DELVING DEEPER

Fibonacci and Related Sequences

Richard A. Askey

IN A COURSE ON PROOFS THAT IS PRIMARILY FOR PRE-service teachers, a number of problems deal with identities for Fibonacci numbers. We consider some of these identities in this article, and we discuss general strategies used to help discover, prove, and generalize these identities.

We start with a definition of Fibonacci numbers. They satisfy

(1) $F_{n+1} = F_n + F_{n-1},\ n = 1, 2, \ldots,\ F_0 = 0,\ F_1 = 1.$

The next few Fibonacci numbers are $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, and $F_5 = 5$. We encourage readers to make a list of Fibonacci numbers up to F_{10} to use as a reference while reading this article.

We notice that

$$\begin{aligned} F_2F_0 - F_1^2 &= -1; \\ F_3F_1 - F_2^2 &= 2 - 1 = 1; \\ F_4F_2 - F_3^2 &= 3 - 4 = -1; \\ F_5F_3 - F_4^2 &= 10 - 9 = 1. \end{aligned}$$

This result suggests that

(2) $$F_{n+2}F_n - F_{n+1}^2 = (-1)^{n+1}.$$

We should be parsimonious if possible

One way to prove result (2) is by mathematical induction. Since $F_2F_0 - F_1^2 = -1$, the statement is true when $n = 0$. To complete the argument, we need to show that if statement (2) is true for n, it is true for $n + 1$. What do we do with

$$F_{n+3}F_{n+1} - F_{n+2}^2?$$

When I give it as a problem, most students use equation (1) in each term, getting

$$(F_{n+2} + F_{n+1})(F_n + F_{n-1}) - (F_{n+1} + F_n)^2.$$

Although carrying through a proof with this start is possible, doing so takes a great deal of calculation. We also notice that this argument introduces F_{n-1}, which we do not want. Making fewer changes is usually better—we should be parsimonious if possible. F_{n+3} has to be removed, so we try using equation (1) just to replace F_{n+3}:

$$\begin{aligned} F_{n+3}F_{n+1} - F_{n+2}^2 &= (F_{n+2} + F_{n+1})F_{n+1} - F_{n+2}^2 \\ &= F_{n+2}F_{n+1} + F_{n+1}^2 - F_{n+2}^2. \end{aligned}$$

The term F_{n+1}^2 is what we want. The other two terms can be combined to get

$$F_{n+2}(F_{n+1} - F_{n+2}) + F_{n+1}^2 = F_{n+1}^2 - F_{n+2}F_n.$$

Thus,

$$\begin{aligned} F_{n+3}F_{n+1} - F_{n+2}^2 &= -[F_{n+2}F_n - F_{n+1}^2] \\ &= -[(-1)^{n+1}] \\ &= (-1)^{n+2}, \end{aligned}$$

which completes the induction, since we have shown that an initial result is true for $n = 0$.

Some interesting curiosities are suggested by equation (2). For example, $n = 5$ gives

$$13 \cdot 5 - 8^2 = 1.$$

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This result is the basis for an interesting geometric puzzle:

> On a piece of graph paper, draw the following figures. First, draw a square with vertices (0, 0), (8, 0), (0, 8), and (8, 8). Then connect the points (0, 5) and (8, 5), (3, 0) and (5, 5), and (0, 8) and (8, 5). Second, draw a rectangle with vertices at (0, 0), (13, 0), (0, 5), and (13, 5).

Now we rearrange the pieces of the square as shown in the top part of **figure 1.** These pieces seem to exactly cover both the square and the rectangle. But the area of the rectangle is 65 and the area of the square is 64. The seeming paradox is resolved when we notice that the four points that seem to be along the diagonal of the rectangle are not collinear and that the two interior points are not actually on the diagonal. Similar figures can be drawn for each n. The larger n is, the smaller the discrepancy is relative to the size of the square and rectangle.

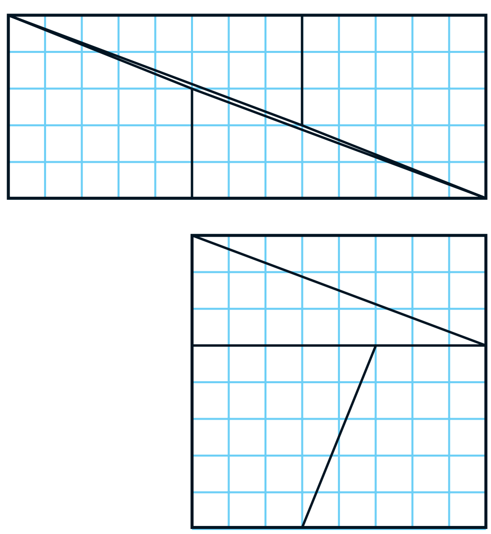

Fig. 1
A seeming paradox

A number of results are analogous to equation (2). Instead of shifting by 1, we can shift by 2. A few examples suggest that

$$(3) \qquad F_{n+4}F_n - F_{n+2}^2 = (-1)^{n+1}.$$

We can use equation (1) on F_{n+4} to get

$$\begin{aligned} F_{n+4}F_n - F_{n+2}^2 &= (F_{n+3} + F_{n+2})F_n - F_{n+2}^2 \\ &= F_{n+3}F_n + F_{n+2}F_n - F_{n+2}^2 \\ &= F_{n+3}F_n + F_{n+2}(-F_{n+1}) \\ &= F_{n+3}F_n - F_{n+2}F_{n+1}. \end{aligned}$$

Thus, equation (3) is true if and only if

$$(4) \qquad F_{n+3}F_n - F_{n+2}F_{n+1} = (-1)^{n+1}.$$

The following is a proof of statement (4):

$$\begin{aligned} F_{n+3}F_n - F_{n+2}F_{n+1} &= (F_{n+2} + F_{n+1})F_n - F_{n+2}F_{n+1} \\ &= F_{n+2}F_n + F_{n+1}(F_n - F_{n+2}) \\ &= F_{n+2}F_n - F_{n+1}^2, \end{aligned}$$

so statement (4) follows from statement (2).

In each of equations (2), (3), and (4), the subscripts in each product add to the same number: $2n + 2$ for (2), $2n + 4$ for (3), and $2n + 3$ for (4). We can use that result to suggest other identities. However, rather than trying to do so now, we start over with a different way of looking at equation (1), which leads to equations (2), (4), and more.

The subscripts in each product add to the same number

Matrices can be used to encode the information in equation (1). One way to do so is as

$$(5) \qquad \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{j+1} \\ F_j \end{bmatrix} = \begin{bmatrix} F_{j+1} + F_j \\ F_{j+1} \end{bmatrix} = \begin{bmatrix} F_{j+2} \\ F_{j+1} \end{bmatrix}.$$

This equation can be iterated to give

$$(6) \qquad \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} F_{j+1} \\ F_j \end{bmatrix} = \begin{bmatrix} F_{n+j+1} \\ F_{n+j} \end{bmatrix}.$$

Next, we use (6) for two columns to get

$$(7) \qquad \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} F_{j+1} & F_1 \\ F_j & F_0 \end{bmatrix} = \begin{bmatrix} F_{n+j+1} & F_{n+1} \\ F_{n+j} & F_n \end{bmatrix}.$$

The determinant of the product of square matrices of the same size is the product of the determinants, so equation (7) implies that

$$(8) \qquad (-1)^n(0 - F_j) = (-1)^{n+1}F_j = F_{n+j+1}F_n - F_{n+j}F_{n+1}.$$

When $j = 1$, the rightmost equation is (2); and when $j = 2$, it is (4).

A further generalization of equation (8) exists. I was not able to prove it by using matrices, but I was able to prove it by a completely different method. It is

$$(9) \qquad F_{n+j+k}F_n - F_{n+j}F_{n+k} = (-1)^{n+1} F_jF_k.$$

I encourage readers to establish this result (or to give it to students as a problem), perhaps by induction. A note giving the details would be a nice contribution to "Delving Deeper."

There are many other identities for Fibonacci numbers. One that is easy to derive is

$$(10) \qquad \begin{aligned} F_{n+1}^2 - F_n^2 &= (F_{n+1} - F_n)(F_{n+1} + F_n) \\ &= (F_{n-1})(F_{n+2}). \end{aligned}$$

If we compute the sum of the squares rather than the difference, a surprise emerges:

n	$F_{n+1}^2 + F_n^2$
0	1
1	2
2	5
3	13

The second column contains Fibonacci numbers, but only half of them. These data suggest that

$$F_{n+1}^2 + F_n^2 = F_{2n+1}. \tag{11}$$

This outcome can be proved using the still-unproved result (9) in two different ways. The easier way is to consider

$$F_{2n+1} - F_{n+1}^2 = F_{2n+1}F_1 - F_{n+1}^2.$$

We next make some substitutions in result (9). We first replace n by 1 and then replace j and k by n. The result is

$$F_{2n+1}F_1 - F_{n+1}^2 = F_n^2.$$

Hence,

$$F_{2n+1} - F_{n+1}^2 = F_n^2$$

and

$$F_{2n+1} = F_n^2 + F_{n+1}^2.$$

A more interesting way to prove equation (11) starts with the left-hand side. We first need to extend the definition of Fibonacci numbers to negative values of n.

One way to do so is to force definition (1) to work for negative values of n. For example, if we require equation (1), that is,

$$F_{n+1} = F_n + F_{n-1},\ n = 1, 2, \ldots, F_0 = 0, F_1 = 1,$$

for $n = 0$, we get $F_1 = F_0 + F_{-1}$, or $1 = 0 + F_{-1}$, making $F_{-1} = 1$. Then applying equation (1) to $n = -1$, we have $F_0 = F_{-1} + F_{-2}$, or $0 = 1 + F_{-2}$, so that $F_{-2} = -1$. Next, we let $n = -2$, and we get $F_{-1} = F_{-2} + F_{-3}$, or $1 = -1 + F_{-3}$, so that $F_{-3} = 2$. If we keep working down in this way, defining F_{-n} by

$$F_{-n} = (-1)^{n+1}F_n,\ n = 0, 1, \ldots \tag{12}$$

seems to extend equation (1) to all negative integers. This result is valid, and it can be proved by induction.

Using (12) as a definition, we have

$$\begin{aligned} F_{n+1}^2 + F_n^2 &= (-1)^n F_{n+1}F_{-n-1} + (-1)^{n+1}F_nF_{-n} \\ &= (-1)^{n+1}(F_{-n}F_n - F_{n+1}F_{-n-1}). \end{aligned}$$

Then we use result (9) when $j = 1$ and $k = -2n - 1$ to get

$$\begin{aligned} F_{n+1}^2 + F_n^2 &= (-1)^{n+1}(-1)^{n+1}F_1F_{-2n-1} \\ &= F_1F_{2n+1} \\ &= F_{2n+1}. \end{aligned}$$

We used result (9) when $k < 0$. Readers should verify that this use is valid.

Lucas numbers were mentioned in the title of this article, but we have not yet introduced them. Lucas numbers are related to Fibonacci numbers. They are denoted by L_n, and they satisfy

$$\begin{aligned} L_{n+1} &= L_n + L_{n-1},\ n = 1, 2, \ldots, \\ L_0 &= 2, \\ L_1 &= 1. \end{aligned} \tag{13}$$

Everything that we have done has analogues for Lucas numbers, and five results analogous to (9) involve Lucas numbers and Fibonacci numbers. They will be discussed in a future article, but readers might try to figure out what they are and how to prove them. As a hint, one extension of expression (2) involves

$$L_{n+2}L_n - 5F_{n+1}^2.$$

EDITORS' NOTE

The whole topic of Fibonacci numbers is full of interesting facts and identities that have fascinated people for centuries. Richard Askey has provided us with a glimpse of some of these beautiful facts and, more important, with a collection of strategies for discovering and proving some of them. He promises further articles on this topic that will make connections with trigonometry, complex numbers, and more. And we invite other readers to write up their own Fibonacci investigations for publication in "Delving Deeper." In addition to the topics suggested in Askey's article, here are some others to think about:

- The Fibonacci numbers are integers, so we can investigate arithmetic properties. For example, what can we say about the greatest common divisor of two Fibonacci numbers?
- Askey defines the Fibonacci numbers recursively. Is there a closed form that produces them?
- Do any connections exist between Fibonacci numbers and other topics that have been discussed in "Delving Deeper"? Pythagorean triples? Pascal's triangle?

Another question is raised by a statement in Askey's article: "The determinant of the product of square matrices of the same size is the product of the determinants." An article that delves into this fact could be interesting.

References about Fibonacci numbers abound. Some good ones are the following:

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The June 2003 issue of *Mathematics Magazine* has three interesting articles about Fibonacci numbers, and the book *Concrete Mathematics* by Donald Knuth (Addison Wesley, 1989) has an extensive chapter on Fibonacci numbers. Once again, we invite readers to send us their favorite Fibonacci references. The following is an abbreviated bibliography of materials that have appeared in the *Mathematics Teacher*.

FURTHER READINGS

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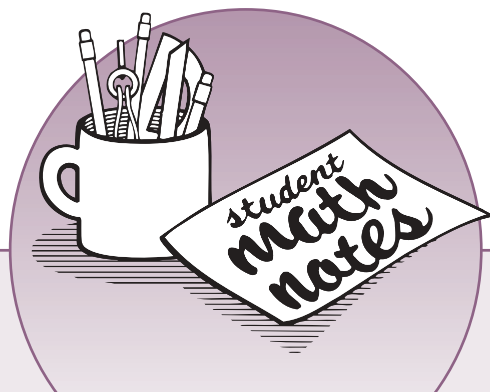

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