M. Kathleen Heid

Computer Algebra Systems in Secondary Mathematics Classes: The Time to Act Is Now!

TECHNOLOGY IS GIVING US AN OPPORTUNITY TO open new doors to mathematical understanding for our students, and we are failing to take advantage of that opportunity. Computer algebra systems (CASs)—and in particular, CAS-capable calculators—provide ready classroom access to automated graphical, numerical, and symbolic-manipulation capabilities; and they should be as much a part of our students' mathematical repertoires as paper-and-pencil strategies or mental arithmetic. However, very few students in the United States have ever been afforded the opportunity to learn mathematics by using these tools.

Fewer trials are being carried out in secondary mathematics classrooms in the United States than in many other countries

Some of the capabilities—for example, graphing and curve-fitting—of the CAS have been welcome in classrooms in the United States for more than a decade, and few would argue for classrooms that ban the use of calculator graphing or curve-fitting. Countless classrooms have taken advantage of the numerical, table, and spreadsheet capacity of today's calculators and computers. Make no mistake about it—the argument mounted is not against CASs per se but against a defining capability of those systems—the symbolic-manipulation capability.

Opponents of allowing the CAS in mathematics classrooms rail against putting automated symbolic manipulation in the hands of students. As a consequence, CASs have thus far not had a widespread effect on mathematics learning in secondary classrooms in the United States. The primary focus of CAS curricular ventures in the United States has for years been on mathematics at the college level with computers and was precipitated by early efforts to incorporate the CAS in the calculus curriculum. Developers of reformed approaches to calculus encouraged students to use CASs to calculate derivatives and integrals and to solve equations so that they could refocus the curriculum on concepts and applications (e.g., Heid [1988]; Palmiter [1991]). In spite of widespread trials in U.S. college classrooms, far fewer trials are being carried out in secondary mathematics classrooms in the United States than in such other countries as France, Austria, Great Britain, and Australia.

WHAT ARE THE ARGUMENTS AGAINST CAS USE IN THE SECONDARY MATHEMATICS CURRICULUM?

With widespread trials of CASs in other countries, what arguments have been in the way of bringing CASs into U.S. classrooms? I have heard a range of arguments, including the following four, listed along with their counterarguments.

Argument 1:

The CAS is too expensive and difficult to use.

Early arguments (Waits and Demana 1992) against incorporating CASs in classrooms centered on the impracticality and cost of equipping each student with the necessary technology. The technology then consisted of CASs on desktop computers, in computer laboratories, or in expensive pocket comput-

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ers and software. This argument is temporal at best. A CAS in a handheld calculator is only slightly more expensive than a graphing calculator, and graphing calculators are now required in many schools. CAS technology continues to come down in price, and newer versions are friendlier and more accessible. The increasing availability of Web-based tools may also make a difference; and as more classrooms adopt CAS technology, market demand can drive costs down.

Argument 2:

New curricula minimize the importance of symbolic work or exact answers.

Mathematics educators in the early 1990s envisioned the development of mathematics curricula that placed far less emphasis on exact symbolic manipulation than previous curricula. Reflecting on this curricular stance, Waits and Demana (1992, p. 180) pointed out that "Exact answers produced by computer symbolic manipulators are often of no real use and sometimes furnish little insight into the problem modeled by the algebraic representations." More so now than in the recent past, however, advocates of curriculum reform are crafting curricula that value mathematics, not just for its practicality as a problem-solving tool but for its value as an example of a finely developed intellectual structure. A need exists for symbolic work, and exposure to the exact answers generated on a CAS may help students understand when exact answers are needed.

Argument 3:

Since CAS calculators are not allowed on standardized tests or in college classrooms, they should not be allowed in the K–12 mathematics classroom.

A common argument against CASs relies on the belief that since CAS calculators are typically not allowed on standardized tests or in college mathematics classrooms, secondary mathematics classrooms should not allow students to become accustomed to their use. But the conditions of testing are changing, and standardized tests are increasingly allowing students to use a range of calculators, including CAS calculators. For example, the TI-89 is allowed on parts of the AP calculus examination. Besides, tests should not drive instruction, and the CAS can be an excellent learning tool. Moreover, if CAS-equipped calculators are good for high school students, maybe colleges should follow suit.

Argument 4:

Use of CASs leads to loss of by-hand symbolic-manipulation skills.

Finally, teachers hesitate to loosen up on requiring students to complete all symbolic manipulations by hand, for fear that such a stance would mean the death of manual symbolic-manipulation skills. And many teachers are unwilling to entertain a future in which by-hand symbolic skills would not be an essential goal: "no one can be sure at this time how much paper-and-pencil algebraic manipulation is really necessary for success in college and in a workplace that requires increasing technological and scientific know-how" (Waits and Demana 1992, p. 180). More than ever before, mathematics teachers are coming to realize that symbolic reasoning is not just something to do when numbers and graphs are not available—it has the potential to enrich students' understandings of the mathematical and real-world quantities that are being represented. Today, more than fifteen years after the first fairly accessible CAS was introduced, the focus is on enhancing students' understanding of the symbolic aspects of algebra—a focus that can be ably assisted by CASs.

Many teachers did not experience CAS use as part of their own learning, so they do not use it in their classrooms. Many teachers are unfamiliar with what the CAS can do and with how it can assist students' learning. Teachers will never learn about the promise of the CAS until they try it in their own classrooms. Just because we may have learned our mathematics in a paper-and-pencil-only environment does not mean that our students need to be confined to such an environment. At one time, that kind of argument was leveled against the use of automobiles, electric lights, and other technologies that we now take for granted.

Teachers will never learn about the promise of the CAS until they try it in their classrooms

Discuss:

- Can students learn something about symbolic manipulation if they start from CAS-generated results?

WHY HAVE SO FEW U.S. STUDENTS HAD THE OPPORTUNITY TO LEARN MATHEMATICS USING A CAS?

The arguments for including CASs in the school curriculum have been offered for almost twenty years (Heid [1983]; Waits and Demana [1996]; McMullin [2001]), and the arguments against them are not insurmountable. So why have U.S. classrooms done so little with CAS calculators?

One reason for so little CAS use in classrooms in the United States is that the public perceives that the work of school mathematics *is* symbolic manipulation. People believe that if the real work of mathematics is executing the skills and if the skills work is relegated to technological assistants, no mathematics would remain for students to learn. The public believes that by-hand symbolic manipulation is important because it is good mental discipline, because it is part of what defines a well-educated person, because it is the next thing after arithmetic, because it is the thing before calculus, because it is the gateway to good jobs and careers, and perhaps even because it sorts those who can do mathematics from those who cannot. The public believes that allowing CASs would eliminate the raison d'être for school algebra.

The CAS can provide automated access to the results of symbolic manipulation so that students can investigate symbolic patterns

Discuss:

- Must by-hand symbolic manipulation precede use of a CAS? What does your answer assume about learning?

A second reason for so little CAS use in U.S. classrooms is that teachers lack examples of curricula that take advantage of CASs. This problem is exacerbated by the shrinking number of conventional publishing companies. One might think that this problem would be mitigated by the continued publication of new curricula that embrace a broader view of mathematics, including but not limited to curricula like the NSF-funded multiyear integrated curriculum projects. Most of these new curricula, however, did not initially view technology as central and are just now reexamining the role of technology in their efforts. The result is that CAS-friendly curricula are not widely available on the U.S. textbook scene.

A third reason for so little CAS use in U.S. classrooms is that finding real trials of curricula that take advantage of CASs is difficult. With little understanding of mathematics or of the potential of CAS calculators to enhance mathematical understanding, some influential politicians eschew their use and establish regulations that prohibit using them on statewide and national assessments. Because of the pressure of tests that prohibit CAS calculator use, teachers are reluctant to accustom their students to using CAS calculators during instruction, as would be necessary if they engaged in robust curriculum trials that used CASs.

Finally, many teachers believe that students will not learn the skills of symbolic algebra if they have regular access to CASs. Studies of the *Concepts in Algebra* (Fey et al. 1995, 1999, 2000) curriculum indicate that students can learn symbolic-manipulation skills quickly when they have a conceptual algebra background but that they do not absorb symbolic manipulation without any study of it. Many mathematics teachers fear the changes that they envision with CAS-enriched classrooms. If the CAS does the symbolic manipulation, they wonder what is left for them to teach.

Discuss:

- How would my algebra curriculum look if my students had constant access to a CAS?

WHY THEN IS CAS NEEDED IN THE SECONDARY MATHEMATICS CURRICULUM?

The overarching rationale for incorporating CASs into school mathematics is the unprecedented learning opportunities that such use would offer students. Algebra is more than symbolic manipulation; it is interpreting algebraic expressions and using algebraic language to describe real and mathematical worlds, it is understanding and using symbols and it is appreciating structure and using symbolic tools to enhance that appreciation. The CAS can help students develop that appreciation and understanding. The CAS can provide automated access to the results of symbolic manipulation so that students can investigate symbolic patterns at a greater level of generality than they have previously encountered in a regular basis.

In "Soundoff! Algemetic," Lin McMullin (2001, p. 85) reminded us to

> Use technology not as a quick way to the answer but as a tool for investigation. Use technology to delve deeper into the problem, not to drive straight through to the answer. Use it as a way to keep focused on the mathematics by avoiding the algemetic. Use the technology correctly and to its fullest extent, not just to add algorithms.

McMullin coined the term *algemetic* to indicate "all the symbol manipulation that the Casio Algebra FX 2.0, HP38, HP48, TI-89, TI-92, Derive, Mathematica, Maple, MathCad, and the like, can do" (2001, p. 84).

The CAS links symbolic, graphical, and numeric representations, thereby offering students new opportunities for exploring concepts through connections among representations.

HOW CAN THE CAS BE USED IN THE SECONDARY MATHEMATICS CURRICULUM?

CASs should be used in U.S. secondary mathematics classrooms for many good reasons; however, the question of how they should be used remains. Several rationales that arise are based on trials worldwide of CAS-enriched secondary school curriculum materials.

Using the CAS to save time

The argument most frequently offered for incorporating CASs as a regular part of school mathematics curricula is that CASs can do more quickly and more accurately the mathematical routines that students spend their mathematical careers mastering, although some of the symbolic manipulation is clearly easier by hand. This statement is even truer today than when the CASs were first introduced. In 1983, I conducted an experiment assessing CAS performance on a standard two-hour calculus examination on which no calculator use was anticipated. The examination required eighteen minutes of computer processing time—and a total of forty-eight minutes of my time—on an Apple II Plus with CP/M card to produce a correct set of responses. I conducted that same experiment recently using a TI-92 Plus calculator and completed the whole examination with similar results in less than fourteen minutes. The calculator processing time was negligible—the vast majority of the time was in deciding on and entering the appropriate expressions and equations.

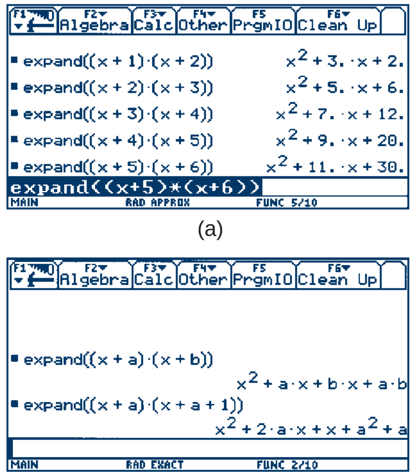
To whatever extent routine symbolic manipulation can be successfully offloaded to the CAS, the time saved can be spent on developing solid conceptual understanding and facility with using mathematics to model and understand mathematical aspects of the real world. Introductory algebra students using the *Computer-Intensive Algebra* curriculum (Fey et al. 1995, 1999, 2000), for example, use a CAS for equation solving, curve fitting, table generating, and graphing so that they can spend their time investigating real-world problems (including problems about exponential growth and decay; about viewing distance; and about revenue, cost, and profit) that they model using families of functions (linear, exponential, and so on). Their understandings of such fundamental algebra concepts as function, equation, variable, equivalence, and systems of equations are solidified in the process.

Discuss:

- What is the role of by-hand symbolic manipulation when students have access to CAS calculators?

Using the CAS to shift the focus to symbolic understanding

CASs can be used as tools for developing a more refined ability to reason about symbols and a deeper understanding of symbolic manipulation. Counting on the presence of CAS calculators in the classroom is not abandoning symbolic reasoning—it is providing a conceptual place at the algebra table for reasoning about symbolic expressions and equations rather than merely changing their form. **Figure 1,** for example, shows a sequence of CAS commands that offers students opportunities to explore the structure of the product of binomials and to examine the relationship of the particular products to the general case. Examining this example, students might first see the pattern in the numerical coefficients, where the constant is the product of two numbers whose sum is the coefficient of the linear term. Using the symbolic capacity of the CAS, students can then generate a proof indicating why this pattern holds. As shown in **figure 1,** a CAS



(b)

Fig. 1
Using a sequence of CAS commands like those shown, students can investigate the structure of binomial products by examining both particular examples and the general case.

expansion of $(x + a)(x + b)$ makes this pattern clear. Further generalization could result from specifying the product as $(x + a)(x + (a + 1))$ and noticing the general form of the coefficients.

Discuss:

- What can students learn from a CAS that produces an unexpected or unintelligible result?

CASs can be used as tools for developing a more refined ability to reason about symbols and a deeper understanding of symbolic manipulation

Over the past few years, colleagues at Pennsylvania State University and University of Iowa have been developing CAS-Intensive Mathematics curriculum materials. The CAS-Intensive Mathematics project is a curriculum development/research project supported by the National Science Foundation under grant No. ESI-96-18029. These materials assume that students have regular access to a CAS and an interactive geometry tool and build on assumed students' understanding of families of functions. They engage students in reasoning about the forms of symbolic expressions and in reasoning about applied situations through their symbolic representations. Students reason, for example, about the effects of the parameters a, b, c, and d on the shape of the graph of a function of the form

$$f(x) = \frac{a}{b + ce^{x}} + d$$

by reasoning about the symbolic expression.

Using the CAS to illuminate symbolic manipulation

The CAS can be used to help students develop a deeper understanding of symbolic manipulation.

Figure 2 illustrates the capacity of a CAS to generate the products of several binomials, thereby giving students the opportunity to notice that the constant term of the product is the product of the constant terms of the factors and that the coefficient of the third term of the product is the sum of the pairwise products of the constants in the binomial factors ($11 = 1 \cdot 2 + 1 \cdot 3 + 2 \cdot 3$; $35 = 1 \cdot 2 + 1 \cdot 3 + 1 \cdot 4 + 2 \cdot 3 + 2 \cdot 4 + 3 \cdot 4$). Students' time can be spent on understanding the origin of the coefficients instead of on producing them.

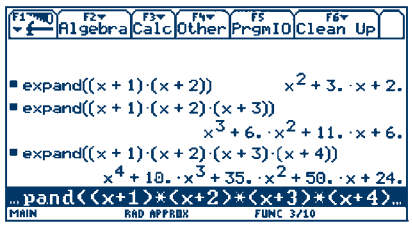
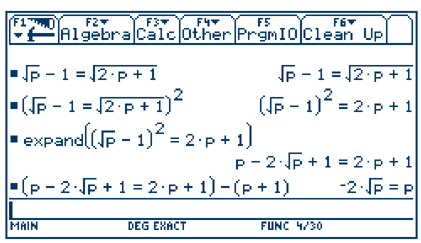


Fig. 2
The CAS can be used to generate the products of several binomials, helping students notice patterns that relate the coefficients of the factors to the coefficients of the product.



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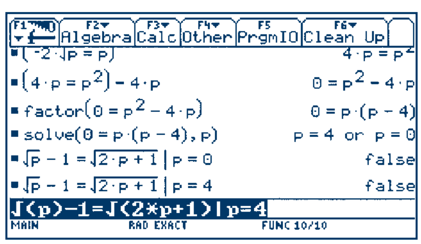

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Fig. 3
The CAS can be used to solve an equation step-by-step. Identifying a sequence of CAS commands for solving an equation helps students learn about the structure of symbolic-manipulation procedures.

Students develop conceptual underpinnings for understanding the role of proof in algebra

Using the CAS to reinforce traditional symbolic manipulation

Individual curriculum trials like that of Edwards (2001) use the CAS to assist students in understanding traditional symbolic manipulation (Heid and Edwards 2001). In those curriculum trials, algebra students used the symbolic calculator to investigate equivalent forms of expressions (for example, by generating equivalent forms of function rules and comparing their graphs), to search for symbolic patterns (for example, by examining "simplified forms" of expressions like x^5x^6 or x^ax^b), and to learn how to solve equations step-by-step. **Figure 3** illustrates the step-by-step solution of a radical equation, including the necessary step of checking the potential roots of 0 and 4. By needing to make decisions about the sequences of commands that provide legitimate steps in a step-by-step technological solution, students can become more aware of the structure of the symbolic-manipulation procedures. Students can learn which procedures to perform without always being required to perform those procedures by hand. At the same time, they develop conceptual underpinnings for understanding the role of proof in algebra.

Using the CAS to enhance students' understanding within a balanced program of work with skills and concepts

The CAS can be used within a balanced program of work with skills and concepts. In large regional efforts like that of the Australian Computer Algebra Systems in Schools, Curriculum, Assessment and Technology Project (CAS-CAT), mathematics educators are investigating the changes that regular access to CAS calculators will have on fourth-year high school mathematics subjects, including curriculum, teaching methods, and associated assessment. These curriculum materials reflect a balanced and progressive development of concepts and skills from each of four areas of study (functions and graphs, algebra, calculus, and probability), with connections among and between the areas.

CONCLUSIONS

The rationales for choosing not to use CASs in classrooms in the United States are weak at best. Many, however, view the entry of the CAS into the classroom as threatening in spite of its considerable promise. The question is, How can we make progress when CAS use is so threatening to so many? Some possible avenues are as follows:

- The public (including politicians) needs to develop a richer understanding of the nature of mathematics. We can imagine the very positive effects of public pressure used to argue for developing our students' ability to think quantitatively. We need to start by identifying the facets of symbolic manipulation that we really want all students to know (for example, understanding how different forms give different information and recognizing different forms of the same expression).

- Throughout the mathematics education community, support is needed for developing and testing curricula that take full advantage of CASs and other technologies.
- Those responsible for mathematics instruction must take on the task of developing a richer understanding of what mathematics can offer—and how students can use technology to learn it.

The time to act is now. The arguments offered year after year against the use of CASs do not hold water. We must quit debating and move ahead with creating CAS-enriched mathematics classrooms for the twenty-first century.

BIBLIOGRAPHY

Edwards, M. Todd. "Computer Symbolic Algebra: Catalysts for Change in Secondary School Classrooms." Unpublished manuscript. Ohio State University, 2000.

Edwards, Michael Todd. "The Electronic 'Other': A Study of Calculator-Based Symbolic Manipulation Utilities with Secondary School Mathematics Students." Doctoral diss., Ohio State University, 2001.

Fey, James T., M. Kathleen Heid, Richard Good, Glendon Blume, Charlene Sheets, and Rose Mary Zbiek. *Concepts in Algebra: A Technological Approach*. Chicago, Ill.: Everyday Learning, 1995, 1999, 2000.

Heid, M. Kathleen. "Calculus with muMath: Implications for Curriculum Reform." *The Computing Teacher* 11 (November 1983): 46–49.

———. "Resequencing Skills and Concepts in Applied Calculus Using the Computer as a Tool." *Journal for Research in Mathematics Education* 19 (January 1988): 3–25.

Heid, M. Kathleen, and M. Todd Edwards. "Computer Algebra Systems: Revolution or Retrofit for Today's Classrooms?" *Theory into Practice* 40 (spring 2001): 128–36.

McMullin, Lin. "Soundoff! Algemetic." *Mathematics Teacher* 94 (February 2001): 84–85.

Palmiter, Jeanette. "Effects of Computer Algebra Systems on Concept and Skill Acquisition in Calculus." *Journal for Research in Mathematics Education* 22 (March 1991): 151–56.

Waits, Bert K., and Franklin Demana. "A Case against Computer Symbolic Manipulation in School Mathematics Today." *Mathematics Teacher* 85 (March 1992): 180–83.

———. "A Computer for All Students—Revisited." *Mathematics Teacher* 89 (December 1996): 712–14.

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