A SNOWFLAKE PROJECT:
CALCULATING, ANALYZING, AND OPTIMIZING WITH THE KOCH SNOWFLAKE

Students calculate the perimeter and area of the Koch snowflake

Although secondary mathematics and calculus students are accustomed to calculating the perimeter and area of triangles, this snowflake project enables them to develop a greater understanding of these familiar topics. In this activity, they calculate the perimeter and area of the Koch snowflake, a fractal generated by simple recursion of an equilateral triangle.

As described here, the project specifically addresses several components of the Algebra and Communication Standards for grades 9–12 presented in *Principles and Standards for School Mathematics* (NCTM 2000), including doing mathematical modeling and using the language of mathematics to express a recursive relationship. The project lends itself to group work and can be used at a variety of levels, from first-year algebra through calculus. The complexity of the students' work depends on their familiarity with infinite series and can easily be adjusted for less advanced students. Students who have not formally explored infinite series can generate the perimeter and area of the snowflakes by directly calculating them at each successive stage, and students can use a table to compare relevant quantities at successive stages. Younger students may need some initial assistance to generate the pattern for successive stages, but they can complete the analysis by using a table. Students with a better understanding of functions can generate discrete functions to calculate the perimeter and area for the *n*th stage and can rely on the graphs of these functions to predict the long-term behavior of the functions. Students who have studied convergence of infinite series can determine the long-term behavior of the functions by analyzing the convergence or divergence of the respective infinite series. Students who have access to graphing calculators and who are familiar with the sequential mode can create a table and a graph directly on their calculators.

THE ACTIVITY

The situation

Mandelbrot Design Company, a new business, plans to produce decorative gold plates that will be marketed as "Snowflake Curves." Each plate begins as a template in the shape of an equilateral triangle, the sides of which are one foot long. In step 1 of the construction, each side of the triangle is divided into three equal parts. An equilateral triangle that faces outward is constructed on the middle part of each section. If the middle part of each original side is deleted, the result is a six-pointed star. In step 2, each side of the polygon that resulted from step 1 is divided into three equal parts and an equilateral triangle is constructed on the middle part of each side. This process is repeated until the desired number of steps has been completed, as shown in **figure 1.** After the design has been determined and the mold has been created, molten gold is poured into the mold to form a plate.

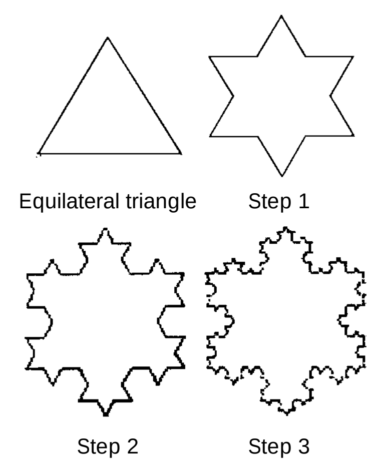

Fig. 1
Koch snowflakes

"Sharing Teaching Ideas" offers practical tips on teaching topics related to the secondary school mathematics curriculum. We hope to include classroom-tested approaches that offer new slants on familiar subjects for the beginning and the experienced teacher. Of particular interest are alternative forms of classroom assessment. See the masthead page for details on submitting manuscripts for review.

The assignment

Part I: As a consultant for Mandelbrot Design Company, you must analyze the profit level for plates at each step. Because production costs reflect the labor needed to cut the template, the cost of production is directly proportional to the perimeter of the plate. On an average, each plate costs $10 per foot of perimeter to manufacture. In contrast, the selling price is based on the amount of gold used; the plates are sold for $160 per square foot of area. Write a memo to your supervisor explaining the amount of profit from various types of plates. Your memo should include a table of production costs and revenue from sales of different plates, a graph comparing production costs and revenue, and an explanation of what happens to production costs and revenue as the level of complexity of the plate, that is, the steps in its construction, increases without bound. In addition, explain the significance of the company's name.

Part II: Explain in detail how you arrived at the mathematical solution to the problem. Include diagrams of the first three Koch snowflakes, the functions used to generate the graphs of the production costs and revenue, and any computations or relevant information—for example, whether you used a calculator or computer in solving the problem.

ASSESSING THE PROJECTS

Assessing group work for an open-ended project can sometimes be difficult. Limiting the group's size to two or three students, delineating clear expectations for the group work, and relying on holistic scoring criteria allow me to direct and assess group work and to assign a numerical grade for the project.

I give students the following guidelines for working on a group project in which all work is completed outside of class:

- Plan your first group meeting as soon as possible. Before your first meeting, carefully read the problem and give it some thought. At the first meeting, plan a method of attack; you may wish to divide the labor among the members of the group. Different members may perform different tasks, or everyone may work on all the tasks. All group members are expected to contribute to the project and to understand all parts of the solution. Regular group meetings to discuss the progress of the solution are important.
- You may ask me for help, but you must have a strategy and some concrete progress to discuss. Meeting with me is probably a good idea even if you think that the project is going well, because doing so may help you avoid dead ends.
- When your group turns in its final product, include an assessment of the relative contributions of yourself and the other members of your group. These peer evaluations will be used to rate each student's participation as excellent, good, adequate, poor, or unacceptable. Members of a group do not necessarily receive the same rating.
- For a group to be successful, all members must contribute their fair share to the project and must attend all scheduled meetings.

Students may initially be uncomfortable with the way that their work is assessed, especially if they are unfamiliar with open-ended assignments that must be completed independently. To alleviate this concern, they can be given specific criteria for assessment. **Figure 2** outlines potential categories and descriptors of levels of proficiency. If necessary, the teacher can give more detailed descriptions of each level of proficiency to convey expectations to students; I assign points to reflect the relative importance of each area and to reflect the overall importance of the assignment.

Assessing group work can be difficult

SOLVING THE PROBLEM

The production costs for each snowflake are directly proportional to the perimeter of the snowflake. An equilateral triangle with one-foot-long sides has a perimeter of three feet, or $P_0 = 3$. Each of the twelve sides of the stage-1 snowflake is one-third foot. Therefore, the perimeter of the stage-1 snowflake is

	Problem Solving	Mathematical Accuracy	Communication of Ideas	Organization	Mechanics
Excellent	Extensive analysis; elegant solution	No errors	Exemplary	Clear, systematic, organized; appropriate transitions	No violations in grammar or punctuation
Fluent	Thorough analysis; efficient solution		Effective		
Good	General analysis; adequate solution	Some minor errors	Proficient	Understandable; some transitions unclear	Some violations in grammar or punctuation
Fair	Partial analysis; partial solution		Acceptable		
Weak	Minimal analysis; attempts solution	Numerous errors	Inappropriate	Disorganized; lacks coherence	Errors interfere with communication
Not acceptable	Little analysis; no solution		Inaccurate		

Fig. 2
Scoring criteria

$12 \cdot 1/3$, or 4, feet, which is one-third larger than the perimeter at stage 0: $P_1 = P_0(4/3)$, or $P_1 = 3(4/3)$. Likewise, the perimeter at stage 2 is one-third larger than the perimeter at stage 1: $P_2 = P_1(4/3)$, or $P_2 = 3(4/3)^2$. This pattern continues, and each successive perimeter is one-third larger than the previous one. The generalized perimeter function is $P_n = 3(4/3)^n$. The first four stages of the snowflakes are shown in **figure 1.** Since P_n forms a geometric sequence with a ratio greater than 1, that is, $r = 4/3$, the perimeter, and hence the production costs, of the snowflake diverge.

Production costs increased without bound, but revenue was bounded

The revenue generated by each snowflake is directly proportional to the area of the snowflake. The stage-0 snowflake has an area $A_0 = \sqrt{3}/4$ square foot. The area of the stage-1 snowflake is the area of the original triangle plus the areas of three smaller triangles that are each one-ninth the area of the original triangle. Therefore,

$$A_1 = A_0 + \frac{\sqrt{3}}{4}\left(\frac{1}{9}\right)(3),$$

or

$$A_1 = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4}\left(\frac{1}{9}\right)(3).$$

At stage 2, twelve new triangles are added to the stage-1 snowflake, and each new triangle has $(1/9)^2$ of the area of the original triangle, that is,

$$A_2 = A_1 + A_0\left(\frac{1}{9}\right)^2(12).$$

Stage 2 can also be viewed as adding four times as many new triangles as were added at stage 1, with each of these new triangles having an area that is one-ninth the area of the triangles added at stage 1, or

$$A_2 = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4}\left(\frac{1}{9}\right)(3) + \frac{\sqrt{3}}{4}\left(\frac{1}{9}\right)^2(3)(4).$$

This pattern continues, with each stage adding four times as many new triangles and each new triangle having an area that is one-ninth the area of the triangles added at the previous stage. The resulting generalized function,

$$A_n = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4}\left(\frac{1}{9}\right)(3) + \frac{\sqrt{3}}{4}\left(\frac{1}{9}\right)^2(3)(4) + \frac{\sqrt{3}}{4}\left(\frac{1}{9}\right)^3(3)(4)^2 + \cdots + \frac{\sqrt{3}}{4}\left(\frac{1}{9}\right)^n(3)(4)^{n-1},$$

simplifies to

$$A_n = \frac{\sqrt{3}}{4}\left[1 + \left(\frac{1}{9}\right)(3) + \left(\frac{1}{9}\right)^2(3)(4) + \cdots + \left(\frac{1}{9}\right)^n(3)(4)^{n-1}\right].$$

Further simplification yields a geometric sequence with a ratio less than 1:

$$A_n = \frac{\sqrt{3}}{4}\left\{1 + \left[\frac{1}{3} + \frac{1}{3}\left(\frac{4}{9}\right) + \frac{1}{3}\left(\frac{4}{9}\right)^2 + \cdots + \frac{1}{3}\left(\frac{4}{9}\right)^{n-1}\right]\right\}$$

This result indicates that A_n converges, and the sum can be calculated directly by using the formula

$$\text{sum} = \frac{a}{1-r},$$

where $a = 1/3$ and $r = 4/9$. Therefore, since

$$A_n = \frac{\sqrt{3}}{4}\left(1 + \frac{\frac{1}{3}}{1-\frac{4}{9}}\right) = \frac{2\sqrt{3}}{5},$$

a finite amount of revenue is generated by the production of the snowflakes.

Figures 3 and **4** show portions of the work from two groups. All groups were able to determine that the production costs increased without bound and that the revenue was bounded. Some groups relied solely on the graphs to predict long-term behavior, whereas others analyzed the convergence or divergence of the geometric sequence. The most common error was graphing production costs and revenue as continuous functions rather than as discrete functions. Each group determined the significance of the company's name: the snowflake is a fractal; fractal geometry was introduced by mathematician Benoit Mandelbrot (1924–). Several groups included such graphics as a picture of Mandelbrot or examples of other fractals in their papers. Peer evaluations appeared honest; individuals indicated the contributions of each group member and noted whether one person contributed more than the others.

CONCLUSION

The final papers were creative, and students enjoyed working in small groups on the open-ended activity. The project not only involved demonstrating knowledge of specific mathematical content, but it also involved communicating that knowledge symbolically and through writing. Mathematical content included calculating the perimeter and area of successive stages of the snowflake; representing production costs and revenue as generalized infinite series; and analyzing the long-term behavior of these quantities to optimize profits. The written component required students to explain the solution and the methods used to obtain it. This activity provided me with a valuable means of assessing the depth of students' understanding of the problem and the solution.

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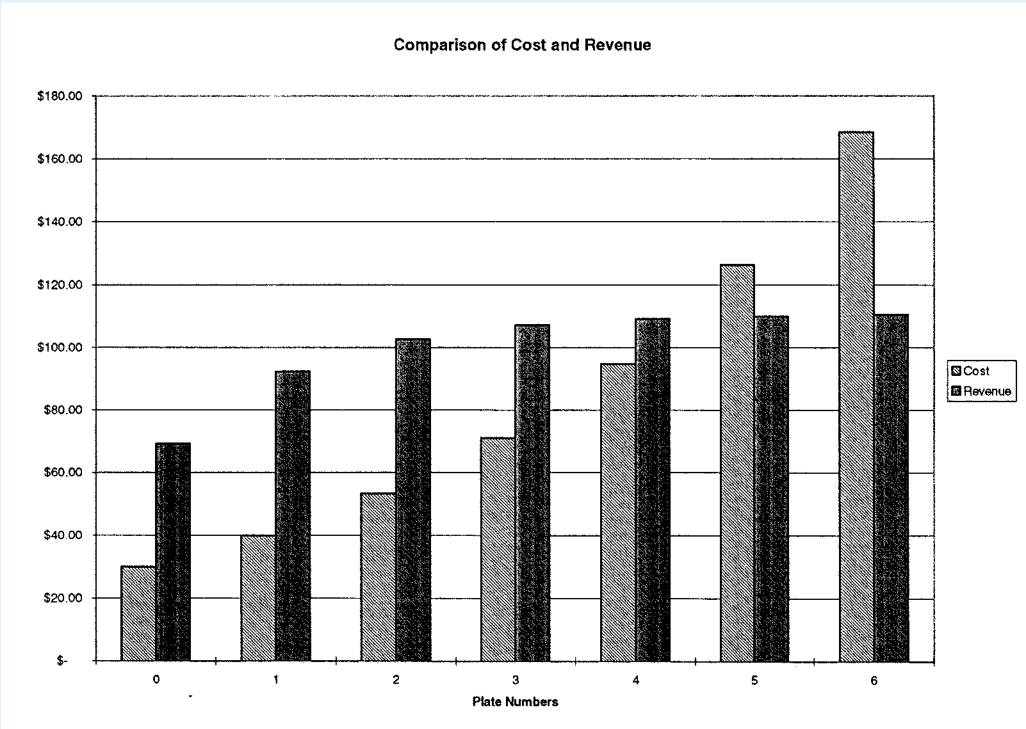


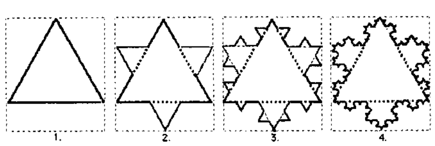
C & G Consultants

Reliable, Dependable, and Accurate

Mandelbrot Design Company
1924 E. Julia Set Drive
Koch, CA 16301

Dear Mandelbrot Design Company:

We are writing in regards to the information you requested concerning the Snowflake Curve for your decorative gold plates. From the information you gave us concerning your cost of production and selling price, we have made a table and graphs comparing the two on the next four pages. The table tells the profit for the first seven plates. As you can see, the greatest profit can be made by plate one, but note this is a very basic plate. Less profit per plate can be made by plates two and three; however, they have a much more attractive design, as shown below.



After the forth plate no profit is made, because the cost of production exceeds the revenue. This can be shown graphically on page four; note the negative profit values by that starts around the fifth and sixth plate and continues infinitely. We calculated the cost, profit, and revenue the following ways: You told us that the cost for each plate is $10 per square foot, so we found the perimeter (distance) around the plate and multiplied it by ten. This shows that the company's cost increase as the plates increase. The company will pay the less cost at plate one. The revenue is calculated by taking the whole area of the plate and multiplying it by $160, that's the sum of money you plan to sell each decorative gold plate for. The revenue increases as the number of sides on the plate increases until it approaches plate 11, from this plate on the revenue is $110.85. The maximum revenue therefore is at plate 11 and this trend continues. A bar graph of both the cost and revenue functions are perfectly illustrated on page five.

We are sure that your company wants to make the highest amount of money possible. The revenue at plate eleven is $110.85, but the cost to your business is $710.31, which means your business would lose $599.46. This loss is called the profit and there is a bar graph on the third page that shows you exactly what happens as the plate number increases. The Mandelbrot Design Company can maximize your profits at plate one.

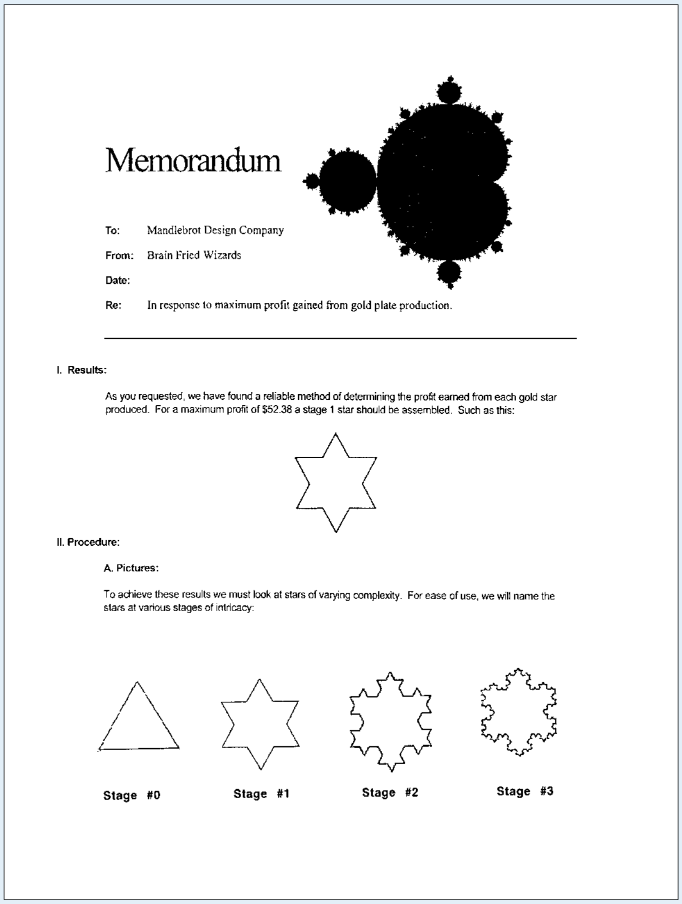
From the information above and on the next page you can clearly decide which plate will fit your interest. If there are any questions or if you would like more information please feel free to give us a call at 1-800-FRACTAL.

Good luck with your decisions and with producing the decorative plates.

Sincerely,

C & G Consultants

Fig. 3
Sample of student work



Memorandum

To: Mandlebrot Design Company

From: Brain Fried Wizards

Date:

Re: In response to maximum profit gained from gold plate production.

I. Results:

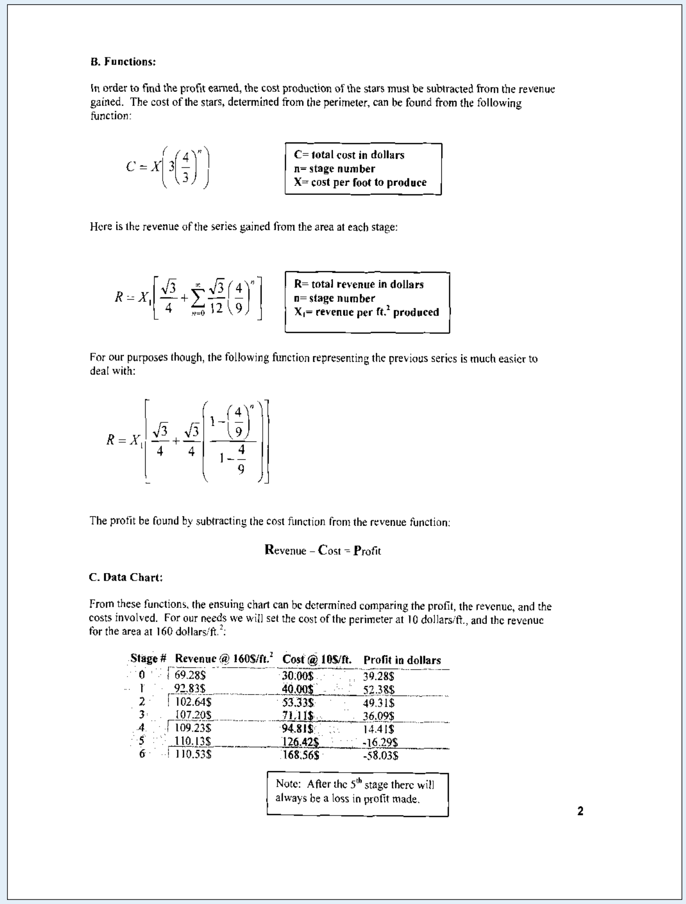
As you requested, we have found a reliable method of determining the profit earned from each gold star produced. For a maximum profit of $52.38 a stage 1 star should be assembled. Such as this:

II. Procedure:

A. Pictures:

To achieve these results we must look at stars of varying complexity. For ease of use, we will name the stars at various stages of intricacy:

Stage #0 **Stage #1** **Stage #2** **Stage #3**



B. Functions:

In order to find the profit earned, the cost production of the stars must be subtracted from the revenue gained. The cost of the stars, determined from the perimeter, can be found from the following function:

$$C = X\left(3\left(\frac{4}{3}\right)^n\right)$$

C= total cost in dollars
n= stage number
X= cost per foot to produce

Here is the revenue of the series gained from the area at each stage:

$$R = X_1\left[\frac{\sqrt{3}}{4} + \sum_{n=0}^{\infty}\frac{\sqrt{3}}{12}\left(\frac{4}{9}\right)^n\right]$$

R= total revenue in dollars
n= stage number
X_1= revenue per ft.^{2} produced

For our purposes though, the following function representing the previous series is much easier to deal with:

$$R = X_1\left[\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4}\left(\frac{1-\left(\frac{4}{9}\right)^n}{1-\frac{4}{9}}\right)\right]$$

The profit be found by subtracting the cost function from the revenue function:

Revenue – **C**ost = **P**rofit

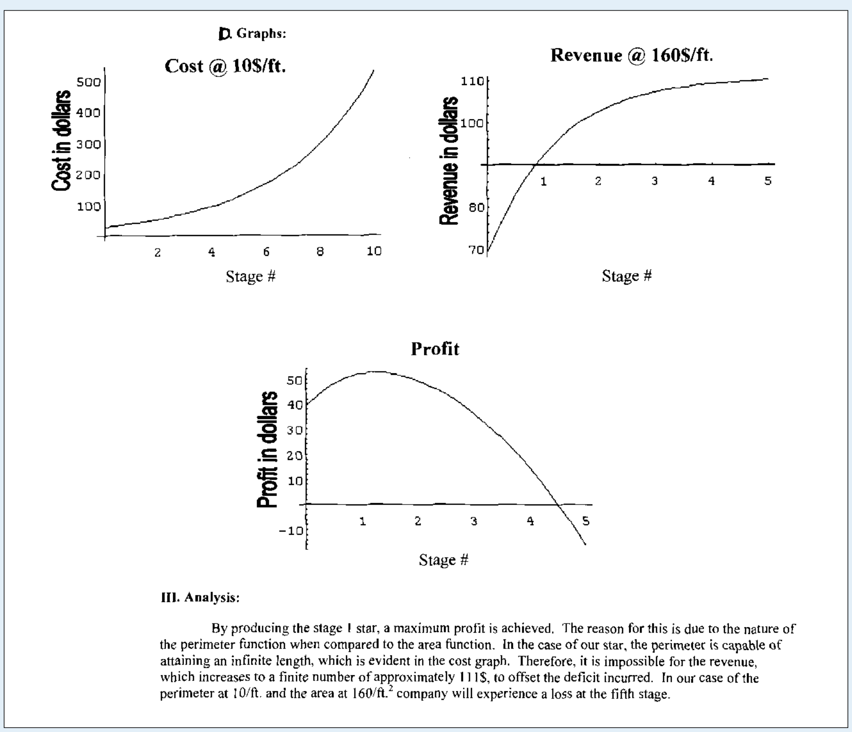
C. Data Chart:

From these functions, the ensuing chart can be determined comparing the profit, the revenue, and the costs involved. For our needs we will set the cost of the perimeter at 10 dollars/ft., and the revenue for the area at 160 dollars/ft.^{2}:

Stage #	Revenue @ 160$/ft.^{2}	Cost @ 10$/ft.	Profit in dollars
0	69.28$	30.00$	39.28$
1	92.83$	40.00$	52.38$
2	102.64$	53.33$	49.31$
3	107.20$	71.11$	36.09$
4	109.23$	94.81$	14.41$
5	110.13$	126.42$	-16.29$
6	110.53$	168.56$	-58.03$

Note: After the 5th stage there will always be a loss in profit made.

2



D. Graphs:

III. Analysis:

By producing the stage 1 star, a maximum profit is achieved. The reason for this is due to the nature of the perimeter function when compared to the area function. In the case of our star, the perimeter is capable of attaining an infinite length, which is evident in the cost graph. Therefore, it is impossible for the revenue, which increases to a finite number of approximately 111$, to offset the deficit incurred. In our case of the perimeter at 10/ft. and the area at 160/ft.^{2} company will experience a loss at the fifth stage.

Fig. 4
Sample of student work

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