basic geometry theorems asks students to compare the length of the short leg to the length of the hypotenuse. The empirical evidence shows that the ratio is 1 : 2. Applying the Pythagorean theorem supplies the length of the long leg. Students who have studied geometry may be asked to follow a rigorous proof, such as the following:

PROOF THAT THE CREATED TRIANGLE IS A 30-60-90 TRIANGLE. See **figure 2.** Triangle ABC and its reflection, triangle ABC', are congruent right triangles, with $\angle BAC \cong \angle BAC'$. Since foldline $\overline{MC'} \parallel \overline{AC}$, $\angle BAC$ is also congruent to $\angle AMC'$. Thus, triangle MAC' has at least two congruent sides, $\overline{AC'}$ and $\overline{MC'}$. M is the midpoint of $\overline{AB}$. If three parallel lines cut off congruent segments on $\overline{CB}$, then they cut off congruent segments on $\overline{AB}$. $\overline{MA} \cong \overline{MC'}$ because the midpoint of the hypotenuse is equidistant from the three vertices. Thus, triangle MAC' is equilateral and equiangular and $m\angle BAC'$ is 60 degrees.

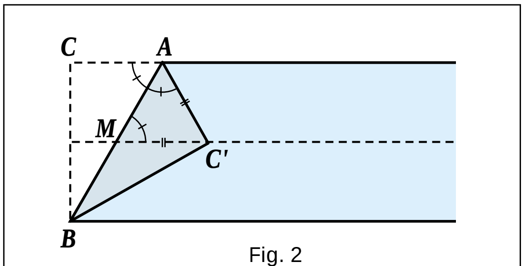

Fig. 2

Students regard such numbers as unfortunate "answers"

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"Sharing Teaching Ideas" offers practical tips on teaching topics related to the secondary school mathematics curriculum. We hope to include classroom-tested approaches that offer new slants on familiar subjects for the beginning and the experienced teacher. Of particular interest are alternative forms of classroom assessment. See the masthead page for details on submitting manuscripts for review.

Students should label the sides of the triangle as 1, $\sqrt{3}$, and 2. They can select any point on the adding-machine tape as 0 and use the short leg of the triangle to mark and label the positive and negative integers. The next step is pure pleasure: students do not need to estimate $\sqrt{3}$ on the number line; they can mark it as precisely as they marked 1, –3, or 6. I like to be sure that the students can correctly find and label the following set: $2 + \sqrt{3}$, $2 - \sqrt{3}$, $-2 + \sqrt{3}$, $-2 - \sqrt{3}$, $2\sqrt{3}$, $-2\sqrt{3}$, and $\sqrt{3}/2$. This activity naturally illustrates the commutative property of addition. The associative and distributive properties can be demonstrated, as well.

Finding the location of such a number as

$$\frac{2+\sqrt{3}}{2}$$

offers a slight challenge. The students can find $2 + \sqrt{3}$ and then fold the adding-machine tape to find half the distance from 0 to $2 + \sqrt{3}$. They can also find $1 + \sqrt{3}/2$ by using half the long leg. This process gives concrete evidence that

$$\frac{2+\sqrt{3}}{2} \neq 1+\sqrt{3},$$

and strong evidence is needed if students are to resist the ever-present temptation to cancel the 2s.

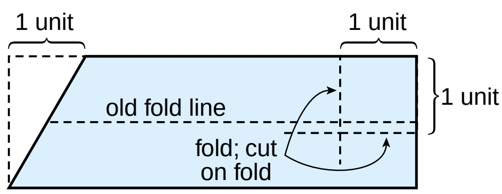


Fig. 3

I next direct the students to use the short leg of the right triangle to mark one unit on two consecutive sides of the unused half of the colored paper to fold and cut off a 1-unit square. See **figure 3.** They fold the square to create a diagonal crease, which is exactly $\sqrt{2}$ units. This time, the students can suggest and find numbers.

I use several types of homework questions. One type, shown in **figure 4a,** gives the students irrational and rational numbers to mark on the line and to list in increasing order. I want students to notice that their number line can become crowded with irrational numbers, as shown in **figure 4b.** Students may find that some numbers appear to occupy the same location because of inaccuracies in cutting or marking. The numbers $2\sqrt{3}$ and $2 + \sqrt{2}$ differ by less than 0.05. Since I wish to avoid decimal approximations in this activity, I demonstrate the inequalities algebraically when a question of order arises; for example,

$$\begin{aligned} 2 &< \frac{9}{4}, \\ \sqrt{2} &< \frac{3}{2}, \\ 4\sqrt{2} &< 6, \\ 4 + 4\sqrt{2} + 2 &< 12, \\ (2+\sqrt{2})(2+\sqrt{2}) &< 12, \\ 2+\sqrt{2} &< 2\sqrt{3}. \end{aligned}$$

A second type of homework question asks the students to give the dimensions of two different rectangles that they could create to obtain a given irrational number as the diagonal. For example, if they had to obtain $\sqrt{5}$ as the diagonal, then they could create a 1×2 rectangle or a $\sqrt{3} \times \sqrt{2}$ rectangle; to obtain $\sqrt{6}$ as the diagonal, they could create a rectangle with dimensions $\sqrt{3} \times \sqrt{3}$, $\sqrt{2} \times 2$, or $1 \times \sqrt{5}$. This problem leads the students on the second day to a recursive method for constructing square roots: if n is a positive integer greater than 1, we find $\sqrt{n}$ when we create a rectangle with dimensions $\sqrt{n-1} \times 1$. This result follows directly from the Pythagorean theorem, since $1^2 + (\sqrt{n-1})^2 = n$.

To emphasize the recursive nature of this construction, I sometimes attempt the following month-long activity. On the first day, I give one student an isosceles right triangle; I define the length of each leg as one unit. That student constructs another right triangle with the long leg equal to the

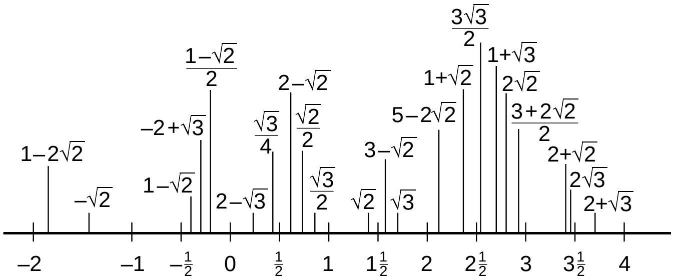
You may wish to start with a fresh piece of adding-machine tape. Mark on your number line the rational and irrational numbers listed below. After you have found the location of each number, list all the numbers in increasing order. Work as accurately as you can; some numbers will be quite close to each other.

Rational numbers: –2, –1 1/2, –1, –1/2, 0, 1/2, 1, 1 1/2, 2, 2 1/2, 3, 3 1/2, 4

Irrational numbers: $\sqrt{2}$, $\sqrt{3}$, $\sqrt{2}/2$, $\sqrt{3}/2$, $-\sqrt{2}$, $1+\sqrt{2}$, $2+\sqrt{2}$, $2\sqrt{2}$, $1-2\sqrt{2}$, $3-\sqrt{2}$, $2-\sqrt{2}$, $1-\sqrt{2}$, $(1-\sqrt{2})/2$, $5-2\sqrt{2}$, $1+\sqrt{3}$, $2\sqrt{3}$, $2+\sqrt{3}$, $-2+\sqrt{3}$, $2-\sqrt{3}$, $(3+2\sqrt{2})/2$, $(3\sqrt{3})/2$, $\sqrt{3}/4$

Do any numbers seem to occupy the same location on the number line?

Sample homework question
(a)



Answer to sample homework question
(b)

Fig. 4

hypotenuse of my triangle and the short leg equal to 1. The next day, he or she passes both triangles to a second student, who constructs a third right triangle with the long leg equal to the hypotenuse of the second triangle and the short leg again equal to 1. This process is repeated until every student has made a triangle. The collection of triangles can be fitted together to form the spiral of roots, as shown in **figure 5.** The triangles can also be placed next to one another on a number line, as

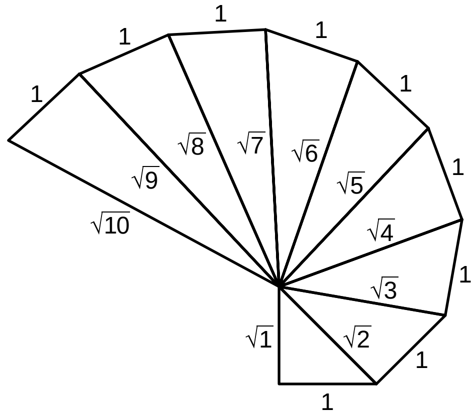


Fig. 5
Spiral of roots

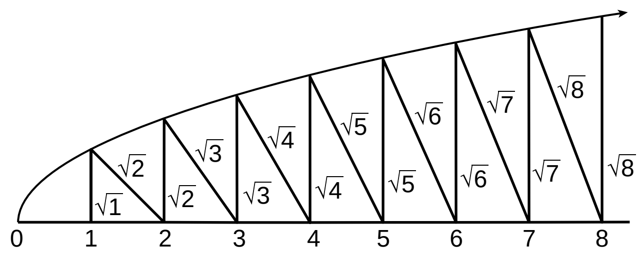


Fig. 6

shown in **figure 6.** When the vertices are connected with a smooth curve, the graph of $y = \sqrt{x}$ appears.

A third type of homework problem asks the students to find an irrational number between two given rational numbers by using only their 30-60-90 triangle and their square. They might be asked to find, for example, an irrational number that is between 1 and 1 1/4. Possible answers include $2 - \sqrt{3}/2$, $2\ 1/2 - \sqrt{2}$, $-3 + 3\sqrt{2}$, or even $-2 + \sqrt{2} + \sqrt{3}$. Finding answers requires a willingness to experiment and a willingness to play with the paper "measuring tools." Students usually enjoy being able to offer so many different correct answers.

This rich activity truly integrates algebra and geometry. My students always enjoy it, and I consider it one of my most successful lessons.

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